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PERFECT FLUID SPACE-TIMES - REVISITED

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Abstract. In this article, we first discuss certain curvature features of perfect fluid space-times and some known findings of perfect fluid space-times. For a Lorentzian manifold to be a perfect fluid space-time, we give two criteria. Furthermore, we discover that a conharmonically flat perfect fluid space-time characterizes the radiation era. We then demonstrate that a stiff matter fluid's vorticity disappears if it complies with Yang's equations. Furthermore, we discover that the perfect fluid space-time is shear-free, vorticity-free, and μ and p are constant if the Ricci tensor is Killing. In addition, we prove that Ricci symmetric or Ricci semi-symmetric perfect fluid space-times are either phantom era or dark matter era. Lastly, we conclude that a perfect fluid space-time that is Ricci symmetric either represents a static space-time or a dark matter era.

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1. Introduction

The most significant difference between Riemannian and semi-Riemannian geometry is the presence of a null vector, that is, a vector v obeying $g_{jk}v^j v^k = 0$. For an n-dimensional Riemannian manifold the signature of the metric tensor is $(+,+,+,\ldots,+,+,+)$, whereas for a semi-Riemannian manifold the signature is $(-,-,-,\ldots,+,+,+)$. Lorentzian manifold is a special class of a semi-Riemannian manifold whose signature is $(-,+,+,\ldots,+)$, that is, index is one.

For instance, in a 4-dimensional semi-Riemannian manifold with the metric

$$ds^2 = -\left(dx^1\right)^2 - \left(dx^2\right)^2 - \left(dx^3\right)^2 + c^2\left(dx^4\right)^2,$$

$\left(-1,-1,1,\frac{\sqrt{3}}{c}\right)$ is a null vector and in a 4-dimensional Lorentzian manifold with the metric

$$ds^2 = -\left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 + c^2\left(dx^4\right)^2,$$

$\left(1,0,0,\frac{1}{c}\right)$ is a null vector.

Some of the most significant ideas in contemporary physics, including string theory and general relativity, are based on Lorentzian geometry.

From an exclusively mathematical perspective, a Lorentzian manifold M is a smooth manifold endowed with a symmetric non-degenerate bilinear form g, called the metric of signature $(-,+,+,+,\ldots,+)$, that is, index of g is 1.

In general, a Lorentzian manifold (M,g) may not have a globally time-like vector field. If (M,g) admits a globally time-like vector field, it is named time oriented Lorentzian manifold, physically known as space-time.

Suppose M is a semi-Riemannian manifold of dimension $n \geq 2$ with a semi-Riemannian metric g of signature (m,p), where $m+p=n$. If g is a Lorentzian metric of signature $(1,n-1)$ or $(n-1,1)$, then M equipped with g is said to be an n-dimensional Lorentzian manifold [18]. If $M = -I \times_{\mathfrak{f}} \mathcal{M}$, where I is an open interval of real numbers $\mathbb{R}$, $\mathcal{M}$ indicates a Riemannian manifold of dimension $n-1$ and $\mathfrak{f} > 0$ stands for a smooth function, named as warping function or scale factor, then M is said to be a generalized Robertson-Walker (GRW) space-time [1]. In particular, if we suppose that $\mathcal{M}$ is a Riemannian manifold of dimension 3 and is of constant sectional curvature, then the GRW space-time becomes a Robertson-Walker (RW) space-time. This states that GRW space-times are the natural extension of RW space-times. It is well-known that the Lorentzian Minkowski space-time, the static Einstein space-time, the Friedmann cosmological models, the Einstein-de Sitter space-time, the de Sitter space-time are included in the GRW space-times [21].

In general relativity, idealized distributions of matter, like the interior of a star or an isotropic cosmos, are modeled using perfect fluids. In the latter scenario, the perfect fluid's equation of state might be applied to the Friedman-Lemaitre-Robertson-Walker (FLRW) equations, which characterize the universe's evaluation. A fluid that has no viscosity and is incompressible is referred to as an perfect fluid. The energy momentum tensor's perfect fluid form is widely used and very significant. The energy momentum tensor T_{hk} is given by

$$T_{jk} = (\mu + p)u_j u_k + p g_{jk}, \tag{1.1}$$

where g is the Lorentzian metric and p and μ denote the perfect fluid's isotropic pressure and energy density, respectively [18]. In the last equation, the velocity vector is defined by $g_{jk}u^j u^k = -1$ and $u_j = g_{jk}u^k$.

For a gravitational constant κ, the Einstein's field equations without a cosmological constant is described by

$$R_{jk} - \frac{R}{2} g_{jk} = \kappa T_{jk}, \tag{1.2}$$

where R^l_{jki} is the curvature tensor of type (1,3), $R_{jk} = R^i_{jki}$ and $R = g^{jk}R_{jk}$ denote the Ricci tensor and the Ricci scalar, respectively.

A space-time M is named a perfect fluid space-time if the non-vanishing Ricci tensor R_{jk} obeys

$$R_{jk} = \alpha g_{jk} + \beta u_j u_k, \tag{1.3}$$

where α and β are smooth functions. The foregoing equation is obtained from the equations (1.1) and (1.2) (see, [15]).

Combining the equations (1.1), (1.2) and (1.3), we acquire

$$\beta = k^2(p+\mu),\ \alpha = \frac{k^2(p-\mu)}{2-n}. \tag{1.4}$$

Moreover, an equation of state with the form $p = p(\mu)$ connects p and μ, and the perfect fluid space-time is known as isentropic. Furthermore, if $p = \mu$, the perfect fluid space-time is referred to as stiff matter [2]. The perfect fluid space-time is named the dark matter era if $p + \mu = 0$, the dust matter fluid if $p = 0$, and the radiation era if $p = \frac{\mu}{3}$ [2]. The universe is represented as accelerating phase when $\frac{p}{\mu} < -\frac{1}{3}$. It covers the quintessence phase if $-1 < \frac{p}{\mu} < 0$ and phantom era if $\frac{p}{\mu} < -1$.

It is well-known that every RW-space-time is a perfect fluid space-time [18]. Also, a 4-dimensional GRW-space-time is a perfect fluid space-time if and only if it is a RW-space-time [9]. In [24], Shepley and Taub established that a 4-dimensional perfect fluid space-time with $\nabla_h C^h_{ijk} = 0$, in which ∇ denotes the covariant differentiation and subject to a state equation $p = p(\mu)$ is conformally flat, and the metric is RW, the flow is irrotational, shear-free, and geodesic. Any perfect fluid solution of Einstein's field equation with $p = p(\mu)$, $p + \mu \neq 0$, admitting a conformal Killing vector parallel to the velocity vector u_j, is locally a FRW model[4]. In [22], Sharma established that if a perfect fluid space-time with divergence free weyl tensor admits a proper conformal symmetry, then it is conformally flat. The existence of a concircular vector field in a conformally flat perfect fluid space-time with closed u_h was established by De and Ghosh in [6]. The characteristics of perfect fluid space-times have been found in ([15], [16]).

In local coordinates, the conformal curvature tensor, represented by C, is given by

$$\begin{aligned} C_{hijk} &= R_{hijk} - \frac{1}{n-2}(g_{hk}R_{ij} - g_{hj}R_{ik} + g_{ij}R_{hk} - g_{ik}R_{hj}) \\ &\quad + \frac{R}{(n-1)(n-2)}(g_{hk}g_{ij} - g_{hj}g_{ik}), \end{aligned} \tag{1.5}$$

where R_{hijk} indicates the curvature tensor of type (0,4).

If the metric of a Lorentzian manifold satisfies the relation

$$\pounds_u g_{jk} + 2R_{jk} + 2\lambda g_{jk} = 0, \tag{1.6}$$

then it is called a Ricci soliton [11], where $\pounds_u$ is the Lie derivative operator and λ denotes a real constant. Here, u is named the potential vector field of the solitons. The solitons are known as almost Ricci solitons if λ is a function [19].

For a non vanishing 1-form ω_k and a scalar function ϕ if the relation $\nabla_k u_h = \omega_k u_h + \phi g_{kh}$ holds, then the vector field u is called torse-forming. This notion was introduced by Yano [29] on a Riemannian manifold. It is noted that the foregoing torse-forming condition becomes $\nabla_k u_h = \phi(u_k u_h + g_{kh})$, for a unit time-like vector.

To investigate a conformally flat hypersurfaces of a Euclidean space, Chen [3] acquire the ensuing expression of the curvature tensor

$$\begin{aligned} R_{hijk} &= \gamma(g_{hk}g_{ij} - g_{hj}g_{ik}) \\ &\quad + \mu(g_{hk}u_i u_j + g_{ij}u_h u_k - g_{hj}u_i u_k - g_{ik}u_h u_j), \end{aligned} \tag{1.7}$$

where u_i is a unit vector, named the generator and γ, μ are scalars. An n-dimensional conformally flat space obeying (1.7) is called a space of quasi-constant sectional curvature and denoted by $(QC)_n$. However, if the equation (1.7) of the curvature tensor holds, then it can be easily verified that the space is conformally flat. So in the definition conformally flatness is not required. A Lorentzian manifold is said to be a space-time of quasi constant sectional curvature if u_j is a unit time-like vector.

The tensor D_{jk} is named Killing [28] if it satisfies the following condition

$$\nabla_l D_{jk} + \nabla_k D_{lj} + \nabla_j D_{kl} = 0.$$

Hall described Ricci recurrent ($\nabla_l R_{jk} = A_l R_{jk}$, A_l is a covariant vector.) space-times in [12]. The Ricci semi-symmetry is well known to be weaker than the Ricci recurrent space-time. In this article, we are interested in looking into the Ricci semi-symmetric perfect fluid space-times.

A space-time is said to be semi-symmetric [26] if it obeys the relation

$$\nabla_l \nabla_m R^h_{ijk} - \nabla_m \nabla_l R^h_{ijk} = 0, \tag{1.8}$$

where ∇ indicates the covariant differentiation. Semi-symmetric space-times have been considered in [13]. It is to be noted that the class of locally symmetric spaces ($\nabla_l R^h_{ijk} = 0$) due to Cartan is a proper subset of semi-symmetric spaces.

A space-time is called Ricci semi-symmetric [17] if it satisfies the relation

$$\nabla_l \nabla_m R_{ij} - \nabla_m \nabla_l R_{ij} = 0. \tag{1.9}$$

If a Lorentzian manifold admits a time-like Killing vector field ρ, it is referred to as a stationary space-time and static ([20], [25], p. 283) if, additionally, ρ is irrotational. We will refer to ρ in this context as the static vector field, where it is assumed that space-time is time-oriented. The product $\mathbb{R} \times S$ is called a static space-time if it is equipped with the metric

$$g[(t, y)] = -\beta(y)dt^2 + g_S[y], \tag{1.10}$$

where g_S denotes a Riemannian metric on S. Any static space-time behaves like a standard one locally, with ρ identifiable to ∂t. A spherically symmetric vacuum solution is necessarily static, according to Birkhoff's theorem [14].

2. Examples of perfect fluid space-times

1. In [18], O'Neill established that every RW-space-time represents a perfect fluid space-time. Dark matter era refers to perfect fluid space-time with the equation of state $p + \sigma = 0$ [2]. However, so far, according to [10] a four-dimensional perfect fluid space-time with $p + \sigma \neq 0$ is RW-space-time if and only if it is a Yang Pure space-time.

2. Any GRW-space-time of dimension four is also a perfect fluid space-time if the space-time is a RW-space-time [9].

3. Multiplying (1.7) with g^{ij}, we acquire

$$\begin{aligned} R_{hk} &= \gamma(4g_{hk} - g_{hk}) + \mu(-g_{hk} + 4A_h A_k - A_h A_k - A_h A_k) \\ &= (3\gamma - \mu)g_{hk} + 2\mu A_h A_k, \end{aligned} \tag{2.1}$$

which represents a perfect fluid space-time. Hence, a space-time of quasi constant sectional curvature is a perfect fluid space-time.

4. In [15], Mantica et al established that with the condition $C^m_{jkl,m} = 0$, GRW-space-times represent perfect fluid space-times.

5. Zhao et al [30] proved that every pseudo-symmetric GRW-space-time is a perfect fluid space-time.

3. Proof of the Theorems

In this paper, we find two necessary criterion for a Lorentzian manifold to be a perfect fluid space-time and prove the following:

Theorem 3.1. *If in a Lorentzian manifold the Ricci tensor R_{hk} satisfies the relation*

$$R_{ij}R_{hk} - R_{hj}R_{ik} = f[g_{ij}g_{hk} - g_{hj}g_{ik}], \tag{3.1}$$

for a smooth function f, then the space-time becomes a perfect fluid space-time.

Proof. Multiplying (3.1) with u^iu^j, we acquire

$$u^iu^jR_{ij}R_{hk} - u^iu^jR_{hj}R_{ik} = f[-g_{hk} - u^iu^jg_{hj}g_{ik}]. \tag{3.2}$$

Let $\sigma = u^iu^jR_{ij}$ and putting this value in the foregoing equation, we infer

$$\sigma R_{hk} - u^iR_{ik}u^jR_{hj} = f[-g_{hk} - u_hu_k]. \tag{3.3}$$

Choose $B_h = u^jR_{hj}$ and $\widetilde{\sigma} = \frac{1}{\sigma}$. Then the above equation yields

$$R_{hk} = \widetilde{\sigma}[B_hB_k - fg_{hk} - fu_hu_k]. \tag{3.4}$$

Again, multiplying (3.1) with u^i, we obtain

$$u^iR_{ij}R_{hk} - u^iR_{hj}R_{ik} = f[u_jg_{hk} - u_kg_{hj}]$$

which implies

$$B_jR_{hk} - B_kR_{hj} = f[u_jg_{hk} - u_kg_{hj}]. \tag{3.5}$$

Using (3.4) in (3.5), we get

$$\widetilde{\sigma}B_j[B_hB_k - fg_{hk} - fu_hu_k] - \widetilde{\sigma}B_k[B_hB_j - fg_{hj} - fu_hu_j] = f[u_jg_{hk} - u_kg_{hj}]. \tag{3.6}$$

Multiplying both sides by g^{hk} gives

$$\widetilde{\sigma}B_j[-4f + f] - \widetilde{\sigma}B_k[-f\delta^k_j - fu^ku_j] = f[4u_j - u_j].$$

which yields

$$(-3f + f)\widetilde{\sigma}B_j + f\widetilde{\sigma}B_ku^ku_j = f[4u_j - u_j]. \tag{3.7}$$

Since $B_h = u^jR_{hj}$, we acquire $B_hu^h = u^hu^jR_{hj} = \sigma$. Hence, the last equation produces

$$-2f\widetilde{\sigma}B_j + f\widetilde{\sigma}\sigma u_j = 3fu_j. \tag{3.8}$$

Since $\sigma\widetilde{\sigma} = \sigma\frac{1}{\sigma} = 1$, we acquire from the previous equation

$$-2f\widetilde{\sigma}B_j + fu_j = 3fu_j,$$

which implies

$$\widetilde{\sigma}B_j = -u_j, \tag{3.9}$$

Using (3.9) in (3.4), we infer

$$R_{hk} = -\frac{1}{\widetilde{\sigma}}u_h u_k - f\widetilde{\sigma}g_{hk} - f\widetilde{\sigma}u_h u_k.$$

which implies

$$R_{hk} = -(f\widetilde{\sigma} + \frac{1}{\widetilde{\sigma}})u_h u_k - f\widetilde{\sigma}g_{hk}. \tag{3.10}$$

Therefore, the spacetime becomes a PF-spacetime.

Hence, the proof is complete. □

In order to arrive to the following conclusion, we examine a Lorentzian manifold that obeys an almost Ricci solitons.

Theorem 3.2. *If a Lorentzian manifold admits an almost Ricci soliton whose potential vector field is a unit time-like torse-forming vector field, then it becomes a perfect fluid space-time.*

Proof. Suppose that a Lorentzian manifold admits an almost Ricci soliton. Therefore the equation (1.6) reveals

$$\nabla_h u_k + \nabla_k u_h + 2R_{hk} + 2\lambda g_{hk} = 0. \tag{3.11}$$

If u is a unit time-like torse-forming vector field, then we infer $\nabla_k u_h = \phi(u_k u_h + g_{kh})$. Using this result in (3.11), we get

$$\phi(u_k u_h + g_{kh}) + \phi(u_h u_k + g_{hk}) + 2R_{hk} + 2\lambda g_{hk} = 0,$$

which implies

$$R_{hk} = -(\phi + \lambda)g_{hk} - \phi u_h u_k. \tag{3.12}$$

Then, the space-time becomes a perfect fluid space-time.

Thus, the proof is complete. □

Here, we have established a number of theorems on perfect fluid space-times in dimension 4.

A space-time of quasi constant sectional curvature is a perfect fluid space-time. Is the converse true? Here, we prove that the converse is not true, in general. Also, we know that every RW-space-time represents a perfect fluid space-time. Is the converse valid? In this article, we establish that the converse is usually not valid and state the following result:

Theorem 3.3. *A conformally flat perfect fluid space-time is a space-time of quasi constant sectional curvature and a RW-space-time with* $p + \mu \neq 0$*.*

Proof. In a conformally flat space-time, the curvature tensor R_{hijk} is written by

$$\begin{aligned} R_{hijk} &= \frac{1}{2}(g_{hk}R_{ij} - g_{hj}R_{ik} + g_{ij}R_{hk} - g_{ik}R_{hj}) \\ &\quad - \frac{R}{6}(g_{hk}g_{ij} - g_{hj}g_{ik}). \end{aligned} \tag{3.13}$$

Let us consider a conformally flat perfect fluid space-time. Then using (1.3) in (3.13), we acquire

$$\begin{aligned} R_{hijk} &= \left(\frac{R}{6} + \alpha\right)(g_{hk}g_{ij} - g_{hj}g_{ik}) \\ &\quad + \frac{\beta}{2}(g_{hk}u_i u_j + g_{ij}u_h u_k - g_{hj}u_i u_k - g_{ik}u_h u_j). \end{aligned} \tag{3.14}$$

Therefore, the space-time represents a space-time of quasi constant sectional curvature.

We know that in a conformally flat space-time $divC = 0$ ('div' denotes the divergence). In [16], it is established that perfect fluid space-times with $p + \mu \neq 0$ and $div\, C = 0$, represent GRW-space-times.

Again, in four dimensions, every GRW-space-time is a perfect fluid space-time if and only if it is a RW-space-time [9] and thus the space-time becomes RW.

Hence, the proof is complete. □

Remark 3.4. One example of a conformally flat perfect fluid solution is the generalized interior Schwarzschild solutions with zero expansion [25]. The equation of state with the shape $p = p(\mu)$ is only accepted by the FRW models.

In [10], Yang Pure Space is defined as a Lorentzian manifold of dimension four whose metric tensor solves Yang's equations: $\nabla_l R_{hk} - \nabla_k R_{hl} = 0$. In dimension four, a perfect fluid space-time with $\mu + p \neq 0$ represents a RW space-time if and only if it is a Yang pure space, according to Guilfoyle and Nolan's proof in their paper [10], whereas here we acquired absolutely different result.

Theorem 3.5. *If a stiff matter fluid obeys Yang's equations, then the vorticity of the fluid vanishes.*

Proof. Let the perfect fluid space-time obey the Yang's equations, which entails

$$\nabla_l R_{hk} = \nabla_k R_{hl}. \tag{3.15}$$

Differentiating (1.3) covariantly gives

$$\nabla_l R_{hk} = (\nabla_l \alpha) g_{hk} + (\nabla_l \beta) u_h u_k + \beta(u_k \nabla_l u_h + u_h \nabla_l u_k). \tag{3.16}$$

Similarly, we infer

$$\nabla_k R_{hl} = (\nabla_k \alpha) g_{hl} + (\nabla_k \beta) u_h u_l + \beta(u_l \nabla_k u_h + u_h \nabla_k u_l). \tag{3.17}$$

Hence, using the foregoing equations in (3.15), we acquire

$$\begin{aligned} 0 &= \nabla_l R_{hk} - \nabla_k R_{hl} \\ &= (\nabla_l \alpha) g_{hk} + (\nabla_l \beta) u_h u_k + \beta(u_k \nabla_l u_h + u_h \nabla_l u_k) \\ &\quad -(\nabla_k \alpha) g_{hl} - (\nabla_k \beta) u_h u_l - \beta(u_l \nabla_k u_h + u_h \nabla_k u_l). \end{aligned} \tag{3.18}$$

Multiplying the last equation by g^{hk} yields

$$\begin{aligned} &4(\nabla_l \alpha) - (\nabla_l \beta) + \beta(u^h \nabla_l u_h + u^k \nabla_l u_k) \\ &= (\nabla_l \alpha) + (\nabla_k \beta) u^k u_l + \beta(u_l \nabla_h u^h + u^k \nabla_h u_l), \end{aligned}$$

which entails

$$3(\nabla_l \alpha) - (\nabla_l \beta) = (\nabla_k \beta) u^k u_l + \beta(u_l \nabla_h u^h + u^k \nabla_h u_l). \tag{3.19}$$

Multiplying (3.19) by u^l, we get

$$[3(\nabla_l \alpha) - (\nabla_l \beta)] u^l = -(\nabla_k \beta) u^k - \beta \nabla_h u^h,$$

which implies

$$3(\nabla_l \alpha) u^l = -\beta \nabla_h u^h. \tag{3.20}$$

We assume the perfect fluid space-time satisfies the stiff matter fluid, that is, $p = \mu$. Then we have $\nabla_l \alpha = 0$, where we have used the equation (1.4). Then the previous equation tells that either $\beta = 0$, or $div\, u^h = 0$.

If $\beta = 0$, then $p + \mu = 0$ which implies $\mu = 0$, since $p = \mu$. Hence, the fluid is vacuum. This is not a physically significant scenario bearing in mind that the universe contains matter.

If $div\, u^h = 0$, then the velocity vector field is conservative. The nature of a conservative vector field is always irrotational, thus we conclude that the perfect fluid has zero vorticity.

This concludes the proof. □

Here, we address the influence of Killing Ricci tensor in a perfect fluid space-time and prove the following theorem:

Theorem 3.6. *If in a perfect fluid space-time the Ricci tensor is Killing, then the perfect fluid space-time is vorticity-free, shear-free, and p and μ are constant.*

To establish the theorem we first state the following Lemma:

Lemma 3.7. *In a space-time obeying Einstein's field equations the Ricci tensor R_{hk} is Killing if and only if the energy momentum tensor is Killing.*

Proof of Theorem 3.6. In [23], Sharma and Ghosh established that in a perfect fluid space-time if T_{hk} is Killing, then the perfect fluid space-time is vorticity-free, shear-free, and p and μ are constant. Now applying this result and the above Lemma, we can state that the perfect fluid space-time is vorticity-free, shear-free, and p and μ are constant.

This accomplishes the proof. □

Now we consider a Ricci semi-symmetric perfect fluid space-time and state the subsequent result:

Theorem 3.8. *If a perfect fluid space-time is Ricci semi-symmetric, then the space-time represents either dark matter era, or phantom era.*

The proof of the above theorem is given in [5].

A space-time is called conformally semi-symmetric if it fulfills the relation

$$\nabla_l \nabla_m C^h_{ijk} - \nabla_m \nabla_l C^h_{ijk} = 0. \tag{3.21}$$

In [8], it is established that in dimension 4, conformally semi-symmetric space-times are Ricci semi-symmetric space-times. Hence, we state the following:

Corollary 3.9. *If a perfect fluid space-time is conformally semi-symmetric, then either the space-time represents dark matter era, or phantom era.*

Again, the class of Ricci symmetric spaces ($\nabla_l R_{ij} = 0$) is a proper subset of Ricci semi-symmetric spaces. Every semi-symmetric space is known to be Ricci semi-symmetric, but the converse is not usually true. In a Riemannian space they are equivalent for dimension three. In [27], it has been shown that for $n \geq 3$, the foregoing stated relations are equivalent for hypersurfaces having non negative Ricci scalar in a Euclidean space E^{n+1}.

We derive the following conclusion from the preceding studies:

Corollary 3.10. *A Ricci symmetric perfect fluid space-time represents either phantom era, or dark matter era.*

Here, we also consider a Ricci symmetric perfect fluid space-time to state a different result.

Theorem 3.11. *If a perfect fluid space-time is Ricci symmetric, then either the space-time represents a dark matter era, or a static space-time.*

The above theorem's proof may be found in [5].

It is well-known ([7], Section 10.7) that any static space-time is everywhere of Petrov type I, D or O. As a result, the space-time under consideration is of Petrov type I, D or O.

Hence, we write:

Corollary 3.12. *If a perfect fluid space-time is Ricci symmetric, then either the space-time represents a dark matter era, or the space-time is of Petrov type I, D or O.*

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