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A NOTE ON LEVI-FLAT HYPERSURFACES IN $\mathbb{C}P^3$

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Abstract. We investigate Levi-flat hypersurfaces in $\mathbb{C}P^3$, focusing on existence results rather than the more commonly studied non-existence theorems. We show that every Levi-flat hypersurface in $\mathbb{C}P^3$ induces an associated Lagrangian submanifold compatible with the two standard almost complex structures. Additionally, we demonstrate that each Levi-flat hypersurface is an example of a hypersurface with non-constant angle function. Finally, we provide a concrete example of a Levi-flat hypersurface to illustrate the theory.

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1. Introduction

Levi-flat hypersurfaces link real and complex geometry and are essential in the study of functions on complex manifolds, in a sense that will become clear later. Consider an almost complex manifold (M, J), that is, J is a $(1,1)$-tensor on M that satisfies $J^2 = -\mathbb{1}$. Given a (real) hypersurface $\mathcal{H}$ in (M, J), the *holomorphic tangent space* or *CR-structure* $H_p\mathcal{H}$ at a point p of $\mathcal{H}$ is defined to be $H_p\mathcal{H} = T_p\mathcal{H} \cap JT_p\mathcal{H}$. Note that the term "CR" comes from Cauchy-Riemann or Complex-Real.

Definition 1.1. A (real) hypersurface $\mathcal{H}$ of an almost complex manifold (M, J) is called *Levi-flat* if its CR-structure is integrable.

In other words, a Levi-flat hypersurface is foliated with (almost) complex submanifolds of (real) dimension $\dim(M) - 2$. We call this the *Levi-foliation* of $\mathcal{H}$. Just as minimal submanifolds have vanishing mean curvature, Levi-flat hypersurfaces have a vanishing complex curvature, hence the name. We give two easy examples of such hypersurfaces from the literature [8].

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The first example is the hypersurface $\{\mathrm{im}(z_n) = 0\}$ of $\mathbb{C}^n = \{(z_1, \ldots, z_n) \mid z_i \in \mathbb{C}\}$ which is foliated by copies of $\mathbb{C}^{n-1}$. The second, compact, example lies inside the space $\mathbb{C}P^1 \times \mathbb{C}P^1$: for any simple smooth closed curve γ, $\mathbb{C}P^1 \times \gamma$ is a Levi-flat hypersurface with Levi-foliation $\{\mathbb{C}P^1 \times \{p\} \mid p \in \gamma\}$.

To understand the significance of Levi-flat hypersurfaces, we look at some standard results about holomorphic functions on $\mathbb{C}^n$. Recall the concept of a *domain of holomorphy* on a complex manifold: a connected open set Ω such that a non-constant holomorphic function on Ω can never be extended to an open U with $U \cap \Omega \neq \emptyset$ and $U \setminus \Omega \neq \emptyset$. In this sense, they are maximal domains of holomorphic functions. When one wants to analytically continue a given holomorphic function, these domains of holomorphy are the limiting factor. Unfortunately, to determine domains of holomorphy is not easy from the definition and we therefore turn to equivalent notions that are easier to check for $\mathbb{C}^n$.

One particularly useful such notion is Levi-pseudoconvexity: given an open with boundary $\{\rho = 0\}$, we say that it is Levi-pseudoconvex if the quadratic form

$$\mathbb{C}^n \to \mathbb{C} : w \mapsto \sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial z_j} w_i \bar{w}_j \tag{1.1}$$

is non-negative everywhere. This quadratic form is known as the *Levi-form.* A special case occurs when the Levi-form vanishes completely, which is equivalent with the Definition 1.1 of a Levi-flat hypersurface [3]. As such, open connected sets in $\mathbb{C}^n$ that have a Levi-flat hypersurface as boundary form an important class of domains of holomorphy.

Since a complex manifold is locally $\mathbb{C}^n$, we can locally view a hypersurface as the zero locus of a function $\rho : M \to \mathbb{R}$. The quadratic form associated to ρ is the local description of the so-called *Levi-form* on M. Again, one can show that if this Levi-form vanishes, then and only then the hypersurface is Levi-flat. On complex manifolds, however, not all open sets with a Levi-flat hypersurface as boundary are domains of holomorphy. Ohsawa [7] recalls examples U of which the boundary is a Levi-flat hypersurface, and where 1) non-constant holomorphic functions do not exist on U, or 2) where U is a *Stein manifold.* Note that one of the defining properties of a Stein manifold is holomorphic seperability: for distinct points p and q, there exists a holomorphic function f such that $f(p) \neq f(q)$. Clearly, on complex manifolds, Levi-flat hypersurfaces have a richer interaction with domains of holomorphy, than on $\mathbb{C}^n$. The tension between these two types of examples motivated researchers to start studying and classifying Levi-flat hypersurfaces [7].

There are already many (non-existence) results of Levi-flat hypersurfaces in $\mathbb{C}P^3$ with its standard complex structure. For instance, there are no closed and real analytic examples [7], which can be generalised to the statement that there are no closed Levi-flat hypersurfaces of class $\mathcal{C}^{12}$ [7]. Another instance is the non-existence of smooth real algebraic Levi-flat hypersurfaces [8]. There do exist, however, non-closed smooth Levi-flat hypersurfaces, as we show in this note.

Even though the concept of a Levi-flat hypersurface is independent of any metric, a metric can still help with the analysis. For example, we show that the existence of a strict nearly Kähler metric on the ambient space is an obstruction to the existence of Levi-flat hypersurfaces. Given a metric g, we may also consider the foliated 2-form

$$H\mathcal{H} \times H\mathcal{H} \to \mathbb{R} : (X, Y) \mapsto g([X, Y], JN) \tag{1.2}$$

on the Levi-foliation, where N is a unit normal on the hypersurface. In other words, this object behaves as a 2-form on the holomorphic tangent space. Checking that this foliated 2-form vanishes completely on $H\mathcal{H}$ is then equivalent to checking that $\mathcal{H}$ is a Levi-flat hypersurface. Up to possible constants, this foliated real 2-form must coincide with the Levi-form evaluated on real vector fields.

The structure of this work is as follows. We start with outlining the basic structures of the Kähler and nearly Kähler $\mathbb{C}P^3$ in Section 2. Moreover, we also recall the basic description of hypersurfaces in the nearly Kähler $\mathbb{C}P^3$ in this section. In Section 3, we collect some results about Levi-flat hypersurfaces in $\mathbb{C}P^3$ with its two standard almost complex structures. Here, we interpret a result of the literature to see that the existence of a strictly nearly Kähler metric obstructs the existence of Levi-flat hypersurfaces. As such, there are none in $\mathbb{C}P^3$ equipped with one of its two almost complex structures. We move to $\mathbb{C}P^3$ with its complex structure inherited from $\mathbb{C}^4$ and prove that any Levi-flat hypersurface must have a non-constant angle function. We proceed by giving a specific example of a Levi-flat hypersurface in $\mathbb{C}P^3$, and generalising it to a large family of examples. We conclude by associating a family of Lagrangian immersions (for both the Kähler and nearly Kähler $\mathbb{C}P^3$) to each Levi-flat hypersurface.

2. Hypersurfaces in the nearly Kähler $\mathbb{C}P^3$

We first recall some basic notions of the description of Kähler and nearly Kähler $\mathbb{C}P^3$, and then recall a local frame on hypersurfaces in $\mathbb{C}P^3$.

The round $\mathbb{S}^7$ embedded in $\mathbb{C}^4$ is equipped with the Sasakian structure coming from the multiplication with i in $\mathbb{C}^4$. When considering the standard projection $\mathbb{C}^4 \to \mathbb{C}P^3$, we can also restrict to the seven-sphere, to obtain the Hopf fibration $\pi : \mathbb{S}^7 \to \mathbb{C}P^3$. There is a unique metric $g_\circ$ on $\mathbb{C}P^3$, called the Fubini-Study metric, such that π is a Riemannian submersion. The vertical space is given by $V(p) = \mathrm{Span}\{ip\}$. The Sasakian structure then induces an almost complex structure $J_\circ$ on $\mathbb{C}P^3$ which is Kähler, i.e. $\nabla^\circ J_\circ = 0$, with ∇° the Levi-Civita connection of $g_\circ$. Concretely, the complex structure $J_\circ$ acts on a vector field $X \in \mathfrak{X}(\mathbb{C}P^3)$ by taking the unique horizontal lift $\tilde{X}$ under π, multiplying by i and then projecting back to $\mathbb{C}P^3$: $J_\circ X = \mathrm{d}\pi(i\tilde{X})$. The link with the round $\mathbb{S}^7 \subset \mathbb{C}^4$ makes it easy to work with Kähler $(\mathbb{C}P^3, g_\circ, J_\circ)$. Finally, we recall that the full isometry group of the Kähler $\mathbb{C}P^3$ is given by $\mathrm{PU}(4) \rtimes \mathbb{Z}_2$.

The nearly Kähler $\mathbb{C}P^3$ is defined as the twistor space $\tau : \mathbb{C}P^3 \to \mathbb{S}^4$ over $\mathbb{S}^4 \cong \mathbb{H}P^1$. Alternatively, yet equivalently, we [5] can define the nearly Kähler $\mathbb{C}P^3$ more similar to the Kähler $\mathbb{C}P^3$. To this end, identify $\mathbb{C}^4$ with $\mathbb{H}^2$ and consider $\mathbb{S}^7 \subset \mathbb{H}^2$. The advantage of this, is to have the extra structures j, k on $\mathbb{S}^7$. Consider the two dimensional distribution $\tilde{\mathcal{D}}_1^2(p) = \mathrm{Span}\{jp, kp\}$ and let $\tilde{\mathcal{D}}_2^4$ be such that $T\mathbb{S}^7 = V \oplus \tilde{\mathcal{D}}_1^2 \oplus \tilde{\mathcal{D}}_2^4$ orthogonally. It was shown [5] that $\mathcal{D}_1^2 = \mathrm{d}\pi(\tilde{\mathcal{D}}_1^2)$ is well-defined, and similarly, $\mathcal{D}_2^4$ is well-defined. Define the almost product structure P by demanding it is the identity on $\mathcal{D}_2^4$ and minus the identity on $\mathcal{D}_1^2$. Then, $J = PJ_\circ = J_\circ P$ is a new almost complex structure, that together with the metric

$$g(X, Y) = \frac{3}{2} g_\circ(X, Y) + \frac{1}{2} g_\circ(X, PY) \tag{2.1}$$

forms the nearly Kähler $(\mathbb{C}P^3, g, J)$. The full isometry group of $(\mathbb{C}P^3, g)$ is given [5, 1] by $\mathrm{PSp}(2) \rtimes \mathbb{Z}_2$. Its concrete action is outlined in [1].

We now focus on hypersurfaces in $\mathbb{C}P^3$. For later convenience (because it has the smaller isometry group), we view them as isometrically immersed in the nearly Kähler $\mathbb{C}P^3$. We recall [4] the angle function of a hypersurface in $(\mathbb{C}P^3, g, J)$: the function θ such that $g(JN, J_\circ N) = g(PN, N) = \cos(2\theta)$ for a g-unit normal N. At any point, it can be taken to lie between 0 and $\pi/2$, but it cannot be constant and equal to $\pi/2$. If the angle does not vanish everywhere, we can use the result [4, Lemma 3.5] that the following is a g-orthonormal frame of $T\mathbb{C}P^3$ along the hypersurface:

(2.2)
$$e_1 = \frac{JN + J_\circ N}{2\cos\theta}, \quad e_2 = \frac{JN - J_\circ N}{2\sin\theta}, \quad e_3 = \frac{(PN)^\top}{\sin 2\theta}, \quad e_4 = \frac{G(PN, N)}{\sin 2\theta}, \quad e_5 = Je_4, \quad N,$$

where $G = \nabla J$, $(PN)^\top = PN - \cos(2\theta)N$ and $Je_4 = J_\circ e_4$. In this frame, we will express the second fundamental form h with respect to the nearly Kähler metric in terms of its component functions:

$$\alpha_{ij} = g(h(e_i, e_j), N) = \alpha_{ji}. \tag{2.3}$$

Finally, we recall that by using that the nearly Kähler $\mathbb{C}P^3$ is of constant type, i.e. there is a condition on the length of G, we find [4] the derivatives of the angle θ in terms of α_{ij}:

$$e_i(\theta) = \alpha_{3i} - \frac{1}{2}\delta_{i,5}, \tag{2.4}$$

with δ the Kronecker delta.

3. Levi-flat hypersurfaces in $\mathbb{C}P^3$

There are two natural almost complex structures on $\mathbb{C}P^3$: J, and $J_\circ$. These are the ones so that $\mathbb{C}P^3$ with these structure (and the correct metric) becomes a homogeneous nearly Kähler or Kähler manifold, respectively. As such, a priori, there are two almost complex manifolds to consider Levi-flat hypersurfaces in. However, we recall the following result.

Theorem 3.1 (Lin, Vrancken, Wijffels, 2020 [6])**.** *Let M^{2n} be a $2n$-dimensional strictly nearly Kähler manifold. Then there do not exist $2n-2$-dimensional almost complex submanifolds.*

From this, we immediately have the following

Corollary 3.2. *Let (M, J) be an almost complex manifold. If there exists a metric g such that (M, g, J) is strictly nearly Kähler, then there are no Levi-flat hypersurfaces in (M, J).*

In particular, for one of the two almost complex structures, there is a negative answer for the existence of Levi-flat hypersurfaces.

Corollary 3.3. *There is no Levi-flat hypersurface in $(\mathbb{C}P^3, J)$.*

The non-existence of Levi-flat hypersurfaces in $(\mathbb{C}P^3, J)$ is a direct consequence of the manifold admitting a nearly Kähler structure. Moreover, the structure further gives the following non-existence result. To state it, recall that g is the metric such that $(\mathbb{C}P^3, g, J)$ is a homogeneous nearly Kähler manifold.

Proposition 3.4. *There is no Levi-flat hypersurface in $(\mathbb{C}P^3, g, J_\circ)$ with angle $\theta = 0$.*

Proof. We prove this via contradiction. Suppose there is a Levi-flat hypersurface $\mathcal{H}$ with angle $\theta = 0$. Then, the g-unit normal N lies in the four dimensional distribution. In other words, $JN = J_\circ N$. Moreover, $N_\circ = \sqrt{2}N$ is a $g_\circ$-unit normal. As $g_\circ([X,Y], J_\circ N_\circ) = 0$ for all tangent vectors $X, Y \in \mathcal{H} \cap J_\circ\mathcal{H}$, we find $0 = g_\circ([X,Y], J_\circ N_\circ) = \sqrt{2}g_\circ([X,Y], JN) = \frac{\sqrt{2}}{2}g([X,Y], JN)$. In other words, $\mathcal{H}$ is Levi-flat for $(\mathbb{C}P^3, J)$, which is a contradiction with Corollary 3.3. □

With a little more work, we can extend the previous result to all constant angles. We first note the following lemma.

Lemma 3.5. *Given a Levi-flat hypersurface in* $(\mathbb{C}P^3, J_\circ)$, *equipped with the induced metric of* g *and the frame of Equation* (2.2), *then the following relations hold for* $\alpha_{ij} = g(h(e_i, e_j), N)$:

$$\alpha_{44} + \alpha_{55} = 0, \qquad \alpha_{34} + \alpha_{25}\cos\theta + \alpha_{15}\sin\theta = 0,$$
$$\alpha_{35} + \cos^2\theta = \alpha_{24}\cos\theta + \alpha_{14}\sin\theta, \qquad \alpha_{33} + \alpha_{12}\sin 2\theta + \alpha_{11}\sin^2\theta + \alpha_{22}\cos^2\theta = 0.$$

Proof. By Proposition 3.4, we know that $\theta \neq 0$, so that we can take this frame. Then, this is a simple computation of $g([X, Y], J_\circ N)$, and demanding it vanishes for all holomorphic tangent vectors X, Y. □

Proposition 3.6. *All Levi-flat hypersurfaces in any* $(\mathbb{C}P^3, g_a, J_\circ)$ $(a \neq 1)$ *have non-constant angle.*

Proof. We argue by contradiction. So, suppose there is a Levi-flat hypersurface with constant angle. From Proposition 3.4, we know $\theta \neq 0$, and we know $\theta \neq \pi/4$. Lemma 3.5 applies. The conditions of having constant angle amount to $\alpha_{13} = \alpha_{23} = 0$, $\alpha_{25} = -\alpha_{15}\tan\theta$, $\alpha_{24} = \frac{1}{2}(2+\cos(2\theta) - 2\alpha_{14}\sin(\theta))/\cos\theta$ and $\alpha_{12} = -\frac{1}{\sin(2\theta)}(\alpha_{22}\cos^2\theta + \alpha_{11}\sin^2\theta)$. We then look at the Gauss equation applied to (e_3, X, e_4, e_5) for $X = e_4, e_5$. We find $\alpha_{45} = \alpha_{45} = 0$. From the Gauss equation applied to (e_3, e_5, e_4, e_1), we then find $(1 + \cos^2\theta)\sin\theta = 0$. This is never satisfied, so that there cannot be a Levi-flat hypersurface with constant angle. □

As such, the previous result tells us that any Levi-flat hypersurface gives interesting examples of hypersurfaces with a non-constant angle function.

Before continuing with general results, we give an example of a Levi-flat hypersurface in $(\mathbb{C}P^3, J_\circ)$, so that we are not making claims about the empty set.

Remark 3.7. From Corollary 3.2, it follows that the following example shows that there cannot be a metric (homogeneous or not) on $\mathbb{C}P^3$ such that $(\mathbb{C}P^3, J_\circ)$ with this metric is strictly nearly Kähler.

Denote I the open interval $I = (-\pi/2, \pi/2)$, and consider the embedding

$$i_1 : I \times \mathbb{S}^5 \subset I \times \mathbb{C}^3 \to \mathbb{S}^7 \subset \mathbb{C}^4 : \left(t, \begin{pmatrix} u \\ v \\ w \end{pmatrix}\right) \mapsto \begin{pmatrix} u\cos t + iv\sin t \\ v\cos t + iu\sin t \\ w\cos t \\ -iw\sin t \end{pmatrix} \tag{3.1}$$

and one of its submanifolds

$$i_2 : \mathbb{S}^2 \subset \mathbb{R} \times \mathbb{C} \to \mathbb{S}^7 \subset \mathbb{C}^4 : \begin{pmatrix} a \\ v \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ a \\ v \\ 0 \end{pmatrix} = i_1\left(0, \begin{pmatrix} 0 \\ a \\ v \end{pmatrix}\right). \tag{3.2}$$

Composing with the Hopf fibration π, we get the following embeddings into $\mathbb{C}P^3$:

$$\iota_1 = \pi \circ i_1 : I \times \mathbb{C}P^2 \to \mathbb{C}P^3, \qquad \iota_2 = \pi \circ i_2 : \mathbb{C}P^1 \to \mathbb{C}P^3. \tag{3.3}$$

Remark 3.8. The example of ι_1 comes from a general idea: 1) take an embedding $U \subset \mathbb{C}^2 \to \mathbb{C}P^3 : (z, w) \mapsto F(z, w)$, 2) take a regular curve $I \to \mathfrak{u}(4) : t \mapsto G(t)$, such that the curve does not belong to $\mathfrak{su}(3) \times \mathfrak{u}(1)$ and 3) form the immersion $I \times U \to \mathbb{C}P^3 : (t, (z, w)) \mapsto e^{G(t)} \cdot F(z, w)$. By regularity, the parameter t discerns the leaves, which are all copies of the image of F and thus complex submanifolds.

The unit normal on i_1 in the round $\mathbb{S}^7$ is given by $\xi_0 = (0, 0, -i\sin t, \cos t)$. Choosing coordinates on $\mathbb{S}^5$, it is straightforward to show that both $i\xi_0$ and ξ_0 are orthogonal to $T\mathbb{S}^5$ in the round $\mathbb{S}^7$. Another way to see this: take $\frac{\partial i_1}{\partial t}$, use Gram-Schmidt to make it orthogonal to $T\mathbb{S}^5$, and observe that it lies in the span of $i\xi_0$. It follows that ι_1 gives rise to a Levi-flat hypersurface in $\mathbb{C}P^3$ with $\mathbb{C}P^2$ as the foliating complex manifold.

By computing G in this example, we find that with $N_0 = \mathrm{d}\pi\xi_0$, the vector fields $G(N_\circ, J_\circ N_\circ)$ and $J_\circ G(N_\circ, J_\circ N_\circ)$ span the tangent space of ι_2. Moreover, both these vector fields lie in $\mathcal{D}_2^4$, so that they are horizontal with respect to the twistor fibration $\tau : \mathbb{C}P^3 \to \mathbb{S}^4$. In particular, any of the integrating surfaces of these vector fields are horizontal with respect to τ and lie in $\mathbb{C}P^2$, and are congruent to a standard $\mathbb{C}P^1$. Since $\mathbb{C}P^1$ is totally geodesic, the image under τ of any of the integrating surfaces is a totally geodesic two-sphere. Being totally geodesic, they are in particular superminimal in $\mathbb{S}^4$. Hence, by a theorem of Storm [9], there are unique Lagrangian submanifolds (of both Kähler and nearly Kähler $\mathbb{C}P^3$) associated with each of these surfaces in $\mathbb{S}^4$. Because the surfaces are all totally geodesic, all of these Lagrangians have to be congruent to the totally geodesic $\mathbb{R}P^3$. The above all generalises for any Levi-flat hypersurface in $\mathbb{C}P^3$.

Theorem 3.9. *All Levi-flat hypersurfaces in $(\mathbb{C}P^3, J_\circ)$ give rise to a family (possibly with only one member) of superminimal surfaces in the round $\mathbb{S}^4$. As such, each Levi-flat hypersurface has a family of Lagrangian (with respect to both Kähler and nearly Kähler $\mathbb{C}P^3$) immersions associated to it.*

Proof. Suppose a Levi-flat hypersurface $\mathcal{H}$, and equip it with the frame of Equation (2.2). Lemma 3.5 applies. Suppose $\mathcal{H}$ is foliated by complex manifolds as $\mathcal{H} = \sqcup_t M_t$.

Let $U = G(N_\circ, J_\circ N_\circ)$ and $V = J_\circ U$. Computing in the frame, we find that $\{U, V\}$ forms an involutive distribution and that both U and V lie in $\mathcal{D}_2^4$. Moreover, U and V belong to $T\mathcal{H} \cap J_\circ T\mathcal{H}$. Let $\Sigma_t \subset M_t$ be an integrating surface of $\{U\big|_{M_t}, V\big|_{M_t}\}$.

By horizontality, we find that $\tau(\Sigma_t)$ is a surface in $\mathbb{S}^4$. Moreover, when $\mathbb{S}^4$ is equipped with the round metric, we find that $\tau(\Sigma_t)$ is superminimal in $\mathbb{S}^4$. From Storm's theorem [9], we find that there is a unique Lagrangian submanifold $\mathcal{L}_t$ (simultaneously for Kähler and nearly Kähler $\mathbb{C}P^3$) associated to each Σ_t. The family $\{\mathcal{L}_t \mid t\}$ is the requested family of Lagrangian submanifolds. $\square$

Remark 3.10. The superminimal surfaces of Remark 3.8 lie in the image of $\pi \circ F$ and are congruent in the Kähler $\mathbb{C}P^3$, and as such they descend to congruent superminimal surfaces in $\mathbb{S}^4$. Therefore, the associated Lagrangian immersions of all Levi-flat hypersurfaces of Remark 3.8 will all be congruent and the family will only consist of one member.

4. Discussion and concluding remarks

In this work, we have considered Levi-flat hypersurfaces of the complex projective space $\mathbb{C}P^3$. In $\mathbb{C}^n$, Levi-flat hypersurfaces bound domains of holomorphy and as such are crucial in understanding where analytic continuation is possible. Even though this implication does not hold in general on complex manifolds, understanding when it does and when it does not is an active field of research and may give new insights in the detection of regions of holomorphy. As a first step, one currently attempts to classify Levi-flat hypersurfaces in different spaces. As a small contribution to this initiative, we proceeded to study Levi-flat hypersurfaces in $\mathbb{C}P^3$.

One key insight arising from our work is that the existence of a strictly nearly Kähler metric (i.e. a metric that turns the ambient space with almost complex structure into a strictly nearly

Kähler manifold) obstructs the existence of Levi-flat hypersurfaces. Since we constructed an example of a Levi-flat hypersurface in $(\mathbb{C}P^3, J_\circ)$, this also shows that this space can never have a metric (homogeneous or not) that turns it into a strictly nearly Kähler manifold and is compatible with $J_\circ$. Moreover, the nearly Kähler obstruction aided us in the classification of Levi-flat hypersurfaces in $(\mathbb{C}P^3, J_\circ)$, as the link between Kähler and nearly Kähler $\mathbb{C}P^3$ prohibits the existence of a Levi-flat hypersurface with vanishing angle function. These are examples of how a metric can help with the classification endeavour, even though the problem itself is independent of a metric. In a similar vein, a metric makes the Levi-form easily expressible in real terms on a Riemannian manifold.

We finish this work with possible further research questions. We showed that with every Levi-flat hypersurface, there is an associated family of Lagrangian immersions, and an associated family of horizontal (lying in $\mathcal{D}_2^4$, horizontal for the twistor fibration τ) complex surfaces in the leaves of the foliation (and following from this, a family of superminimal surfaces in $\mathbb{S}^4$). Bryant [2] classified all possible horizontal complex surfaces in $(\mathbb{C}P^3, J_\circ)$, see also Xu's work [10]. It is natural to ask what kind of restrictions this classification lays on the existence of Levi-flat hypersurfaces. Another question is whether there exist Levi-flat hypersurfaces whose associated family of Lagrangian immersions contains the so-called Chiang Lagrangian: the Lagrangian corresponding to the Veronese surface in $\mathbb{S}^4$. Apart from $\mathbb{R}P^3$, this is the unique Lagrangian whose associated surface has constant sectional curvature. A final question we pose is whether there exist Levi-flat hypersurfaces whose associated family of Lagrangians contains both the unique (nearly Kähler) totally geodesic Lagrangian $\mathbb{R}P^3$ and the Chiang Lagrangian.

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