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ON φ-INVARIANT SASAKI-LIKE STATISTICAL SUBMERSIONS

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Abstract. We discuss Sasaki-like statistical submersions such that the structure vector ξ is horizontal or vertical, and each fiber is φ-invariant. We give some examples of Sasaki-like statistical manifolds and Sasaki-like statistical submersions.

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1. Introduction

Statistical models in information geometry have a Fisher metric as a Riemannian metric, and admit a torsion-free affine connection which is constructed from expectations of derivatives of a probability density ([3], [4]). This affine connection is called an α-connection, denoted by $\nabla^{(\alpha)}$, where α is a real number, and conjugate relative to the Fisher metric is a $(-\alpha)$-connection. The 0-connection is a Levi-Civita connection with respect to the Fisher metric. Particularly, $\nabla^{(1)}$ (resp. $\nabla^{(-1)}$) is said to be an exponential connection (resp. mixture connection) or e-connection (resp. m-connection) simply and denoted by $\nabla^{(\mathrm{e})}$ (resp. $\nabla^{(\mathrm{m})}$). The statistical model of an exponential family (resp. mixture family) is 1-flat (resp. (-1)-flat). The e-connection and m-connection are dual with respect to the Fisher metric. The concept of dual connection is very important in information geometry.

Let (M, g) and ∇ be a (semi-)Riemannian manifold and a torsion-free affine connection. A statistical manifold is a smooth manifold with a statistical structure (g, ∇), and denoted by (M, g, ∇). We define another affine connection ∇^* with respect to g, and said to be conjugate (or dual). Then (g, ∇^*) is a statistical structure, and (M, g, ∇^*) is a statistical manifold, too. In [13], Noguchi studied statistical manifolds.

Let M and B be two (semi-)Riemannian manifolds of class C^∞. A (semi-)Riemannian submersion $\pi : M \to B$ is a mapping of M onto B such that π has maximal rank and π_* preserves lengths of horizontal vectors ([6], [9], [11]). A (semi-)Riemannian submersion π is said to be an almost Hermitian submersion, if M and B are almost Hermitian manifolds and commutes with almost complex structures. Especially, we say that π is a Kählerian submersion if M is a Kählerian manifold [21].

There are many studies of manifolds with geometric structures such as almost complex structures and almost contact structures. In a sense, we can define dual another geometric structures with respect to these geometric structures. In [15], we defined a Kähler-like statistical manifold similar to Kählerian manifold and studied statistical submersion which the total space is a Kähler-like statistical manifold (M, g, ∇, J) and each fiber is J-invariant submanifold of M. The concept of statistical submersion was defined by Abe and Hasegawa [1]. Also, we defined an analogy of a Sasakian structure on the statistical manifold [16]. We studied the Sasaki-like statistical submersion that the total space is a Sasaki-like statistical manifold (M, g, ∇) with geometric structure (φ, ξ, η), each fiber is φ-invariant submanifold of M and tangent to the vector ξ.

In [8], Furuhata and Hasegawa studied submanifolds of holomorphic statistical manifolds. Recently, we considered anti-holomorphic statistical submersion [10]. Also, we studied locally product-like statistical manifolds and their hypersurfaces [7], locally product-like statistical submersions [17], and generalized Kähler-like statistical submersion [18]. Moreover, the following papers study statistical submersions with other geometric structures: cosymplectic-like statistical submersions [5], quaternionic Kähler-like statistical submersions [19], para-Kähler-like statistical submersions [20], Kenmotsu-like statistical submersions [14], etc.

2. Preliminaries

An m-dimensional semi-Riemannian manifold is a smooth manifold M^m furnished with a metric tensor g, where g is a symmetric nondegenerate tensor field on M of constant index. The common value ν of index g on M is called the index of M $(0 \leq \nu \leq m)$ and we denote a semi-Riemannian manifold by M^m_ν. If $\nu = 0$, then M is a Riemannian manifold. For each $p \in M$, a tangent vector E to M is spacelike (resp. null, timelike) if $g(E, E) > 0$ or $E = 0$ (resp. $g(E, E) = 0$ and $E \neq 0$, $g(E, E) < 0$). Let $\mathbb{R}^m_\nu$ be an m-dimensional real vector space with an inner product of signature $(\nu, m - \nu)$ given by

$$\langle x, x \rangle = -\sum_{i=1}^{\nu} x_i^2 + \sum_{i=\nu+1}^{m} x_i^2, \tag{2.1}$$

where $x = (x_1, \ldots, x_m)$ is the natural coordinate of $\mathbb{R}^m_\nu$. $\mathbb{R}^m_\nu$ is called an m-dimensional semi-Euclidean space. If $\nu = 0$ (resp. $\nu = 1$), then $\mathbb{R}^m$ (resp. $\mathbb{R}^m_1$) is an Euclidean space (resp. a Lorentzian space).

Let M be a semi-Riemannian manifold. Denote a torsion-free affine connection by ∇. The triple (M, g, ∇) is called a statistical manifold if ∇g is symmetric. For the statistical manifold (M, g, ∇), we define another affine connection ∇^* by

$$Eg(F, G) = g(\nabla_E F, G) + g(F, \nabla^*_E G), \tag{2.2}$$

for vector fields E, F and G on M. The affine connection ∇^* is called conjugate (or dual) to ∇ with respect to g. The affine connection ∇^* is torsion-free, $\nabla^* g$ is symmetric and satisfies $(\nabla^*)^* = \nabla$. Clearly, the triple (M, g, ∇^*) is statistical. We denote by R and R^* the curvature tensors on M with respect to the affine connection ∇ and its conjugate ∇^*, respectively. Then we find

$$g(R(E,F)G,H) = -g(G,R^*(E,F)H), \tag{2.3}$$

for any vector fields E, F, G and H on M, where $R(E,F)G = [\nabla_E, \nabla_F]G - \nabla_{[E,F]}G$. Therefore R vanishes identically if and only if so is R^*. We call flat if R vanishes identically. If the curvature tensor R with respect to the affine connection ∇ satisfies

$$R(E,F)G = c\,\{\,g(F,G)E - g(E,G)F\,\}, \tag{2.4}$$

then the statistical manifold (M, g, ∇) is called a space of constant curvaure c. The triple (M, g, ∇) is of constant curvature c if and only if so is (M, g, ∇^*).

We denote by the local orthonomal basis of T_pM for each $p \in M$ by $\{E_1, \ldots, E_m\}$. We define the Ricci tensor of the affine connection ∇ by

$$\mathrm{Ric}(E,F) = \sum_{A=1}^{m} \varepsilon_A g(R(E_A, E)F, E_A),$$

where $\varepsilon_A = g(E_A, E_A) = -1$ or $+1$ according as E_A is timelike or spacelike. If the Ricci tensor satisfies

$$\mathrm{Ric}(E,F) = k\,g(E,F), \tag{2.5}$$

where k is a constant, then (M, g, ∇) is called Einstein.

Let M be a smooth manifold with a tensor field J of type $(1,1)$ on M such that

$$J^2 = -I, \tag{2.6}$$

where I stands for the identity transformation. Then we say that M is an almost complex manifold with almost complex structure J. An almost complex manifold is necessarily orientable and must have an even dimension. We consider the semi-Riemannian manifold on the almost complex manifold M. If J preserves the metric g, that is,

$$g(JE, JF) = g(E,F) \tag{2.7}$$

for vector fields E and F on M, then (M, g, J) is called an almost Hermitian manifold. Now, we consider the semi-Riemannian manifold (M, g) with the almost complex structure J which has another tensor field J^* of type $(1,1)$ satisfying

$$g(JE, F) + g(E, J^*F) = 0 \tag{2.8}$$

for any vector fields E and F. Then the triple (M, g, J) is called an almost Hermite-like manifold.

We see that $(J^*)^* = J$, $(J^*)^2 = -I$ and

$$g(JE, J^*F) = g(E,F). \tag{2.9}$$

Lemma 2.1. [15] *The triple (M, g, J) is an almost Hermite-like manifold if and only if so is (M, g, J^*).*

Next, if J is parallel with respect to the affine connection ∇, then (M, g, ∇, J) is called a Kähler-like statistical manifold. From (2.8), we get

$$g((\nabla_G J)E, F) + g(E, (\nabla^*_G J^*)F) = 0, \tag{2.10}$$

for any vector fields E, F and G on M. Hence we have (see [15]) the following

Lemma 2.2. *(M, g, ∇, J) is a Kähler-like statistical manifold if and only if so is (M, g, ∇^*, J^*).*

Remark 2.3. Let (M, g, ∇, J) be a Kähler-like statistical manifold. If M is of constant curvature c with respect to the affine connection ∇, then $c = 0$ ($\dim M \geq 4$), that is, M is flat [22].

We put
$$S_E F = \nabla_E F - \nabla^*_E F,$$
for $E, F \in TM$. Then $S_E F = S_F E$ and $g(S_E F, G) = g(F, S_E G)$ hold. If the curvature tensor R satisfies

$$R(E,F)G = \frac{c}{4}\,[\, g(F,G)E - g(E,G)F - g(F,JG)JE + g(E,JG)JF + \{\, g(E,JF) - g(JE,F)\}JG\,], \tag{2.11}$$

then the Kähler-like statistical manifold is called a space of constant holomorphic sectional curvature c. The curvature tensor R satisfies $R(E,F)JG = JR(E,F)G$ and the Bianchi's 1st identitiy. We put
$$(\nabla_D R)(E,F)G = \nabla_D\{R(E,F)G\} - R(\nabla_D E, F)G - R(E, \nabla_D F)G - R(E,F)\nabla_D G.$$
Then it is easy to see from (2.11) that
$$(\nabla_D R)(E,F)G = -\frac{c}{4}\,[\, g(S_D F,G)E - g(S_D E,G)F - g(S_D F,JG)JE + g(S_D E,JG)JF + \{\, g(S_D E,JF) - g(JE,S_D F)\}JG\,]$$
holds, which implies that the curvature tensor R satisfies the Bianchi's 2nd identity. Moreover, we have from (2.3)

$$R^*(E,F)G = \frac{c}{4}\,[\, g(F,G)E - g(E,G)F - g(F,J^*G)J^*E + g(E,J^*G)J^*F + \{\, g(E,J^*F) - g(J^*E,F)\}J^*G\,]. \tag{2.12}$$

Then the Kähler-like statistical manifold (M, g, ∇^*, J^*) is called a space of constant holomorphic sectional curvature c. (M, g, ∇, J) is a space of constant holomorphic sectional curvature c if and only if so is (M, g, ∇^*, J^*).

Remark 2.4. If M is a Kählerian manifold, then M satisfying (2.11) is a space of constant holomorphic sectional curvature c (see [22]).

Next, let M be a $(2n+1)$-dimensional manifold and φ, ξ, η be a tensor field of type $(1,1)$, a vector field, a 1-form on M respectively. If φ, ξ and η satisfy the following conditions

$$\eta(\xi) = 1, \qquad \varphi^2 E = -E + \eta(E)\xi, \tag{2.13}$$

for any vector field E on M, then M is said to have an almost contact structure (φ, ξ, η) and is called an almost contact manifold. We find

$$\varphi\xi = 0, \qquad \eta(\varphi E) = 0. \tag{2.14}$$

Example 2.5. Let $\mathbb{R}^3$ be a smooth manifold with local coordinate system (x_1, x_2, x_3) and
$$\varphi = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & x_2 & 0 \end{pmatrix}, \qquad \xi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \eta = (-x_2, 0, 1).$$

Then $\mathbb{R}^3$ is an almost contact manifold with an almost contact structure (φ, ξ, η). It is easy to see that $\eta \wedge d\eta = dx_1 \wedge dx_2 \wedge dx_3 \,(\neq 0)$, which means that η is a contact structure.

Example 2.6. Let $\mathbb{R}^5$ be a smooth manifold with local coordinate system $(x_1, x_2, x_3, x_4, x_5)$ and

$$\varphi = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ x_2 & 0 & x_4 & 0 & 0 \end{pmatrix}, \qquad \xi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \eta = (-x_4, 0, x_2, 0, 1).$$

Then $\mathbb{R}^5$ is an almost contact manifold with an almost contact structure (φ, ξ, η). It is easy to see that

$$\eta \wedge (d\eta)^2 = 2dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5 \neq 0,$$

which means that η is a contact structure.

Moreover, if we put

$$\eta(E) = \varepsilon g(\xi, E), \tag{2.15}$$

then we get $g(\xi, \xi) = \varepsilon$, where $\varepsilon = -1$ or $+1$ according as ξ is timelike or spacelike, respectively.

Now, we consider the semi-Riemannian manifold (M, g) with the almost contact structure (φ, ξ, η) which has another tensor field φ^* of type $(1, 1)$ satisfying

$$g(\varphi E, F) + g(E, \varphi^* F) = 0, \tag{2.16}$$

for any vector fields E and F. Then the pair (M, g) is called an almost contact metric manifold with almost contact structure (φ, ξ, η). We see that $(\varphi^*)^* = \varphi$, $(\varphi^*)^2 E = -E + \eta(E)\xi$, $\varphi^* \xi = 0$, $\eta(\varphi^* E) = 0$ and

$$g(\varphi E, \varphi^* F) = g(E, F) - \varepsilon \eta(E)\eta(F). \tag{2.17}$$

Lemma 2.7. [16] *The pair (M, g) is an almost contact metric manifold with almost contact structure (φ, ξ, η) if and only if so is (M, g) with (φ^*, ξ, η).*

Next, we give two examples of the almost contact metric manifold.

Example 2.8. We put $M^3 = \{(x_1, x_2, x_3) \mid -\infty < x_i < \infty \, (i = 1, 2, 3)\} = \mathbb{R}^3$ with an almost contact structure (φ, ξ, η) of Example 2.5 and

$$g = \begin{pmatrix} \varepsilon x_2^2 & 1 & -\varepsilon x_2 \\ 1 & 1 & 0 \\ -\varepsilon x_2 & 0 & \varepsilon \end{pmatrix},$$

then (M, g) is an almost contact metric manifold with almost contact structure (φ, ξ, η) and (M, g) is with almost contact structure (φ^*, ξ, η), where

$$\varphi^* = \begin{pmatrix} -1 & -2 & 0 \\ 1 & 1 & 0 \\ -x_2 & -2x_2 & 0 \end{pmatrix}.$$

We notice that $\det g = -\varepsilon$, and the signature of g is $(1, 2)$ if ξ is spacelike, is $(2, 1)$ if ξ is timelike.

Example 2.9. We put $M^5 = \{(x_1, x_2, x_3, x_4, x_5) \,|\, x_2 > 0,\, x_4 > 0\} \subset \mathbb{R}^5$ with an almost contact structure (φ, ξ, η) of Example 2.6 and

$$g = \begin{pmatrix} \varepsilon x_4^2 & 0 & 0 & 0 & -\varepsilon x_4 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon x_2^2 & 1 & \varepsilon x_2 \\ 0 & 0 & 1 & 1 & 0 \\ -\varepsilon x_4 & 0 & \varepsilon x_2 & 0 & \varepsilon \end{pmatrix},$$

then (M, g) is an almost contact metric manifold with almost contact structure (φ, ξ, η) and (M, g) is with almost contact structure (φ^*, ξ, η), where

$$\varphi^* = \begin{pmatrix} \frac{\varepsilon}{x_2 x_4} & -\frac{\varepsilon}{x_2 x_4} & -1 & \frac{1}{x_2^2 x_4^2} & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & \frac{\varepsilon}{x_2 x_4} & 0 \\ -1 & 1 & 0 & -\frac{\varepsilon}{x_2 x_4} & 0 \\ -\frac{x_2^2 - \varepsilon}{x_2} & -\frac{\varepsilon}{x_2} & -x_4 & -\frac{\varepsilon x_2^2 - 1}{x_2^2 x_4} & 0 \end{pmatrix}.$$

Also, we find $\det g = \varepsilon x_2^2 x_4^2$.

The triple (M, g, ∇) is called a Sasaki-like statistical manifold with Sasaki-like structure (φ, ξ, η) satisfying

$$\nabla_E \xi = -\varepsilon \varphi E, \tag{2.18}$$

$$(\nabla_E \varphi) F = g(E, F)\xi - \varepsilon \eta(F) E. \tag{2.19}$$

It is clear from $\eta(\varphi F) = 0$ and (2.19) that $g(\nabla^*_E \xi, \varphi F) + \varepsilon g(E, F) - \eta(E)\eta(F) = 0$, which yields that $\nabla^*_E \xi = -\varepsilon \varphi^* E$. From (2.16), we get

$$g((\nabla_G \varphi) E, F) + g(E, (\nabla^*_G \varphi^*) F) = 0, \tag{2.20}$$

which means that $(\nabla^*_G \varphi^*) F = g(G, F)\xi - \varepsilon \eta(F) G$. Hence we have

Lemma 2.10. *The triple (M, g, ∇) is a Sasaki-like statistical manifold with Sasaki-like structure (φ, ξ, η) if and only if (M, g, ∇^*) is with Sasaki-like structure (φ^*, ξ, η).*

We give two examples of Sasaki-like statistical manifold.

Example 2.11. Let (M, g) be an almost contact metric manifold with almost contact structure (φ, ξ, η) of Example 2.8. We put the affine connection ∇ as follows:

$$\begin{aligned} &\nabla_{\partial_1} \partial_1 = -2\varepsilon x_2 \partial_2 + \partial_3, \\ &\nabla_{\partial_1} \partial_2 = \nabla_{\partial_2} \partial_1 = \varepsilon x_2 \partial_1 + \varepsilon x_2^2 \partial_3, \\ &\nabla_{\partial_1} \partial_3 = \nabla_{\partial_3} \partial_1 = \varepsilon \partial_2, \\ &\nabla_{\partial_2} \partial_2 = -\partial_3, \\ &\nabla_{\partial_2} \partial_3 = \nabla_{\partial_3} \partial_2 = -\varepsilon \partial_1 - \varepsilon x_2 \partial_3, \\ &\nabla_{\partial_3} \partial_3 = 0, \end{aligned}$$

where $\partial_i = \partial/\partial x_i \, (i = 1, 2, 3)$ and $\xi = \partial_3$. Then we find

$$\begin{aligned}
&\nabla^*_{\partial_1}\partial_1 = -2\varepsilon x_2\partial_1 + 2\varepsilon x_2\partial_2 - (2\varepsilon x_2^2 + 1)\partial_3,\\
&\nabla^*_{\partial_1}\partial_2 = \nabla^*_{\partial_2}\partial_1 = -2\varepsilon x_2\partial_1 + \varepsilon x_2\partial_2 - (2\varepsilon x_2^2 + 1)\partial_3,\\
&\nabla^*_{\partial_1}\partial_3 = \nabla^*_{\partial_3}\partial_1 = \varepsilon\partial_1 - \varepsilon\partial_2 + \varepsilon x_2\partial_3,\\
&\nabla^*_{\partial_2}\partial_2 = \partial_3,\\
&\nabla^*_{\partial_2}\partial_3 = \nabla^*_{\partial_3}\partial_2 = 2\varepsilon\partial_1 - \varepsilon\partial_2 + 2\varepsilon x_2\partial_3,\\
&\nabla^*_{\partial_3}\partial_3 = 0.
\end{aligned}$$

Therefore (M, g, ∇) is a Sasaki-like statistical manifold with Sasaki-like structure (φ, ξ, η) and (M, g, ∇^*) is with Sasaki-like structure (φ^*, ξ, η).

In a Sasaki-like statistical manifold (M, g, ∇) of Example 2.11, if we put

$$X_1 = \partial_1 - \partial_2 + x_2\partial_3, \qquad X_2 = \partial_2, \qquad X_3 = \xi = \partial_3,$$

then $\{X_1, X_2, X_3\}$ is an orthonormal basis such that $g(X_1, X_1) = -1$, $g(X_2, X_2) = 1$, $g(X_3, X_3) = \varepsilon$, that is, X_1 is timelike and X_2 is spacelike. Thus we have

Example 2.12. The affine connections ∇ and ∇^* are rewritten as follows:

$$\begin{aligned}
&\nabla_{X_1}X_1 = \nabla_{X_2}X_2 = -X_3,\\
&2\nabla_{X_1}X_2 = \nabla_{X_2}X_1 = 2X_3,\\
&\nabla_{X_1}X_3 = \nabla_{X_3}X_1 = \varepsilon(X_1 + 2X_2),\\
&\nabla_{X_2}X_3 = \nabla_{X_3}X_2 = -\varepsilon(X_1 + X_2),\\
&\nabla_{X_3}X_3 = 0
\end{aligned}$$

and

$$\begin{aligned}
&\nabla^*_{X_1}X_1 = \nabla^*_{X_2}X_2 = X_3,\\
&\nabla^*_{X_1}X_2 = 2\nabla^*_{X_2}X_1 = -2X_3,\\
&\nabla^*_{X_1}X_3 = \nabla^*_{X_3}X_1 = -\varepsilon(X_1 + X_2),\\
&\nabla^*_{X_2}X_3 = \nabla^*_{X_3}X_2 = \varepsilon(2X_1 + X_2),\\
&\nabla^*_{X_3}X_3 = 0.
\end{aligned}$$

Also, we get $\varphi X_1 = -X_1 - 2X_2$, $\varphi X_2 = X_1 + X_2$ and $\varphi X_3 = 0$.

Example 2.13. Let (M, g) be an almost contact metric manifold with almost contact structure (φ, ξ, η) of Example 2.9. We put the affine connection ∇ as follows:

$$
\begin{aligned}
&\nabla_{\partial_1}\partial_1 = -\varepsilon x_2\partial_1 - \varepsilon x_4\partial_3 - \varepsilon x_4\partial_4 + \varepsilon x_2 x_4\partial_5,\\
&\nabla_{\partial_1}\partial_2 = \nabla_{\partial_2}\partial_1 = -\varepsilon x_4\partial_4,\\
&\nabla_{\partial_1}\partial_3 = \nabla_{\partial_3}\partial_1 = \nabla_{\partial_1}\partial_4 = \nabla_{\partial_4}\partial_1 = \varepsilon x_4\partial_2,\\
&\nabla_{\partial_1}\partial_5 = \nabla_{\partial_5}\partial_1 = \varepsilon\partial_3 - \varepsilon x_2\partial_5,\\
&\nabla_{\partial_2}\partial_2 = \nabla_{\partial_4}\partial_4 = \nabla_{\partial_5}\partial_5 = 0,\\
&\nabla_{\partial_2}\partial_3 = \nabla_{\partial_3}\partial_2 = \varepsilon x_2\partial_4 + \partial_5,\\
&\nabla_{\partial_2}\partial_4 = \nabla_{\partial_4}\partial_2 = \partial_5,\\
&\nabla_{\partial_2}\partial_5 = \nabla_{\partial_5}\partial_2 = \varepsilon\partial_4,\\
&\nabla_{\partial_3}\partial_3 = -\varepsilon x_2\partial_1 - \varepsilon x_4\partial_3 + \varepsilon x_4\partial_4 - \varepsilon x_2 x_4\partial_5,\\
&\nabla_{\partial_3}\partial_4 = \nabla_{\partial_4}\partial_3 = -\varepsilon x_2\partial_2,\\
&\nabla_{\partial_3}\partial_5 = \nabla_{\partial_5}\partial_3 = -\varepsilon\partial_1 - \varepsilon x_4\partial_5,\\
&\nabla_{\partial_4}\partial_5 = \nabla_{\partial_5}\partial_4 = -\varepsilon\partial_2,
\end{aligned}
$$

where $\partial_i = \partial/\partial x_i\,(i = 1, 2, 3, 4, 5)$ and $\xi = \partial_5$. Then we find

$$
\begin{aligned}
&\nabla^*_{\partial_1}\partial_1 = \frac{\varepsilon(x_2^2+\varepsilon)}{x_2}\partial_1 + \varepsilon x_4\partial_3 - \varepsilon x_4\partial_4 - \frac{\varepsilon x_4(x_2^2-\varepsilon)}{x_2}\partial_5,\\
&\nabla^*_{\partial_1}\partial_2 = \nabla^*_{\partial_2}\partial_1 = \varepsilon x_4\partial_4,\\
&\nabla^*_{\partial_1}\partial_3 = \nabla^*_{\partial_3}\partial_1 = \frac{\varepsilon}{x_2^2 x_4}\partial_1 - \varepsilon x_4\partial_2 + \frac{1}{x_2}\partial_3 - \frac{1}{x_2}\partial_4 - \frac{x_2^2-\varepsilon}{x_2^2}\partial_5,\\
&\nabla^*_{\partial_1}\partial_4 = \nabla^*_{\partial_4}\partial_1 = \frac{x_2^2+\varepsilon}{x_2^2 x_4}\partial_1 - \varepsilon x_4\partial_2 + \frac{1}{x_2}\partial_3 - \frac{1}{x_2}\partial_4 - \frac{x_2^2-\varepsilon}{x_2^2}\partial_5,\\
&\nabla^*_{\partial_1}\partial_5 = \nabla^*_{\partial_5}\partial_1 = -\frac{1}{x_2 x_4}\partial_1 - \varepsilon\partial_3 + \varepsilon\partial_4 + \frac{\varepsilon(x_2^2-\varepsilon)}{x_2}\partial_5,\\
&\nabla^*_{\partial_2}\partial_2 = \nabla^*_{\partial_4}\partial_4 = \nabla^*_{\partial_5}\partial_5 = 0,\\
&\nabla^*_{\partial_2}\partial_3 = \nabla^*_{\partial_3}\partial_2 = \frac{x_2^2+\varepsilon}{x_2^2 x_4}\partial_1 + \frac{1}{x_2}\partial_3 - \frac{\varepsilon(x_2^2+\varepsilon)}{x_2}\partial_4 + \frac{\varepsilon}{x_2^2}\partial_5,\\
&\nabla^*_{\partial_2}\partial_4 = \nabla^*_{\partial_4}\partial_2 = -\partial_5,\\
&\nabla^*_{\partial_2}\partial_5 = \nabla^*_{\partial_5}\partial_2 = \frac{1}{x_2 x_4}\partial_1 - \varepsilon\partial_4 + \frac{1}{x_2}\partial_5,\\
&\nabla^*_{\partial_3}\partial_3 = \varepsilon x_2\partial_1 + 2\varepsilon x_2\partial_2 + \varepsilon x_4\partial_3 - \varepsilon x_4\partial_4 + \varepsilon x_2 x_4\partial_5,
\end{aligned}
$$

$$\nabla^*_{\partial_3}\partial_4 = \nabla^*_{\partial_4}\partial_3 = \varepsilon x_2\partial_2,$$

$$\nabla^*_{\partial_3}\partial_5 = \nabla^*_{\partial_5}\partial_3 = \varepsilon\partial_1 + \varepsilon\partial_2 + \varepsilon x_4\partial_5,$$

$$\nabla^*_{\partial_4}\partial_5 = \nabla^*_{\partial_5}\partial_4 = -\frac{\varepsilon}{x_2^2x_4^2}\partial_1 + \varepsilon\partial_2 - \frac{1}{x_2x_4}\partial_3 + \frac{1}{x_2x_4}\partial_4 + \frac{x_2^2-\varepsilon}{x_2^2x_4}\partial_5.$$

Therefore (M, g, ∇) is a Sasaki-like statistical manifold with Sasaki-like structure (φ, ξ, η) and (M, g, ∇^*) is with Sasaki-like structure (φ^*, ξ, η).

For any vector fields E, F, G on the Sasaki-like statistical manifold, we obtain

(2.21) $\quad R(E,F)\xi = \eta(F)E - \eta(E)F,$

(2.22) $\quad R(E,F)\varphi G - \varphi R(E,F)G = \varepsilon\{g(F,\varphi G)E - g(E,\varphi G)F - g(F,G)\varphi E + g(E,G)\varphi F\},$

where we used $\eta(S_E F) = -g(\varphi E, F) - g(E, \varphi F)$. From (2.21) or (2.22), we have

Lemma 2.14. *Let (M, g, ∇) be a Sasaki-like statistical manifold with Sasaki-like structure (φ, ξ, η). If (M, g, ∇) is of constant curvature c, then $c = \varepsilon$, that is,*

$$R(E,F)G = \varepsilon\{g(F,G)E - g(E,G)F\}.$$

On the Sasaki-like statistical manifold, we consider

$$\begin{aligned}
R(E,F)G = {} & \frac{1}{4}(c+3\varepsilon)\{g(F,G)E - g(E,G)F\} \\
& + \frac{1}{4}(c-\varepsilon)[\varepsilon\eta(G)\{\eta(E)F - \eta(F)E\} + \{g(E,G)\eta(F) - g(F,G)\eta(E)\}\xi \\
& \quad - g(F,\varphi G)\varphi E + g(E,\varphi G)\varphi F + \{g(E,\varphi F) - g(\varphi E, F)\}\varphi G],
\end{aligned} \tag{2.23}$$

where c is a constant [2]. If the curvature tensor R satisfies (2.24), then the Sasaki-like statistical manifold (M, g, ∇) with Sasaki-like structure (φ, ξ, η), or (M, g, ∇) simply is called a space of constant φ-holomorphic sectional curvature c. The curvature tensor R satisfies (2.21), (2.22) and the Bianchi's 1st identity. If $c = \varepsilon$, then the Sasaki-like statistical manifold is of constant curvature ε. It is easy to see from (2.23) and $\eta(S_D E) = -g(\varphi D, E) - g(D, \varphi E)$ that

$$\begin{aligned}
& (\nabla_D R)(E,F)G \\
= {} & \frac{1}{4}(c+3\varepsilon)\{g(S_D E, G)F - g(S_D F, G)E\} \\
& + \frac{1}{4}(c-\varepsilon)[\varepsilon g(D,\varphi G)\{\eta(E)F - \eta(F)E\} + \varepsilon\{g(F,\varphi G)\eta(E) - g(E,\varphi G)\eta(F)\}D \\
& \quad + \varepsilon\eta(G)\{g(D,\varphi E)F - g(D,\varphi F)E - g(E,\varphi F)D + g(F,\varphi E)D\} \\
& \quad + \varepsilon g(D,G)\{\eta(E)\varphi F - \eta(F)\varphi E\} + \varepsilon\{g(F,G)\eta(E) - g(E,G)\eta(F)\}\varphi D \\
& \quad + \varepsilon\eta(G)\{g(D,F)\varphi E - g(D,E)\varphi F\} + 2\varepsilon\{g(D,F)\eta(E) - g(D,E)\eta(F)\}\varphi G \\
& \quad + \{g(E,G)g(D,\varphi F) - g(F,G)g(D,\varphi E) + g(D,G)g(E,\varphi F) \\
& \qquad - g(D,G)g(F,\varphi E) - g(D,E)g(F,\varphi G) + g(D,F)g(E,\varphi G)\}\xi \\
& \quad - g(S_D E, \varphi G)\varphi F + g(S_D F, \varphi G)\varphi E - \{g(S_D E, \varphi F) - g(S_D F, \varphi E)\}\varphi G \\
& \quad - \{g(S_D E, G)\eta(F) - g(S_D F, G)\eta(E)\}\xi\,]
\end{aligned}$$

holds, which denotes that the curvature tensor R satisfies the Bianchi's 2nd identity. Also, we obtain from (2.3)

$$\begin{aligned}(2.24)\quad R^*(E,F)G = &\frac{1}{4}(c+3\varepsilon)\{g(F,G)E - g(E,G)F\}\\ &+\frac{1}{4}(c-\varepsilon)[\varepsilon\eta(G)\{\eta(E)F-\eta(F)E\} + \{g(E,G)\eta(F) - g(F,G)\eta(E)\}\xi\\ &\quad -g(F,\varphi^*G)\varphi^*E + g(E,\varphi^*G)\varphi^*F\\ &\quad +\{g(E,\varphi^*F) - g(\varphi^*E,F)\}\varphi^*G].\end{aligned}$$

Then the Sasaki-like statistical manifold (M, g, ∇^*) with Sasaki-like structure (φ^*, ξ, η), or (M, g, ∇^*) simply is called a space of constant φ^*-holomorphic sectional curvature c. The triple (M, g, ∇) is of constant φ-holomorphic sectional curvature c if and only if so is (M, g, ∇^*).

Example 2.15. Let (M, g, ∇) be a Sasaki-like statistical manifold with Sasaki-like structure (φ, ξ, η) of Example 2.11. Then (M, g, ∇) is a space of constant φ-holomorphic sectional curvature $c = -3\varepsilon$.

Remark 2.16. If M is a Sasakian manifold and $\varepsilon = 1$, then M satisfying (2.23) is a space of constant φ-holomorphic sectional curvature c [22].

Remark 2.17. Let $H(X) = K(X, \varphi X) = g(R(X, \varphi X)\varphi X, X)$ be a φ-sectional curvature for φ-section in the Sasakian manifold. If M is a Sasakian, then we get $H(X) = c$ for (2.23).

3. Statistical submersions

Let M and B be semi-Riemannian manifolds. A surjective mapping $\pi : M \to B$ is called a semi-Riemannian submersion if π has maximal rank and π_* preserves lengths of horizontal vectors. Let $\pi : M \to B$ be a semi-Riemannian submersion. We put $\dim M = m$ and $\dim B = n$. For each point $x \in B$, semi-Riemannian submanifold $\pi^{-1}(x)$ with the induced metric $\overline{g}$ is called a fiber and denoted by $\overline{M}_x$ or $\overline{M}$ simply. We notice that the dimension of each fiber is always $m - n\,(= s)$. A vector field on M is vertical if it is always tangent to fibers, horizontal if always orthogonal to fibers. We denote the vertical and horizontal subspace in the tangent space T_pM of the total space M by $\mathcal{V}_p(M)$ and $\mathcal{H}_p(M)$ for each point $p \in M$, and the vertical and horizontal distributions in the tangent bundle TM of M by $\mathcal{V}(M)$ and $\mathcal{H}(M)$, respectively. Then TM is the direct sum of $\mathcal{V}(M)$ and $\mathcal{H}(M)$. The projection mappings are denoted $\mathcal{V} : TM \to \mathcal{V}(M)$ and $\mathcal{H} : TM \to \mathcal{H}(M)$ respectively. We call a vector field X on M projectable if there exists a vector field X_* on B such that $\pi_*(X_p) = X_{*\pi(p)}$ for each $p \in M$, and say that X and X_* are π-related. Also, a vector field X on M is called basic if it is projectable and horizontal. Then we have ([6], [9], [11], [12], [22], etc.)

Lemma 3.1. *If X and Y are basic vector fields on M which are π-related to X_* and Y_* on B, then*

(1) *$g(X, Y) = g_B(X_*, Y_*) \circ \pi$, where g is the metric on M and g_B the metric on B,*

(2) *$\mathcal{H}[X, Y]$ is basic and is π-related to $[X_*, Y_*]$,*

(3) *$\mathcal{H}\nabla'_X Y$ is basic and π-related to $\widehat{\nabla}'_{X_*} Y_*$, where ∇' and $\widehat{\nabla}'$ are the Levi-Civita connections of M and B, respectively.*

Let (M, g, ∇) be a statistical manifold and $\pi : M \to B$ be a semi-Riemannian submersion. We denote the affine connections of $\overline{M}$ be $\overline{\nabla}$ and $\overline{\nabla}^*$. Notice that $\overline{\nabla}_U V$ and $\overline{\nabla}^*_U V$ are well-defined vertical vector fields on M for vertical vector fields U and V on M, more precisely $\overline{\nabla}_U V = \mathcal{V}\nabla_U V$ and $\overline{\nabla}^*_U V = \mathcal{V}\nabla^*_U V$. Moreover, $\overline{\nabla}$ and $\overline{\nabla}^*$ are torsion-free and conjugate to each other with respect to $\overline{g}$. The triple $(\overline{M}, \overline{g}, \overline{\nabla})$ is a statistical manifold and so is $(\overline{M}, \overline{g}, \overline{\nabla}^*)$.

We call that $\pi : (M, g, \nabla) \to (B, g_B, \widehat{\nabla})$ is a statistical submersion [1] if $\pi : M \to B$ satisfies

$$\pi_*(\nabla_X Y)_p = (\widehat{\nabla}_{X_*} Y_*)_{\pi(p)} \tag{3.1}$$

for basic vector fields X, Y and $p \in M$. The tensor fields T and A of type (1,2) defined by

$$T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \qquad A_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F \tag{3.2}$$

for any vector fields E and F on M. Changing ∇ to ∇^* in the above equations, we set T^* and A^*, respectively. Then we find $T^{**} = T$ and $A^{**} = A$. For vertical vector fields, T and T^* have the symmetry property. For $X, Y \in \mathcal{H}(M)$ and $U, V \in \mathcal{V}(M)$, we obtain

$$g(T_U V, X) = -g(V, T^*_U X), \qquad g(A_X Y, U) = -g(Y, A^*_X U). \tag{3.3}$$

Thus $T_U V$ (resp. $T_U X$) vanishes identically if and only if $T^*_U X$ (resp. $T^*_U V$) vanishes identically. If $T_U V$ (resp. $T^*_U V$) vanishes identically, then π is called with isometric fiber with respect to ∇ (resp. ∇^*). It is known that

Theorem 3.2. [1] *Let $\pi : M \to B$ be a semi-Riemannian submersion. Then (M, g, ∇) is a statistical manifold if and only if the following conditions hold:*

(1) $\mathcal{H}S_V X = A_X V - A^*_X V$,
(2) $\mathcal{V}S_X V = T_V X - T^*_V X$,
(3) $(\overline{M}, \overline{g}, \overline{\nabla})$ *is a statistical manifold for each* $x \in B$,
(4) $(B, g_B, \widehat{\nabla})$ *is a statistical manifold.*

For the statistical submersion $\pi : (M, g, \nabla) \to (B, g_B, \widehat{\nabla})$, we have the following Lemmas:.

Lemma 3.3. [15] *If X and Y are horizontal vector fields, then $A_X Y = -A^*_Y X$.*

From (3.3) and Lemma 3.3, the tensor field A vanishes identically if and only if A^* vanishes identically. Since A is related to the integrability of $\mathcal{H}(M)$, it is identically zero if and only if $\mathcal{H}(M)$ is integrable.

Lemma 3.4. [15] *For $X, Y \in \mathcal{H}(M)$ and $U, V \in \mathcal{V}(M)$ we have*

$$\begin{aligned} \nabla_U V &= T_U V + \overline{\nabla}_U V, & \nabla^*_U V &= T^*_U V + \overline{\nabla}^*_U V, \\ \nabla_U X &= \mathcal{H}\nabla_U X + T_U X, & \nabla^*_U X &= \mathcal{H}\nabla^*_U X + T^*_U X, \\ \nabla_X U &= A_X U + \mathcal{V}\nabla_X U, & \nabla^*_X U &= A^*_X U + \mathcal{V}\nabla^*_X U, \\ \nabla_X Y &= \mathcal{H}\nabla_X Y + A_X Y, & \nabla^*_X Y &= \mathcal{H}\nabla^*_X Y + A^*_X Y. \end{aligned}$$

Furthermore, if X is basic, then $\mathcal{H}\nabla_U X = A_X U$ and $\mathcal{H}\nabla^*_U X = A^*_X U$.

We define the covariant derivatives ∇T and ∇A by

$$(\nabla_E T)_F G = \nabla_E(T_F G) - T_{\nabla_E F} G - T_F(\nabla_E G),$$

$$(\nabla_E A)_F G = \nabla_E(A_F G) - A_{\nabla_E F} G - A_F(\nabla_E G)$$

for any $E, F, G \in TM$. We change ∇ to ∇^*, then the covariant derivatives $\nabla^* T^*$ and $\nabla^* A^*$ are defined simiraly. We consider the curvature tensor on the statistical submersion. Let $\overline{R}$ (resp. $\overline{R}^*$) be the curvature tensor with respect to the induced affine connection $\overline{\nabla}$ (resp. $\overline{\nabla}^*$) of each fiber. Also, let $\widehat{R}(X,Y)Z$ (resp. $\widehat{R}^*(X,Y)Z$) be horizontal vector field such that $\pi_*(\widehat{R}(X,Y)Z) = \widehat{R}(\pi_*X, \pi_*Y)\pi_*Z$ (resp. $\pi_*(\widehat{R}^*(X,Y)Z) = \widehat{R}^*(\pi_*X, \pi_*Y)\pi_*Z$) at each $p \in M$, where $\widehat{R}$ (resp. $\widehat{R}^*$) is the curvature tensor on B of the affine connection $\widehat{\nabla}$ (resp. $\widehat{\nabla}^*$). Then we have

Theorem 3.5. [15] *If $\pi : (M, g, \nabla) \to (B, g_B, \widehat{\nabla})$ is a statistical submersion, then we get for $X, Y, Z, Z' \in \mathcal{H}(M)$ and $U, V, W, W' \in \mathcal{V}(M)$*

$$
\begin{aligned}
g(R(U,V)W,W') &= g(\overline{R}(U,V)W,W') + g(T_U W, T_V^* W') - g(T_V W, T_U^* W'),\\
g(R(U,V)W,X) &= g((\nabla_U T)_V W, X) - g((\nabla_V T)_U W, X),\\
g(R(U,V)X,W) &= g((\nabla_U T)_V X, W) - g((\nabla_V T)_U X, W),\\
g(R(U,V)X,Y) &= g((\nabla_U A)_X V, Y) - g((\nabla_V A)_X U, Y) + g(T_U X, T_V^* Y) - g(T_V X, T_U^* Y)\\
&\quad -g(A_X U, A_Y^* V) + g(A_X V, A_Y^* U),\\
g(R(X,U)V,W) &= g([\mathcal{V}\nabla_X, \overline{\nabla}_U]V, W) - g(\nabla_{[X,U]}V, W) - g(T_U V, A_X^* W) + g(T_U^* W, A_X V),\\
g(R(X,U)V,Y) &= g((\nabla_X T)_U V, Y) - g((\nabla_U A)_X V, Y) + g(A_X U, A_Y^* V) - g(T_U X, T_V^* Y),\\
g(R(X,U)Y,V) &= g((\nabla_X T)_U Y, V) - g((\nabla_U A)_X Y, V) + g(T_U X, T_V Y) - g(A_X U, A_Y V),\\
g(R(X,U)Y,Z) &= g((\nabla_X A)_Y U, Z) - g(T_U X, A_Y^* Z) - g(T_U Y, A_X^* Z) + g(A_X Y, T_U^* Z),\\
g(R(X,Y)U,V) &= g([\mathcal{V}\nabla_X, \mathcal{V}\nabla_Y]U, V) - g(\nabla_{[X,Y]}U, V) + g(A_X U, A_Y^* V) - g(A_Y U, A_X^* V),\\
g(R(X,Y)U,Z) &= g((\nabla_X A)_Y U, Z) - g((\nabla_Y A)_X U, Z) + g(T_U^* Z, \theta_X Y),\\
g(R(X,Y)Z,U) &= g((\nabla_X A)_Y Z, U) - g((\nabla_Y A)_X Z, U) - g(T_U Z, \theta_X Y),\\
g(R(X,Y)Z,Z') &= g(\widehat{R}(X,Y)Z, Z') - g(A_Y Z, A_X^* Z') + g(A_X Z, A_Y^* Z') + g(\theta_X Y, A_Z^* Z'),
\end{aligned}
$$

where we put $\theta_X Y = A_X Y + A_X^* Y = \mathcal{V}[X,Y]$.

For each $p \in M$, we denote by $\{E_1, \ldots, E_m\}$, $\{X_1, \ldots, X_n\}$ and $\{U_1, \ldots, U_s\}$ local orthonormal bases of T_pM, $\mathcal{H}_p(M)$ and $\mathcal{V}_p(M)$, respectively such that $E_i = X_i$ $(i = 1, \ldots, n)$ and $E_{n+\alpha} = U_\alpha$ $(\alpha = 1, \ldots, s)$. Denote respectively by ω_a^b and ω_a^{*b} the connection forms in terms of local coordinates with respect to $\{E_1, \ldots, E_m\}$ of the affine connection ∇ and its conjugate ∇^*, where a, b run over the range $\{1, \ldots, m\}$. Set $\varepsilon_a = g(E_a, E_a) = -1$ or $+1$ according as E_a is timelike or spacelike. Also, mean curvature vectors of the affine connections are given by the horizontal vector field $N = \sum \varepsilon_\alpha T_{U_\alpha} U_\alpha$ and $N^* = \sum \varepsilon_\alpha T^*_{U_\alpha} U_\alpha$. If $T_U V = \frac{1}{s} g(U,V)N$ (resp. $T_U^* V = \frac{1}{s} g(U,V)N^*$) holds, then π is called with conformal fiber with respect to ∇ (resp. ∇^*). Moreover, we put $\sigma = \sum \varepsilon_i A_{X_i} X_i$.

Lemma 3.6. [15] *$g(N,N)$ and $g(N,N^*)$ are constants on each fiber.*

Next, we define the Ricci tensor $\mathrm{Ric}(E,F)$ of the affine connection ∇ for E, $F \in TM$ by

$$\mathrm{Ric}(E,F) = \sum_{i=1}^{n} \varepsilon_i g(R(X_i, E)F, X_i) + \sum_{\alpha=1}^{s} \varepsilon_\alpha g(R(U_\alpha, E)F, U_\alpha),$$

moreover, we put for $X, Y \in \mathcal{H}(M)$ and $U, V \in \mathcal{V}(M)$

$$\widehat{\mathrm{Ric}}(X,Y) = \sum_{i=1}^{n} \varepsilon_i g(\widehat{R}(X_i, X)Y, X_i), \qquad \overline{\mathrm{Ric}}(U,V) = \sum_{\alpha=1}^{s} \varepsilon_\alpha g(\overline{R}(U_\alpha, U)V, U_\alpha).$$

Changing R (resp. $\widehat{R}, \overline{R}$) to R^* (resp. $\widehat{R}^*, \overline{R}^*$) in the above equations, we set Ric^* (resp. $\widehat{\mathrm{Ric}}^*, \overline{\mathrm{Ric}}^*$). Then $\widehat{\mathrm{Ric}}$ (resp. $\widehat{\mathrm{Ric}}^*$) is the horizontal 2-form on M such that $\widehat{\mathrm{Ric}}(X,Y) = \widehat{\mathrm{Ric}}(\pi_* X, \pi_* Y)$ (resp. $\widehat{\mathrm{Ric}}^*(X,Y) = \widehat{\mathrm{Ric}}^*(\pi_* X, \pi_* Y)$), and $\overline{\mathrm{Ric}}$ (resp. $\overline{\mathrm{Ric}}^*$) is the Ricci tensor of each fiber with respect to the induced affine connection $\overline{\nabla}$ (resp. conjugate $\overline{\nabla}^*$ of $\overline{\nabla}$).

4. Sasaki-like statistical submersions

Let (M, g) be an almost contact metric manifold with almost contact structure (φ, ξ, η), and (B, g_B) be a semi-Riemannian manifold. The semi-Riemannian submersion $\pi : (M, g) \to (B, g_B)$ is called an almost contact metric submersion. For $X \in \mathcal{H}(M)$, we put ([22])

$$\varphi X = PX + FX, \qquad \varphi^* X = P^* X + F^* X, \tag{4.1}$$

where $PX, P^*X \in \mathcal{H}(M)$ and $FX, F^*X \in \mathcal{V}(M)$. For $V \in \mathcal{V}(M)$ we set

$$\varphi V = tV + fV, \qquad \varphi^* V = t^* V + f^* V, \tag{4.2}$$

where $tV, t^*V \in \mathcal{H}(M)$ and $fV, f^*V \in \mathcal{V}(M)$. From $(\varphi^*)^* = \varphi$, we find $(P^*)^* = P, (F^*)^* = F, (t^*)^* = t$ and $(f^*)^* = f$. Because of $\varphi^2 = -I + \eta \otimes \xi$, we get

Lemma 4.1. *In an almost contact metric submersion, we find*

(1) *if $\xi \in \mathcal{H}(M)$, then*

$$P^2 = -I - tF + \eta \otimes \xi, \quad FP + fF = 0, \quad Pt + tf = 0, \quad f^2 = -I - Ft.$$

(2) *if $\xi \in \mathcal{V}(M)$, then*

$$P^2 = -I - tF, \quad FP + fF = 0, \quad Pt + tf = 0, \quad f^2 = -I - Ft + \eta \otimes \xi.$$

From $\varphi\xi = 0$ and $\eta(\varphi E) = 0$, we have

Lemma 4.2. *In an almost contact metric submersion, we find*

(1) *if $\xi \in \mathcal{H}(M)$, then $P\xi = 0$, $F\xi = 0$, $\eta(PX) = 0$ and $\eta(tV) = 0$.*
(2) *if $\xi \in \mathcal{V}(M)$, then $t\xi = 0$, $f\xi = 0$, $\eta(FX) = 0$ and $\eta(fV) = 0$.*

Because of $g(\varphi E, F) + g(E, \varphi^* F) = 0$ for any vector fields E and F on M, we find

$$g(PX, Y) + g(X, P^*Y) = 0, \tag{4.3}$$

$$g(FX, V) + g(X, t^*V) = 0, \tag{4.4}$$

$$g(tV, Y) + g(V, F^*Y) = 0, \tag{4.5}$$

$$g(fV, W) + g(V, f^*W) = 0. \tag{4.6}$$

Thus P (resp. F) vanishes idenitically if and only if so is P^* (resp. t^*), and t (resp. f) vanishes identically is and only if so is F^* (resp. f^*). Thus we get

Lemma 4.3. *In an almost contact metric submersion, we find*

(1) *if* $\xi \in \mathcal{H}(M)$*, then*

$$g(PX, P^*Y) = g(X,Y) - g(FX, F^*Y) - \varepsilon\eta(X)\eta(Y),$$
$$g(fU, f^*V) = g(U,V) - g(tU, t^*V).$$

(2) *if* $\xi \in \mathcal{V}(M)$*, then*

$$g(PX, P^*Y) = g(X,Y) - g(FX, F^*Y),$$
$$g(fU, f^*V) = g(U,V) - g(tU, t^*V) - \varepsilon\eta(U)\eta(V).$$

Lemma 4.4. *In an almost contact metric submersion, we find for each* $p \in M$

(1) $\varphi(\mathcal{V}_p(M)) \subset \mathcal{V}_p(M)$ *if and only if* $\varphi^*(\mathcal{H}_p(M)) \subset \mathcal{H}_p(M)$.

(2) $\varphi(\mathcal{H}_p(M)) \subset \mathcal{H}_p(M)$ *if and only if* $\varphi^*(\mathcal{V}_p(M)) \subset \mathcal{V}_p(M)$.

(3) $\varphi(\mathcal{V}_p(M)) \subset \mathcal{H}_p(M)$ *if and only if* $\varphi^*(\mathcal{V}_p(M)) \subset \mathcal{H}_p(M)$.

(4) $\varphi(\mathcal{H}_p(M)) \subset \mathcal{V}_p(M)$ *if and only if* $\varphi^*(\mathcal{H}_p(M)) \subset \mathcal{V}_p(M)$.

If $\varphi(\mathcal{V}_p(M)) \subset \mathcal{V}_p(M)$ (resp. $\varphi^*(\mathcal{V}_p(M)) \subset \mathcal{V}_p(M)$) for each $p \in M$, then $\overline{M}$ is said to be a φ-invariant (resp. φ^*-invariant) submanifold of M. Then t and F^* (resp. F and t^*) vanish identically. If $\varphi(\mathcal{V}_p(M)) \subset \mathcal{H}_p(M)$ for each $p \in M$, then $\overline{M}$ is said to be a φ-anti-invariant submanifold of M. Since $f = 0$ is equivalent to $f^* = 0$, $\overline{M}$ is φ-anti-invariant if and only if $\overline{M}$ is φ^*-anti-invariant. Thus, in this paper, it is simply referred to as anti-invariant. Let $\overline{f}$, $\overline{\xi}$ and $\overline{\eta}$ be a tensor field of type $(1,1)$, vector field and 1-form such that $\overline{f} = f|_{\overline{M}}$, $\overline{\xi} = \xi|_{\overline{M}}$ and $\overline{\eta} = \eta|_{\overline{M}}$, where $f|_{\overline{M}}$ denote the restriction of f to $\overline{M}$. Also, let $\widehat{P}$, $\widehat{\xi}$ and $\widehat{\eta}$ be a tensor field of type $(1,1)$, vector field and 1-form such that $\pi_*P = \widehat{P}\pi_*$, $\pi_*\xi = \widehat{\xi}$ and $\eta(\pi_*X) = \widehat{\eta}(X_*)$ for basic vector field X. From Lemmas 4.1∼4.3, we obtain

Theorem 4.5. *Let* π *be an almost contact metric submersion, and* $\overline{M}$ *be* φ*-invariant or* φ^**-invariant of* M*. If* $\xi \in \mathcal{H}(M)$*, then*

(1) *each fiber* $(\overline{M}, \overline{g}, \overline{f})$ *is an almost Hermite-like manifold.*

(2) *the base space* (B, g_B) *ia an almost contact metric manifold with almost contact structure* $(\widehat{P}, \widehat{\xi}, \widehat{\eta})$.

Theorem 4.6. *Let* π *be an almost contact metric submersion, and* $\overline{M}$ *be* φ*-invariant or* φ^**-invariant of* M*. If* $\xi \in \mathcal{V}(M)$*, then*

(1) *each fiber* $(\overline{M}, \overline{g})$ *is an almost contact metric manifold with almost contact structure* $(\overline{f}, \overline{\xi}, \overline{\eta})$.

(2) *the base space* $(B, g_B, \widehat{P})$ *is an almost Hermite-like manifold.*

Let (M, g, ∇) be a Sasaki-like statistical manifold with a Sasaki-like structure (φ, ξ, η) and $(B, g_B, \widehat{\nabla})$ be a statistical manifold. The statistical submersion $\pi : (M, g, \nabla) \to (B, g_B, \widehat{\nabla})$ is called a Sasaki-like statistical submersion. We put

$$(\mathcal{H}\nabla_X P)Y = \mathcal{H}\nabla_X(PY) - P(\mathcal{H}\nabla_X Y), \qquad (\mathcal{H}\nabla_U P)Y = \mathcal{H}\nabla_U(PY) - P(\mathcal{H}\nabla_U Y),$$
$$(\mathcal{V}\nabla_X F)Y = \mathcal{V}\nabla_X(FY) - F(\mathcal{H}\nabla_X Y), \qquad (\mathcal{V}\nabla_U F)Y = \overline{\nabla}_U(FY) - F(\mathcal{H}\nabla_U Y),$$
$$(\mathcal{H}\nabla_X t)V = \mathcal{H}\nabla_X(tV) - t(\mathcal{V}\nabla_X V), \qquad (\mathcal{H}\nabla_U t)V = \mathcal{H}\nabla_U(tV) - t(\overline{\nabla}_U V),$$
$$(\mathcal{V}\nabla_X f)V = \mathcal{V}\nabla_X(fV) - f(\mathcal{V}\nabla_X V), \qquad (\overline{\nabla}_U f)V = \overline{\nabla}_U(fV) - f(\overline{\nabla}_U V),$$

also, we set $(\mathcal{H}\nabla^*_X P^*)Y = \mathcal{H}\nabla^*_X(P^*Y) - P^*(\mathcal{H}\nabla^*_X Y)$, etc. Then we have from (4.3)$\sim$(4.6)

Lemma 4.7. *If $\pi : (M, g, \nabla) \to (B, g_B, \widehat{\nabla})$ is a Sasaki-like statistical submersion, then we find*

$$g((\mathcal{H}\nabla_X P)Y, Z) + g(Y, (\mathcal{H}\nabla^*_X P^*)Z) = 0, \qquad g((\mathcal{H}\nabla_U P)X, Y) + g(X, (\mathcal{H}\nabla^*_U P^*)Y) = 0,$$
$$g(\mathcal{V}\nabla_X F)Y, V) + g(Y, (\mathcal{H}\nabla^*_X t^*)V) = 0, \qquad g((\mathcal{V}\nabla_U F)Y, V) + g(Y, (\mathcal{H}\nabla^*_U t^*)V) = 0,$$
$$g((\mathcal{H}\nabla_X t)V, Y) + g(V, (\mathcal{V}\nabla^*_X F^*)Y) = 0, \qquad g((\mathcal{H}\nabla_U t)V, Y) + g(V, (\mathcal{V}\nabla^*_U F^*)Y) = 0,$$
$$g((\mathcal{V}\nabla_X f)V, W) + g(V, (\mathcal{V}\nabla^*_X f^*)W) = 0, \qquad g((\overline{\nabla}_U f)V, W) + g(V, (\overline{\nabla}^*_U f^*)W) = 0.$$

Hence we have

Corollary 4.8. *Let $\pi : (M, g, \nabla) \to (B, g_B, \widehat{\nabla})$ be a Sasaki-like statistical submersion. We get*

(1) $\mathcal{H}\nabla P = 0$ *is equivalent to* $\mathcal{H}\nabla^* P^* = 0$.
(2) $\mathcal{V}\nabla F = 0$ *is equivalent to* $\mathcal{H}\nabla^* t^* = 0$.
(3) $\mathcal{H}\nabla t = 0$ *is equivalent to* $\mathcal{V}\nabla^* F^* = 0$.
(4) $\mathcal{V}\nabla f = 0$ *is equivalent to* $\mathcal{V}\nabla^* f^* = 0$, *where* $\mathcal{V}\nabla_U f = \overline{\nabla}_U f$ *and* $\mathcal{V}\nabla^*_U f = \overline{\nabla}^*_U f$.

Because of $\nabla_E \xi = -\varepsilon\varphi E$ and $(\nabla_E \varphi)G = g(E, G)\xi - \varepsilon\eta(G)E$, we get

Proposition 4.9. *Let $\pi : (M, g, \nabla) \to (B, g_B, \widehat{\nabla})$ be a Sasaki-like statistical submersion. We get for any $U \in \mathcal{V}(M)$ and $X \in \mathcal{H}(M)$*

(1) *if $\xi \in \mathcal{H}(M)$, then*

$$\mathcal{H}\nabla_U \xi = -\varepsilon tU, \qquad T_U \xi = -\varepsilon fU, \qquad \mathcal{H}\nabla_X \xi = -\varepsilon PX, \qquad A_X \xi = -\varepsilon FX.$$

(2) *if $\xi \in \mathcal{V}(M)$, then*

$$T_U \xi = -\varepsilon tU, \qquad \overline{\nabla}_U \xi = -\varepsilon fU, \qquad A_X \xi = -\varepsilon PX, \qquad \mathcal{V}\nabla_X \xi = -\varepsilon FX.$$

Proposition 4.10. *Let $\pi : (M, g, \nabla) \to (B, g_B, \widehat{\nabla})$ be a Sasaki-like statistical submersion. We get for any $U, V \in \mathcal{V}(M)$ and $X, Y \in \mathcal{H}(M)$*

(1) *if $\xi \in \mathcal{H}(M)$, then*

$$(\mathcal{H}\nabla_U t)V + T_U(fV) - P(T_U V) = g(U, V)\xi,$$
$$(\overline{\nabla}_U f)V + T_U(tV) - F(T_U V) = 0,$$
$$(\mathcal{H}\nabla_U P)Y + T_U(FY) - t(T_U Y) = 0,$$
$$(\mathcal{V}\nabla_U F)Y + T_U(PY) - f(T_U Y) = -\varepsilon\eta(Y)U,$$
$$(\mathcal{H}\nabla_X t)V + A_X(fV) - P(A_X V) = 0,$$
$$(\mathcal{V}\nabla_X f)V + A_X(tV) - F(A_X V) = 0,$$
$$(\mathcal{H}\nabla_X P)Y + A_X(FY) - t(A_X Y) = g(X, Y)\xi - \varepsilon\eta(Y)X,$$
$$(\mathcal{V}\nabla_X F)Y + A_X(PY) - f(A_X Y) = 0.$$

(2) *if* $\xi \in \mathcal{V}(M)$, *then*

$$\begin{aligned}
&(\mathcal{H}\nabla_U t)V + T_U(fV) - P(T_U V) = 0,\\
&(\overline{\nabla}_U f)V + T_U(tV) - F(T_U V) = g(U,V)\xi - \varepsilon\eta(V)U,\\
&(\mathcal{H}\nabla_U P)Y + T_U(FY) - t(T_U Y) = 0,\\
&(\mathcal{V}\nabla_U F)Y + T_U(PY) - f(T_U Y) = 0,\\
&(\mathcal{H}\nabla_X t)V + A_X(fV) - P(A_X V) = -\varepsilon\eta(V)X,\\
&(\mathcal{V}\nabla_X f)V + A_X(tV) - F(A_X V) = 0,\\
&(\mathcal{H}\nabla_X P)Y + A_X(FY) - t(A_X Y) = 0,\\
&(\mathcal{V}\nabla_X F)Y + A_X(PY) - f(A_X Y) = g(X,Y)\xi.
\end{aligned}$$

By virtue of Lemmas 4.1, 4.2 and Propositions 4.9, 4.10, we have

Lemma 4.11. *Let* $\pi : (M, g, \nabla) \to (B, g_B, \widehat{\nabla})$ *be a Sasaki-like statistical submersion. We get*

(1) *If* $\xi \in \mathcal{H}(M)$, *then* $\eta(T_U V) = -g(U, fV)$ *and* $\eta(A_X V) = -g(X, tV)$ *hold. Moreover, we find* $f^* = -f$.

(2) *If* $\xi \in \mathcal{V}(M)$, *then* $\eta(T_U Y) = -g(U, FY)$ *and* $\eta(A_X Y) = -g(X, PY)$ *hold.*

Theorem 4.12. *Let* $\pi : (M, g, \nabla) \to (B, g_B, \widehat{\nabla})$ *be a Sasaki-like statistical submersion with isometric fiber with respect to* ∇

(1) *If* $\xi \in \mathcal{H}(M)$, *then each fiber is anti-invariant.*

(2) *If* $\xi \in \mathcal{V}(M)$, *then each fiber is* φ*-invariant. Moreover, each fiber* $(\overline{M}, \overline{g}, \overline{\nabla})$ *is a Sasaki-like statistical manifold with Sasaki-like structure* $(\overline{f}, \overline{\xi}, \overline{\eta})$.

Let $\pi : (M, g, \nabla) \to (B, g_B, \widehat{\nabla})$ be a Sasaki-like statistical submersion. If the total space (M, g, ∇) is of constant curvature ε, then we get from Lemma 2.14 and Theorem 3.5

$$\begin{aligned}
(4.7)\quad & g(\overline{R}(U,V)W,W') + g(T_U W, T_V^* W') - g(T_V W, T_U^* W')\\
&= \varepsilon\,\{\, g(V,W)g(U,W') - g(U,W)g(V,W')\,\},
\end{aligned}$$

$$(4.8)\quad g((\nabla_U T)_V W, X) - g((\nabla_V T)_U W, X) = 0,$$

$$(4.9)\quad g((\nabla_U T)_V X, W) - g((\nabla_V T)_U X, W) = 0,$$

$$\begin{aligned}
(4.10)\quad & g((\nabla_U A)_X V, Y) - g((\nabla_V A)_X U, Y) + g(T_U X, T_V^* Y) - g(T_V X, T_U^* Y)\\
& - g(A_X U, A_Y^* V) + g(A_X V, A_Y^* U) = 0,
\end{aligned}$$

$$(4.11)\quad g([\mathcal{V}\nabla_X, \overline{\nabla}_U]V, W) - g(\nabla_{[X,U]}V, W) - g(T_U V, A_X^* W) + g(T_U^* W, A_X V) = 0,$$

$$(4.12)\quad g((\nabla_X T)_U V, Y) - g((\nabla_U A)_X V, Y) + g(A_X U, A_Y^* V) - g(T_U X, T_V^* Y) = \varepsilon\, g(U,V)g(X,Y),$$

$$\begin{aligned}
(4.13)\quad & g((\nabla_X T)_U Y, V) - g((\nabla_U A)_X Y, V) + g(T_U X, T_V Y) - g(A_X U, A_Y V)\\
&= -\varepsilon\, g(U,V)g(X,Y),
\end{aligned}$$

$$(4.14)\quad g((\nabla_X A)_Y U, Z) - g(T_U X, A_Y^* Z) - g(T_U Y, A_X^* Z) + g(A_X Y, T_U^* Z) = 0,$$

$$g([\mathcal{V}\nabla_X, \mathcal{V}\nabla_Y]U, V) - g(\nabla_{[X,Y]}U, V) + g(A_XU, A^*_YV) - g(A_YU, A^*_XV) = 0, \tag{4.15}$$

$$g((\nabla_XA)_YU, Z) - g((\nabla_YA)_XU, Z) + g(T^*_UZ, \theta_XY) = 0, \tag{4.16}$$

$$g((\nabla_XA)_YZ, U) - g((\nabla_YA)_XZ, U) - g(T_UZ, \theta_XY) = 0, \tag{4.17}$$

$$\begin{aligned} &g(\widehat{R}(X, Y)Z, Z') - g(A_YZ, A^*_XZ') + g(A_XZ, A^*_YZ') + g(\theta_XY, A^*_ZZ') \\ &= \varepsilon\,\{\, g(Y, Z)g(X, Z') - g(X, Z)g(Y, Z')\}, \end{aligned} \tag{4.18}$$

for $X, Y, Z, Z' \in \mathcal{H}(M)$ and $U, V, W, W' \in \mathcal{V}(M)$. We discuss a Sasaki-like statistical submersion with conformal fiber with respect to ∇ and ∇^*, that is,

$$T_UV = \frac{1}{s}g(U, V)N, \qquad T^*_UV = \frac{1}{s}g(U, V)N^*.$$

Then we get $T_UX = -\frac{1}{s}g(N^*, X)U$ and $T^*_UX = -\frac{1}{s}g(N, X)U$. It is easy to see from (4.7) that we find

$$\overline{R}(U, V)W = \left\{\varepsilon + \frac{1}{s^2}g(N, N^*)\right\}\{\, g(V, W)U - g(U, W)V\,\}. \tag{4.19}$$

Because of Lemma 3.6, it should be noticed that $\varepsilon + \frac{1}{s^2}g(N, N^*)$ is a constant on each fiber. Thus we have

Theorem 4.13. *Let $\pi : (M, g, \nabla) \to (B, g_B, \widehat{\nabla})$ be a Sasaki-like statistical submersion with conformal fiber with respect to ∇ and ∇^*. If the total space (M, g, ∇) is of constant curvature ε, then each fiber satisfies* (4.19).

Corollary 4.14. *Let π be a Sasaki-like statistical submersion with isometric fiber with respect to ∇ or ∇^*. If the total space (M, g, ∇) is of constant curvature ε, then each fiber is of constant curvature ε.*

By virtue of (4.8), we have

Lemma 4.15. *Let $\pi : (M, g, \nabla) \to (B, g_B, \widehat{\nabla})$ be a Sasaki-like statistical submersion with conformal fiber with respect to ∇. If the total space (M, g, ∇) is of constant curvature ε and $s \geq 2$, then $\mathcal{H}\nabla_UN = 0$ holds.*

If the total space (M, g, ∇) is of constant φ-holomorphic sectional curvature c, then we find from (2.23) and Theorem 3.5 If the total space (M, g, ∇) is of constant φ-holomorphic sectional curvature c, then we find from (2.23) and Theorem 3.5:

$$\begin{aligned}
(4.20)\quad & g(\overline{R}(U,V)W,W') + g(T_UW,T_V^*W') - g(T_VW,T_U^*W') \\
&= \frac{1}{4}(c+3\varepsilon)\,\{\,g(V,W)g(U,W') - g(U,W)g(V,W')\,\} \\
&\quad + \frac{1}{4}(c-\varepsilon)\,[\,\varepsilon\eta(W)\{\eta(U)g(V,W') - \eta(V)g(U,W')\} \\
&\qquad + \varepsilon\{g(U,W)\eta(V) - g(V,W)\eta(U)\}\eta(W') - g(V,fW)g(fU,W') \\
&\qquad + g(U,fW)g(fV,W') + \{g(U,fV) - g(fU,V)\}g(fW,W')\,],
\end{aligned}$$

$$\begin{aligned}
(4.21)\quad & g((\nabla_UT)_VW,X) - g((\nabla_VT)_UW,X) \\
&= \frac{1}{4}(c-\varepsilon)\,[-g(V,fW)g(tU,X) + g(U,fW)g(tV,X) + \{\,g(U,fV) - g(fU,V)\}g(tW,X)\,],
\end{aligned}$$

$$\begin{aligned}
(4.22)\quad & g((\nabla_UT)_VX,W) - g((\nabla_VT)_UX,W) \\
&= \frac{1}{4}(c-\varepsilon)\,[-g(V,FX)g(fU,W) + g(U,FX)g(fV,W) + \{\,g(U,fV) - g(fU,V)\}g(FX,W)\,],
\end{aligned}$$

$$\begin{aligned}
(4.23)\quad & g((\nabla_UA)_XV,Y) - g((\nabla_VA)_XU,Y) + g(T_UX,T_V^*Y) - g(T_VX,T_U^*Y) \\
& - g(A_XU,A_Y^*V) + g(A_XV,A_Y^*U) \\
&= \frac{1}{4}(c-\varepsilon)\,[\,-g(V,FX)g(tU,Y) + g(U,FX)g(tV,Y) \\
&\qquad + \{g(U,fV) - g(fU,V)\}g(PX,Y)\,],
\end{aligned}$$

$$\begin{aligned}
(4.24)\quad & g([\mathcal{V}\nabla_X,\overline{\nabla}_U]V,W) - g(\nabla_{[X,U]}V,W) - g(T_UV,A_X^*W) + g(T_U^*W,A_XV), \\
&= \frac{1}{4}(c-\varepsilon)\,[-g(U,fV)g(FX,W) + g(X,tV)g(fU,W) \\
&\qquad + \{g(X,tU) - g(FX,U)\}g(fV,W)\,],
\end{aligned}$$

$$\begin{aligned}
(4.25)\quad & g((\nabla_XT)_UV,Y) - g((\nabla_UA)_XV,Y) + g(A_XU,A_Y^*V) - g(T_UX,T_V^*Y) \\
&= \frac{1}{4}(c+3\varepsilon)g(U,V)g(X,Y) \\
&\quad + \frac{1}{4}(c-\varepsilon)\,[-\varepsilon\eta(U)\eta(V)g(X,Y) - \varepsilon\eta(X)\eta(Y)g(U,V) - g(U,fV)g(PX,Y) \\
&\qquad + g(X,tV)g(tU,Y) + \{g(X,tU) - g(FX,U)\}g(tV,Y)\,],
\end{aligned}$$

$$\begin{aligned}
(4.26)\quad & g((\nabla_XT)_UY,V) - g((\nabla_UA)_XY,V) + g(T_UX,T_VY) - g(A_XU,A_YV) \\
&= -\frac{1}{4}(c+3\varepsilon)g(U,V)g(X,Y) \\
&\quad + \frac{1}{4}(c-\varepsilon)\,[\,\varepsilon\eta(U)\eta(V)g(X,Y) + \varepsilon\eta(X)\eta(Y)g(U,V) - (U,FY)g(FX,V) \\
&\qquad + g(X,PY)g(fU,V) + \{g(X,tU) - g(FX,U)\}g(FY,V)\,],
\end{aligned}$$

$$\begin{aligned}
(4.27)\quad & g((\nabla_XA)_YU,Z) - g(T_UX,A_Y^*Z) - g(T_UY,A_X^*Z) + g(A_XY,T_U^*Z) \\
&= \frac{1}{4}(c-\varepsilon)\,[\,-g(U,FY)g(PX,Z) + g(X,PY)g(tU,Z) \\
&\qquad + \{g(X,tU) - g(FX,U)\}g(PY,Z)\,],
\end{aligned}$$

$$\text{(4.28)}\quad g((\nabla_X A)_Y Z, U) - g((\nabla_Y A)_X Z, U) - g(T_U Z, \theta_X Y)$$
$$= \frac{1}{4}(c-\varepsilon)\,[\,-g(Y,PZ)g(FX,U) + g(X,PZ)g(FY,U) + \{g(X,PY) - g(PX,Y)\}g(FZ,U)\,],$$

$$\text{(4.29)}\quad g(\widehat{R}(X,Y)Z,Z') - g(A_Y Z, A^*_X Z') + g(A_X Z, A^*_Y Z') + g(\theta_X Y, A^*_Z Z')$$
$$= \frac{1}{4}(c+3\varepsilon)\,\{\,g(Y,Z)g(X,Z') - g(X,Z)g(Y,Z')\}$$
$$+\frac{1}{4}(c-\varepsilon)\,[\,\varepsilon\eta(Z)\{\eta(X)g(Y,Z') - \eta(Y)g(X,Z')\}$$
$$+\varepsilon\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\eta(Z') - g(Y,PZ)g(PX,Z')$$
$$+g(X,PZ)g(PY,Z') + \{g(X,PY) - g(PX,Y)\}g(PZ,Z')\,],$$

for $X, Y, Z, Z' \in \mathcal{H}(M)$ and $U, V, W, W' \in \mathcal{V}(M)$. We assume that π is with conformal fiber with respect to ∇ and ∇^* such that $\xi \in \mathcal{H}(M)$ and each fiber is anti-invariant. From (4.20), we find

$$\text{(4.30)}\quad g(\overline{R}(U,V)W,W')$$
$$= \left\{\frac{1}{4}(c+3\varepsilon) + \frac{1}{s^2}g(N,N^*)\right\}\{\,g(V,W)g(U,W') - g(U,W)g(V,W')\,\}.$$

Hence we have

Theorem 4.16. *Let $\pi : (M, g, \nabla) \to (B, g_B, \widehat{\nabla})$ be a Sasaki-like statistical submersion with conformal fiber with respect to ∇ and ∇^* such that $\xi \in \mathcal{H}(M)$. If the total space is of constant φ-holomorphic sectional curvature c and each fiber is anti-invariant, then each fiber satisfies (4.30).*

Corollary 4.17. *Let $\pi : (M, g, \nabla^*) \to (B, g_B, \widehat{\nabla}^*)$ be a Sasaki-like statistical submersion with isometric fiber with respect to ∇ or ∇^* such that $\xi \in \mathcal{H}(M)$. If the total space is of constant φ-holomorphic sectional curvature c and each fiber is anti-invariant, then each fiber is of constant curvature $\frac{1}{4}(c+3\varepsilon)$.*

Corollary 4.18. *Let $\pi : (M, g, \nabla^*) \to (B, g_B, \widehat{\nabla}^*)$ be a Sasaki-like statistical submersion with conformal fiber with respect to ∇ and ∇^* such that $\xi \in \mathcal{H}(M)$. If the total space is of constant φ^*-holomorphic sectional curvature c and each fiber is anti-invariant, then each fiber satisfies (4.30).*

Corollary 4.19. *Let $\pi : (M, g, \nabla^*) \to (B, g_B, \widehat{\nabla}^*)$ be a Sasaki-like statistical submersion with isometric fiber with respect to ∇ or ∇^* such that $\xi \in \mathcal{H}(M)$. If the total space is of constant φ^*-holomorphic sectional curvature c and each fiber is anti-invariant, then each fiber is of constant curvature $\frac{1}{4}(c+3\varepsilon)$.*

In the case of the Sasaki-like statistical submersion with isometric fiber with respect to ∇, we get from (4.21)

$$(c-\varepsilon)\,[\,-g(V,fW)g(tU,X) + g(U,fW)g(tV,X) + \{\,g(U,fV) - g(fU,V)\}g(tW,X)\,] = 0,$$

which implies that $c = \varepsilon$ or

$$t\,[\,-g(V,fW)U + g(U,fW)V + \{\,g(U,fV) - g(fU,V)\}W\,] = 0.$$

If $-g(V, fW)U + g(U, fW)V + \{g(U, fV) - g(fU, V)\}W = 0$ holds, then $f = 0$ ($s \geq 3$). Thus we get $t = 0$ if $f \neq 0$ and $s \geq 3$. From (4.22), we get $c = \varepsilon$ or

$$t^*[g(fU, W)V - g(fV, W)U - \{g(U, fV) - g(fU, V)\}W] = 0,$$

which yields that $f = 0$, or $t^* = 0$ if $s \geq 3$. Hence we have

Theorem 4.20. *Let $\pi : (M, g, \nabla) \to (B, g_B, \widehat{\nabla})$ be a Sasaki-like statistical submersion with isometric fiber with respect to ∇. If the total space is of constant φ-holomorphic sectional curvature c, then*

(1) *$c = \varepsilon$, that is, the total space and each fiber are of constant curvature ε, or*
(2) *each fiber is anti-invariant if $s \geq 3$, or*
(3) *each fiber is φ-invariant or φ^*-invariant of M if $s \geq 3$.*

Corollary 4.21. *Let $\pi : (M, g, \nabla^*) \to (B, g_B, \widehat{\nabla}^*)$ be a Sasaki-like statistical submersion with isometric fiber with respect to ∇^*. If the total space is of constant φ^*-holomorphic sectional curvature c, then*

(1) *$c = \varepsilon$, that is, the total space and each fiber are of constant curvature ε, or*
(2) *each fiber is anti-invariant if $s \geq 3$, or*
(3) *each fiber is φ-invariant or φ^*-invariant of M if $s \geq 3$.*

Next, we give two examples of Sasaki-like statistical submersion.

Example 4.22. Let π be a Sasaki-like statistical submersion. The total space is a Sasaki-like statistical manifold (M, g, ∇) with Sasaki-like structure (φ, ξ, η) of Example 2.12. For $X_1 \in \mathcal{H}(M)$ and X_2, $X_3 \in \mathcal{V}(M)$, we get

$$\begin{aligned}
&T_{X_2}X_2 = 0, && \overline{\nabla}_{X_2}X_2 = -X_3,\\
&T_{X_2}X_3 = T_{X_3}X_2 = -\varepsilon X_1, && \overline{\nabla}_{X_2}X_3 = \overline{\nabla}_{X_3}X_2 = -\varepsilon X_2,\\
&T_{X_3}X_3 = 0, && \overline{\nabla}_{X_3}X_3 = 0,\\
&\mathcal{H}\nabla_{X_2}X_1 = 0, && T_{X_2}X_1 = 2X_3,\\
&\mathcal{H}\nabla_{X_3}X_1 = \varepsilon X_1, && T_{X_3}X_1 = 2\varepsilon X_2,\\
&A_{X_1}X_2 = 0, && \mathcal{V}\nabla_{X_1}X_2 = X_3,\\
&A_{X_1}X_3 = \varepsilon X_1, && \mathcal{V}\nabla_{X_1}X_3 = 2\varepsilon X_2,\\
&\mathcal{H}\nabla_{X_1}X_1 = 0, && A_{X_1}X_1 = -X_3.
\end{aligned}$$

Thus each fiber $(\overline{M}, \overline{g}, \overline{\nabla})$ is minimal and is of constant curvature $-\varepsilon$. Also, we find

$$\begin{aligned}
&PX_1 = -X_1, \quad FX_1 = -2X_2,\\
&tX_2 = X_1, \quad fX_2 = X_2, \quad tX_3 = 0, \quad fX_3 = 0.
\end{aligned}$$

Example 4.23. Let π be a Sasaki-like statistical submersion. The total space is a Sasaki-like statistical manifold (M, g, ∇) with Sasaki-like structure (φ, ξ, η) of Example 2.12. For X_1, $X_2 \in$

$\mathcal{H}(M)$ and $X_3 \in \mathcal{V}(M)$, we get

$$\begin{aligned} &T_{X_3}X_3 = 0, && \overline{\nabla}_{X_3}X_3 = 0, \\ &\mathcal{H}\nabla_{X_3}X_1 = \varepsilon(X_1 + 2X_2), && T_{X_3}X_1 = 0, \\ &\mathcal{H}\nabla_{X_3}X_2 = -\varepsilon(X_1 + X_2), && T_{X_3}X_2 = 0, \\ &A_{X_1}X_3 = \varepsilon(X_1 + 2X_2), && \mathcal{V}\nabla_{X_3}X_1 = 0, \\ &A_{X_2}X_3 = -\varepsilon(X_1 + X_2), && \mathcal{V}\nabla_{X_2}X_3 = 0, \\ &\mathcal{H}\nabla_{X_1}X_1 = \mathcal{H}\nabla_{X_2}X_2 = 0, && A_{X_1}X_1 = A_{X_2}X_2 = -X_3, \\ &2\mathcal{H}\nabla_{X_1}X_2 = \mathcal{H}\nabla_{X_2}X_1 = 0, && 2A_{X_1}X_2 = A_{X_2}X_1 = 2X_3. \end{aligned}$$

Thus π is with isometric fiber with respect to ∇ and ∇^*, and the base space is flat. Also, we find $F = 0$, namely, π is φ^*-invariant. Moreover, $t = 0$ (φ-invariant) and $f = 0$ (anti-invariant) are trivial.

5. φ-invariant Sasaki-like statistical submersions

The Sasaki-like statistical submersion $\pi : (M, g, \nabla) \to (B, g_B, \widehat{\nabla})$ is called a φ-invariant if $\overline{M}$ is a φ-invariant submanifold of M, that is, $\varphi(\mathcal{V}_p(M)) \subset \mathcal{V}_p(M)$ (see Lemma 4.4 (1)). In this section, we discuss the two cases of $\xi \in \mathcal{H}(M)$ and $\xi \in \mathcal{V}(M)$ in the φ-invariant Sasaki-like statistical submersion. And we give an example such that $t = 0$.

5.1. Case of $\xi \in \mathcal{H}(M)$.

From Lemmas 4.1, 4.2 and 4.3, we find

Lemma 5.1. *Let π be an almost contact metric submersion such that $\xi \in \mathcal{H}(M)$. If $\overline{M}$ is φ-invariant, then we get*

$$\begin{aligned} &P^2 = -I + \eta \otimes \xi, && FP + fF = 0, && f^2 = -I, \\ &(P^*)^2 = -I + \eta \otimes \xi, && P^*t^* + t^*f^* = 0, && (f^*)^2 = -I. \end{aligned}$$

Moreover, each fiber is of even dimension.

Lemma 5.2. *Let π be an almost contact metric submersion such that $\xi \in \mathcal{H}(M)$. If $\overline{M}$ is φ-invariant, then we obtain*

$$\begin{aligned} &P\xi = 0, && F\xi = 0, && \eta(PX) = 0, \\ &P^*\xi = 0, && \eta(P^*X) = 0, && \eta(t^*V) = 0. \end{aligned}$$

Lemma 5.3. *Let π be an almost contact metric submersion such that $\xi \in \mathcal{H}(M)$. If $\overline{M}$ is φ-invariant, then we have $g(PX, P^*Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y)$ and $g(fU, f^*V) = g(U, V)$.*

Moreover, we have from Propositions 4.9, 4.10 and Lemma 4.11

Lemma 5.4. *For the φ-invariant Sasaki-like statistical submersion such that $\xi \in \mathcal{H}(M)$, we get*

$$\begin{aligned} &\mathcal{H}\nabla_U\xi = 0, && T_U\xi = -\varepsilon fU, && \mathcal{H}\nabla_X\xi = -\varepsilon PX, && A_X\xi = -\varepsilon FX, \\ &\mathcal{H}\nabla^*_U\xi = -\varepsilon t^*U, && T^*_U\xi = -\varepsilon f^*U, && \mathcal{H}\nabla^*_X\xi = -\varepsilon P^*X, && A^*_X\xi = 0. \end{aligned}$$

Lemma 5.5. *For the φ-invariant Sasaki-like statistical submersion such that $\xi \in \mathcal{H}(M)$, we find*

$$(\overline{\nabla}_U f)V - F(T_U V) = 0, \tag{5.1}$$

$$T_U(fV) - P(T_U V) = g(U,V)\xi, \tag{5.2}$$

$$(\mathcal{V}\nabla_U F)Y + T_U(PY) - f(T_U Y) = -\varepsilon\eta(Y)U, \tag{5.3}$$

$$(\mathcal{H}\nabla_U P)Y + T_U(FY) = 0, \tag{5.4}$$

$$(\mathcal{V}\nabla_X f)V - F(A_X V) = 0, \tag{5.5}$$

$$A_X(fV) - P(A_X V) = 0, \tag{5.6}$$

$$(\mathcal{V}\nabla_X F)Y + A_X(PY) - f(A_X Y) = 0, \tag{5.7}$$

$$(\mathcal{H}\nabla_X P)Y + A_X(FY) = g(X,Y)\xi - \varepsilon\eta(Y)X. \tag{5.8}$$

Corollary 5.6. *For the φ-invariant Sasaki-like statistical submersion such that $\xi \in \mathcal{H}(M)$, we find*

$$\begin{aligned}
&(\overline{\nabla}^*_U f^*)V + T^*_U(t^*V) = 0,\\
&(\mathcal{H}\nabla^*_U t^*)V + T^*_U(f^*V) - P^*(T^*_U V) = g(U,V)\xi,\\
&T^*_U(P^*Y) - f^*(T^*_U Y) = -\varepsilon\eta(Y)U,\\
&(\mathcal{H}\nabla^*_U P^*)Y - t^*(T^*_U Y) = 0,\\
&(\mathcal{V}\nabla^*_X f^*)V + A^*_X(t^*V) = 0,\\
&(\mathcal{H}\nabla^*_X t^*)V + A^*_X(f^*V) - P^*(A^*_X V) = 0,\\
&A^*_X(P^*Y) - f^*(A^*_X Y) = 0,\\
&(\mathcal{H}\nabla^*_X P^*)Y - t^*(A^*_X Y) = g(X,Y)\xi - \varepsilon\eta(Y)X.
\end{aligned}$$

Lemma 5.7. *If the Sasaki-like statistical submersion is φ-invariant such that $\xi \in \mathcal{H}(M)$, then we find*

$$\begin{aligned}
&\eta(T_U V) = -g(U,fV), \qquad \eta(A_X V) = 0, \qquad f^* = -f,\\
&\eta(T^*_U V) = g(U,fV), \qquad \eta(A^*_X V) = -g(X,t^*V).
\end{aligned}$$

It is easy to see from of Lemmas 5.3 and 5.7 that we find $g(fU,fV) = -g(U,V)$, which implies that $\sum \varepsilon_\alpha g(fU_\alpha, fU_\alpha) = -s$. Thus we have

Proposition 5.8. *If g is a positive definite, then the φ-invariant Sasaki-like statistical submersion such that $\xi \in \mathcal{H}(M)$ does not exist.*

Proposition 5.9. *For the φ-invariant Sasaki-like statistical submersion such that $\xi \in \mathcal{H}(M)$, if $U \in \mathcal{V}(M)$ is timelike (resp. spacelike), then $fU \in \mathcal{V}(M)$ is spacelike (resp. timelike).*

We consider that the case of g is indefinite. We assume $\mathcal{V}\nabla_X F = 0$ holds. It is clear from (5.7) that $A_X(PY) = f(A_X Y)$, which yields that $fFX = 0$, namely, $F = 0$. Hence we have from Lemmas 5.4 and 5.5

Theorem 5.10. *In the φ-invariant Sasaki-like statistical submersion such that $\xi \in \mathcal{H}(M)$, if $\mathcal{V}\nabla_X F = 0$ holds, then*

(1) *each fiber is* φ^**-invariant, moreover,* $(\overline{M}, \overline{g}, \overline{\nabla}, \overline{f})$ *is a Kähler-like statistical manifold.*

(2) *the base space* $(B, g_B, \widehat{\nabla})$ *is a Sasaki-like statistical manifold with Sasaki-like structure* $(\widehat{P}, \widehat{\xi}, \widehat{\eta})$.

We suppose the total space is of constant curvature ε. Changing V to fV in (4.12), we get

$$\begin{aligned} &g((\nabla_X T)_U V, P^*Y) - g((\nabla_U A)_X V, P^*Y) - g((\mathcal{H}\nabla_X P)(T_U V), Y) + g(T_U\{(\mathcal{V}\nabla_X f)V\}, Y) \\ &+g((\mathcal{H}\nabla_U P)(A_X V), Y) - g(A_X\{(\overline{\nabla}_U f)V\}, Y) + \varepsilon g(U, V)g(PX, Y) + \varepsilon\eta(Y)g(S_U X, V) \\ &-g(A_X U, A^*_Y(fV)) + g(T_U X, T^*_{fV}Y) = -\varepsilon g(U, fV)g(X, Y) - \varepsilon g(U, V)g(PX, Y). \end{aligned}$$

Also, if we change Y to P^*Y in (4.12), then we find

$$\begin{aligned} &g((\nabla_X T)_U V, P^*Y) - g((\nabla_U A)_X V, P^*Y) - g(A_X U, A^*_Y(fV)) + g(T_U X, T^*_{fV}Y) \\ &+\varepsilon\eta(Y)g(T_U X, V) = -\varepsilon g(U, V)g(PX, Y). \end{aligned}$$

Thus we obtain from above two equations

$$\begin{aligned} &g((\mathcal{H}\nabla_X P)(T_U V), Y) - g(T_U\{(\mathcal{V}\nabla_X f)V\}, Y) - g((\mathcal{H}\nabla_U P)(A_X V), Y) \\ &+g(A_X\{(\overline{\nabla}_U f)V\}, Y) - \varepsilon\eta(Y)g(T_U V, X) = \varepsilon g(U, fV)g(X, Y). \end{aligned}$$

We assume that $\mathcal{H}\nabla P = 0$ and $\mathcal{V}\nabla f = 0$ hold. Then we have

Lemma 5.11. *Let* M *is of constant curvature* ε *in the* φ*-invariant Sasaki-like statistical submersion such that* $\xi \in \mathcal{H}(M)$*. If* $\mathcal{H}\nabla P = 0$ *and* $\mathcal{V}\nabla f = 0$ *hold, then we find* $T_U V = -g(U, fV)\xi$*. Moreover, the mean curvature vector field* N *is parallel to the structure vector field* ξ *if* $\operatorname{tr} f \neq 0$.

It should be noticed that $N = 0$ is equivalent to $\operatorname{tr} f = 0$. From (4.7) and Lemma 5.11, we get

$$\overline{R}(U, V)W = \varepsilon\{g(V, W)U - g(U, W)V - g(V, fW)fU + g(U, fW)fV\},$$

which denotes that $\overline{\mathrm{Ric}}(V, W) = \varepsilon\{(s-2)g(V, W) - (\operatorname{tr} f)g(V, fW)\}$. Thus we have

Lemma 5.12. *Let* M *be of constant curvature* ε *in the* φ*-invariant Sasaki-like statistical submersion such that* $\xi \in \mathcal{H}(M)$*. If* $\mathcal{H}\nabla P = 0$, $\mathcal{V}\nabla f = 0$, $\operatorname{tr} f = 0$ *and* $s \geq 3$ *hold, then each fiber is Einstein.*

Next, let (M, g, ∇) be of constant φ-holomorphic sectional curvature c. Changing X to PX in (4.22), we get

$$\begin{aligned} &g((\nabla_U T)_V X, f^*W) - g((\nabla_V T)_U X, f^*W) + g(\nabla_U\{(\mathcal{V}\nabla_V F)X\}, W) - g((\mathcal{V}\nabla_{\overline{\nabla}_U V} F)X, W) \\ &-g((\mathcal{V}\nabla_V F)(\mathcal{H}\nabla_U X), W) - g(\nabla_V\{(\mathcal{V}\nabla_U F)X\}, W) + g((\mathcal{V}\nabla_{\overline{\nabla}_V U} F)X, W) \\ &+g((\mathcal{V}\nabla_U F)(\mathcal{H}\nabla_V X), W) - g((\overline{\nabla}_U f)(T_V X), W) + g((\overline{\nabla}_V f)(T_U X), W) \\ &+g(T_V\{(\mathcal{H}\nabla_U P)X\}, W) - g(T_U\{(\mathcal{H}\nabla_V P)X\}, W) + \varepsilon g(U, FX)g(V, W) - \varepsilon g(V, FX)g(U, W) \\ &= \frac{1}{4}(c - \varepsilon)\{g(V, FPX)g(fU, W) - g(U, FPX)g(fV, W)\}. \end{aligned}$$

Also, if we change W to f^*W in (4.22), then we obtain

$$g((\nabla_U T)_V X, f^*W) - g((\nabla_V T)_U X, f^*W) = \frac{1}{4}\{-g(V, FX)g(U, W) + g(U, FX)g(V, W)\}.$$

Therefore we find from above two equations

$$\begin{aligned}&g(\nabla_U\{(\mathcal{V}\nabla_V F)X\}, W) - g((\mathcal{V}\nabla_{\overline{\nabla}_U V} F)X, W) - g((\mathcal{V}\nabla_V F)(\mathcal{H}\nabla_U X), W)\\ &-g(\nabla_V\{(\mathcal{V}\nabla_U F)X\}, W) + g((\mathcal{V}\nabla_{\overline{\nabla}_V U} F)X, W) + g((\mathcal{V}\nabla_U F)(\mathcal{H}\nabla_V X), W)\\ &-g((\overline{\nabla}_U f)(T_V X), W) + g((\overline{\nabla}_V f)(T_U X), W)\\ &+g(T_V\{(\mathcal{H}\nabla_U P)X\}, W) - g(T_U\{(\mathcal{H}\nabla_V P)X\}, W)\\ &= \frac{1}{4}(c-\varepsilon)\{g(V, FPX)g(fU, W) - g(U, FPX)g(fV, W)\}\\ &\quad+\frac{1}{4}(c+3\varepsilon)\{g(V, FX)g(U, W) - g(U, FX)g(V, W)\}.\end{aligned}$$

If $\mathcal{H}\nabla_U P = 0$, $\mathcal{V}\nabla_U F = 0$ and $\overline{\nabla}_U f = 0$, then we get

$$\begin{aligned}&(c-\varepsilon)\{g(V, FPX)g(fU, W) - g(U, FPX)g(fV, W)\}\\ &+(c+3\varepsilon)\{g(V, FX)g(U, W) - g(U, FX)g(V, W)\} = 0;\end{aligned}$$

moreover, if we change W and X to f^*W and PX, respectively, then above equation can be rewritten as follows:

$$\begin{aligned}&(c+3\varepsilon)\{g(V, FPX)g(fU, W) - g(U, FPX)g(fV, W)\}\\ &+(c-\varepsilon)\{g(V, FX)g(U, W) - g(U, FX)g(V, W)\} = 0.\end{aligned}$$

Furthermore, it is easy to see from above two equations that

$$(c+\varepsilon)\{g(V, FX)g(U, W) - g(U, FX)g(V, W)\} = 0,$$

which implies that $c = -\varepsilon$ or $g(V, FX)g(U, W) - g(U, FX)g(V, W) = 0$, that is, $(s-1)FX = 0$.
Hence we have

Theorem 5.13. *Let M be of constant φ-holomorphic sectional curvature c in the φ-invariant Sasaki-like statistical submersion such that $\xi \in \mathcal{H}(M)$. If $\mathcal{H}\nabla_U P = 0$, $\mathcal{V}\nabla_U F = 0$ and $\overline{\nabla}_U f = 0$ hold, then*

(1) *$c = -\varepsilon$ or*
(2) *each fiber is φ^*-invariant if $s \geq 2$.*

Next, changing V to fV in (4.25), we get

$$\begin{aligned}&g((\nabla_X T)_U V, P^*Y) - g((\nabla_U A)_X V, P^*Y) - g((\mathcal{H}\nabla_X P)(T_U V), Y)\\ &+g(T_U\{(\mathcal{V}\nabla_X f)V\}, Y) + g((\mathcal{H}\nabla_U P)(A_X V), Y) - g(A_X\{(\overline{\nabla}_U f)V\}, Y)\\ &+\varepsilon\eta(Y)g(S_U X, V) - g(A_X U, A^*_Y(fV)) + g(T_U X, T^*_{fV} Y)\\ &= \frac{\varepsilon}{4}(c-\varepsilon)\eta(X)\eta(Y)g(U, fV) - \frac{1}{4}(c+3\varepsilon)\{g(U, fV)g(X, Y) + g(U, V)g(PX, Y)\}.\end{aligned}$$

Also, if we change Y to P^*Y in (4.25), then we obtain

$$\begin{aligned}&g((\nabla_X T)_U V, P^*Y) - g((\nabla_U A)_X V, P^*Y) - g(A_X U, A^*_Y(fV))\\ &+g(T_U X, T^*_{fV} Y) + \varepsilon\eta(Y)g(T_U X, V)\\ &= -\frac{1}{4}(c+3\varepsilon)g(U, V)g(PX, Y) - \frac{1}{4}(c-\varepsilon)g(U, fV)\{g(X, Y) - \varepsilon\eta(X)\eta(Y)\}.\end{aligned}$$

It is clear from above two equationa that

$$g((\mathcal{H}\nabla_X P)(T_U V), Y) - g(T_U\{(\mathcal{V}\nabla_X f)V\}, Y) - g((\mathcal{H}\nabla_U P)(A_X V), Y)$$
$$+g(A_X\{(\overline{\nabla}_U f)V\}, Y) - \varepsilon\eta(Y)g(X, T_U V) = \varepsilon g(U, fV)g(X, Y).$$

We assume that $\mathcal{H}\nabla P = 0$ and $\mathcal{V}\nabla f = 0$. Then we find $\eta(Y)g(X, T_U V) = -g(U, fV)g(X, Y)$. Hence we have

Lemma 5.14. *Let M be of constant φ-holomorphic sectional curvature c in the φ-invariant Sasaki-like statistical submersion such that $\xi \in \mathcal{H}(M)$. If $\mathcal{H}\nabla P = 0$ and $\mathcal{V}\nabla f = 0$ hold, then*

$$T_U V = -g(U, fV)\xi.$$

From (4.20) and Lemma 5.14, we get

$$\overline{R}(U, V)W = \frac{1}{4}(c + 3\varepsilon)\{g(V, W)U - g(U, W)V - g(V, fW)fU + g(U, fW)fV\},$$

which yields that $\overline{\mathrm{Ric}}(V, W) = \varepsilon\{(s - 2)g(V, W) - (\mathrm{tr}\, f)g(V, fW)\}$. Thus we have

Theorem 5.15. *Let M be of constant φ-holomorphic sectional curvature c in the φ-invariant Sasaki-like statistical submersion such that $\xi \in \mathcal{H}(M)$. In the case of $\mathcal{H}\nabla P = 0$ and $\mathcal{V}\nabla f = 0$, we get*

(1) *if $c = -3\varepsilon$, then each fiber is flat.*
(2) *if $\mathrm{tr}\, f = 0$ and $s \geq 3$, then each fiber is Einstein.*

5.2. **Case of $\xi \in \mathcal{V}(M)$.** From Lemmas 4.1, 4.2 and 4.3, we find

Lemma 5.16. *Let π be an almost contact metric submersion such that $\xi \in \mathcal{V}(M)$. If $\overline{M}$ is φ-invariant, then we get*

$$P^2 = -I, \qquad FP + fF = 0, \qquad f^2 = -I + \eta \otimes \xi,$$
$$(P^*)^2 = -I, \qquad P^*t^* + t^*f^* = 0, \qquad (f^*)^2 = -I + \eta \otimes \xi.$$

Lemma 5.17. *Let π be an almost contact metric submersion such that $\xi \in \mathcal{V}(M)$. If $\overline{M}$ is φ-invariant, then we obtain*

$$f\xi = 0, \qquad \eta(FX) = 0, \qquad \eta(fV) = 0,$$
$$t^*\xi = 0, \qquad f^*\xi = 0, \qquad \eta(f^*V) = 0.$$

Lemma 5.18. *Let π be an almost contact metric submersion such that $\xi \in \mathcal{V}(M)$. If $\overline{M}$ is φ-invariant, then we have*

$$g(PX, P^*Y) = g(X, Y), \qquad g(fU, f^*V) = g(U, V) - \varepsilon\eta(U)\eta(V).$$

Moreover, we have from Propositions 4.9, 4.10 and Lemma 4.11

Lemma 5.19. *For the φ-invariant Sasaki-like statistical submersion such that $\xi \in \mathcal{V}(M)$, we get*

$$T_U\xi = 0, \qquad \overline{\nabla}_U\xi = -\varepsilon fU, \qquad A_X\xi = -\varepsilon PX, \qquad \mathcal{V}\nabla_X\xi = -\varepsilon FX,$$
$$T^*_U\xi = -\varepsilon t^*U, \qquad \overline{\nabla}^*_U\xi = -\varepsilon f^*U, \qquad A^*_X\xi = -\varepsilon P^*X, \qquad \mathcal{V}\nabla^*_X\xi = 0.$$

Lemma 5.20. *For the φ-invariant Sasaki-like statistical submersion such that $\xi \in \mathcal{V}(M)$, we find*

$$(\overline{\nabla}_U f)V - F(T_U V) = g(U,V)\xi - \varepsilon\eta(V)U, \tag{5.9}$$
$$T_U(fV) - P(T_U V) = 0, \tag{5.10}$$
$$(\mathcal{V}\nabla_U F)Y + T_U(PY) - f(T_U Y) = 0, \tag{5.11}$$
$$(\mathcal{H}\nabla_U P)Y + T_U(FY) = 0, \tag{5.12}$$
$$(\mathcal{V}\nabla_X f)V - F(A_X V) = 0, \tag{5.13}$$
$$A_X(fV) - P(A_X V) = -\varepsilon\eta(V)X, \tag{5.14}$$
$$(\mathcal{V}\nabla_X F)Y + A_X(PY) - f(A_X Y) = g(X,Y)\xi, \tag{5.15}$$
$$(\mathcal{H}\nabla_X P)Y + A_X(FY) = 0. \tag{5.16}$$

Corollary 5.21. *For the φ-invariant Sasaki-like statistical submersion such that $\xi \in \mathcal{V}(M)$, we find*

$$(\overline{\nabla}^*_U f^*)V + T^*_U(t^*V) = g(U,V)\xi - \varepsilon\eta(V)U, \tag{5.17}$$
$$(\mathcal{H}\nabla^*_U t^*)V + T^*_U(f^*V) - P^*(T^*_U V) = 0, \tag{5.18}$$
$$T^*_U(P^*Y) - f^*(T^*_U Y) = 0, \tag{5.19}$$
$$(\mathcal{H}\nabla^*_U P^*)Y - t^*(T^*_U Y) = 0, \tag{5.20}$$
$$(\mathcal{V}\nabla^*_X f^*)V + A^*_X(t^*V) = 0, \tag{5.21}$$
$$(\mathcal{H}\nabla^*_X t^*)V + A^*_X(f^*V) - P^*(A^*_X V) = -\varepsilon\eta(V)X, \tag{5.22}$$
$$A^*_X(P^*Y) - f^*(A^*_X Y) = g(X,Y)\xi, \tag{5.23}$$
$$(\mathcal{H}\nabla^*_X P^*)Y - t^*(A^*_X Y) = 0. \tag{5.24}$$

Lemma 5.22. *If the Sasaki-like statistical submersion is φ-invariant such that $\xi \in \mathcal{V}(M)$, then we find*

$$\eta(T_U Y) = -g(U, FY), \qquad \eta(A_X Y) = -g(X, PY),$$
$$\eta(T^*_U Y) = 0, \qquad \eta(A^*_X Y) = -g(X, P^*Y).$$

From (5.13), $\mathcal{V}\nabla_X f = 0$ if and only if $F(A_X V) = 0$. If we change V to ξ, then we find $FPX = 0$, namely, $FX = 0$. Hence we have

Lemma 5.23. *In the φ-invariant Sasaki-like statistical submersion such that $\xi \in \mathcal{V}(M)$, it is equivalent that $\mathcal{V}\nabla_X f = 0$ holds and each fiber is φ^*-invariant.*

Because of (5.9), (5.16), Lemmas 5.19 and 5.23, we have

Theorem 5.24. *In the φ-invariant Sasaki-like statistical submersion such that $\xi \in \mathcal{V}(M)$, if $\mathcal{V}\nabla_X f = 0$, then we find*

(1) *each fiber $(\overline{M}, \overline{g}, \overline{\nabla})$ is a Sasaki-like statistical manifold with Sasaki-like structure $(\overline{f}, \overline{\xi}, \overline{\eta})$.*
(2) *the base space $(B, g_B, \widehat{\nabla}, \widehat{P})$ is a Kähler-like statistical manifold.*

We assume that $\mathcal{V}\nabla_X f = 0$ holds. It is easy to see from (5.23) that $A^*_Y(P^*X) - f^*(A^*_Y X) = g(X,Y)\xi$, which means that

$$-A_{P^*X}Y + f^*(A_X Y) = g(X,Y)\xi.$$

Moreover, using (5.15), we have

$$(f + f^*)A_X Y = A_X(PY) + A_{P^*X}Y.$$

Also, if PY is basic, then we get $A_{PY}U - P(A_Y U) = 0$ from (5.12). Therefore we have $g(U, A_X(PY)) + g(U, A_{P^*X}Y) = 0$, which implies that $A_X(PY) + A_{P^*X}Y = 0$. Thus $(f + f^*)A_X Y = 0$ holds. When $\operatorname{rank}(f + f^*) = \dim \overline{M} - 1$ holds, we obtain $A_X Y = -g(X, PY)\xi$. Hence we have

Lemma 5.25. *In the φ-invariant Sasaki-like statistical submersion such that $\xi \in \mathcal{V}(M)$, if $\mathcal{V}\nabla_X f = 0$ and $\operatorname{rank}(f + f^*) = \dim \overline{M} - 1$ hold, then we get $A_X Y = -g(X, PY)\xi$.*

We suppose the total space is of constant curvature ε. Because of (4.18) and Lemma 5.25, we get

$$\begin{aligned}\widehat{R}(X,Y)Z = \varepsilon[\,& g(Y,Z)X - g(X,Z)Y - g(Y,PZ)PX + g(X,PZ)PY \\ & + \{g(X,PY) - g(PX,Y)\}PZ\,].\end{aligned}$$

Hence we have

Theorem 5.26. *Let M be of constant curvature ε in the φ-invariant Sasaki-like statistical submersion such that $\xi \in \mathcal{V}(M)$. If $\mathcal{V}\nabla_X f = 0$ and $\operatorname{rank}(f + f^*) = \dim \overline{M} - 1$ hold, then the base space $(B, g_B, \widehat{\nabla}, \widehat{P})$ is of constant holomorphic sectional curvature 4ε.*

Next, when the total space is of constant φ-holomorphic sectional curvature c, equation (4.31) can be rewritten as follows from Lemma 5.25:

$$\begin{aligned}\widehat{R}(X,Y)Z \;=\; & \frac{1}{4}(c + 3\varepsilon)[\,g(Y,Z)X - g(X,Z)Y - g(Y,PZ)PX + g(X,PZ)PY \\ & - \{\,g(X,PY) - g(PX,Y)\,\}PZ\,].\end{aligned}$$

Thus we have

Theorem 5.27. *Let M be of constant φ-holomorphic sectional curvature c in the φ-invariant Sasaki-like statistical submersion such that $\xi \in \mathcal{V}(M)$. If $\mathcal{V}\nabla_X f = 0$ and $\operatorname{rank}(f + f^*) = \dim \overline{M} - 1$ hold, then the base space $(B, g_B, \widehat{\nabla}, \widehat{P})$ is of constant holomorphic sectional curvature $c + 3\varepsilon$.*

Example 5.28. Let π be a Sasaki-like statistical submersion of Example 4.23. It is easy to see from

$$\begin{aligned} PX_1 &= -X_1 - 2X_2, & PX_2 &= X_1 + X_2, \\ P^*X_1 &= X_1 + X_2, & P^*X_2 &= -2X_1 - X_2 \end{aligned}$$

that Theorems 4.6 (2) and 5.24 (2) holds. Moreover, we find $A_{X_i}X_j = -g(X_i, PX_j)\xi$ $(i, j = 1, 2)$ (see Lemma 5.25).

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