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STRUCTURED CONDITION PSEUDOSPECTRA OF MATRICES WITH ENTRIES IN A NON-ARCHIMEDEAN FIELD

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Abstract. In this paper, the structured pseudospectra of matrices with entries in the non-Archimedean field $\mathbb{K}$ and the structured pseudospectra of matrix pencils with entries in the non-Archimedean field $\mathbb{K}$ are introduced. Many results are proved about them and we give a few examples.

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1. Introduction and preliminaries

In the classical setting, L. N. Trefethen [18] developed the pseudospectra of matrices with entries in $\mathbb{C}$ where $\mathbb{C}$ is the field of complex numbers. The pseudospectra of matrices is useful in many fields in applied mathematics such as control theory, Markov chains, structural analysis, numerical solution of differential equations, matrix iterations and economics. For more details, we refer to [12, 13, 16, 17]. Recently, C. R. Johnson [12] studied the numerical determination of the field of values of a general complex matrix. On the other hand, T. Kailath [13] collected several results on linear systems. The concept of approximate eigenvalues and the integral equation of laser theory were studied by H. J. Landau [14].

In non-Archimedean operator theory, J. Ettayb [8] studied the determinant spectrum of matrices with entries in the complex Levi-Civita field $\mathcal{C}$. The pseudospectrum of matrix pencils with entries in $\mathbb{K}$ (where $\mathbb{K}$ is a non-Archimedean field) was studied by [7] and he proved that the intersection of all pseudospectra of a matrix pencil with entries in $\mathbb{K}$ is the spectrum of this matrix pencil and the pseudospectrum of the matrix pencil (A, B) is the collection of numbers in $\mathbb{K}$ that are eigenvalues of some perturbed matrix $A + C$ with $\|C\| < \varepsilon$.

Throughout this paper, $\mathbb{K}$ is a non-Archimedean complete valued field with a nontrivial valuation $|\cdot|$, $\mathcal{M}_n(\mathbb{K})$ denotes the space of all $n \times n$ matrices over $\mathbb{K}$, $\mathbb{Q}_p$ is the field of p-adic numbers, X is non-Archimedean finite dimensional Banach space, X^* denotes the dual space of X and $\mathcal{L}(X)$ is the set of all bounded linear operators on X.

In this article, we introduce and study the structured pseudospectra and the structured condition pseudospectra of matrices with entries in $\mathbb{K}$. We give several results about them and some examples are supplied. We start by remembering some needed results.

Definition 1.1. [5] A field $\mathbb{K}$ is said to be *non-Archimedean* if it is endowed with an application $|\cdot| : \mathbb{K} \to \mathbb{R}^+$ such that

(i) $|\mu| = 0$ if, and only if, $\mu = 0$;
(ii) For all $\alpha, \mu \in \mathbb{K}$, $|\alpha\mu| = |\alpha||\mu|$;
(iii) For each $\alpha, \mu \in \mathbb{K}, |\alpha + \mu| \leq \max\{|\alpha|, |\mu|\}$.

Definition 1.2. [5] Let X be a vector space over $\mathbb{K}$. A mapping $\|\cdot\| : X \to \mathbb{R}_+$ is called a *non-Archimedean norm* if

(i) For each $x \in X$, $\|x\| = 0$ if and only if $x = 0$;
(ii) For all $x \in X$ and $\alpha \in \mathbb{K}$, $\|\alpha x\| = |\alpha|\|x\|$;
(iii) For any $x, y \in X$, $\|x + y\| \leq \max(\|x\|, \|y\|)$.

Definition 1.3. [5] We have

(i) A *non-Archimedean normed space* X is a vector space X endowed with a non-Archimedean norm $\|\cdot\|$.
(ii) A *non-Archimedean Banach space* is a complete non-Archimedean normed space.

Lemma 1.4. [5] *Let X be a non-Archimedean Banach space over $\mathbb{K}$. Let $A \in \mathcal{L}(X)$ such that $\|A\| < 1$, then $(I - A)^{-1} \in \mathcal{L}(X)$ and $\|(I - A)^{-1}\| \leq 1$.*

Definition 1.5. [15] A non-Archimedean field $\mathbb{K}$ is said to be *spherically complete* if each decreasing sequence of closed balls $(B_n)_n$ has nonempty intersection.

Theorem 1.6. [15] *Suppose that $\mathbb{K}$ is spherically complete. Let X be a non-Archimedean Banach space over $\mathbb{K}$. For all $x \in X\backslash\{0\}$, there exists $x^* \in X^*$ such that $x^*(x) = 1$ and $\|x^*\| = \|x\|^{-1}$.*

Definition 1.7. [7] Let $A \in \mathcal{M}_n(\mathbb{K})$, the *spectrum* $\sigma(A)$ *of a matrix* A is defined by

$$\sigma(A) = \{\lambda \in \mathbb{K} : \det(A - \lambda I) = 0\}.$$

The *resolvent set* $\rho(A)$ *of a matrix* A is the complement of $\sigma(A)$ in $\mathbb{K}$ given by

$$\rho(A) = \{\lambda \in \mathbb{K} : \ R_\lambda(A) = (A - \lambda I)^{-1} \text{ exists in } \mathcal{M}_n(\mathbb{K})\}.$$

$R_\lambda(A)$ is called the resolvent of the matrix A.

Proposition 1.8. [9] *Let X be a non-Archimedean Banach space over $\mathbb{K}$. If $A, B \in \mathcal{L}(X)$, then $-1 \notin \sigma(AB)$ if, and only if, $-1 \notin \sigma(BA)$.*

We introduce the following definitions.

Definition 1.9. [7] Let $A, B \in \mathcal{M}_n(\mathbb{K})$, the *spectrum* $\sigma(A, B)$ *of a matrix pencil* (A, B) of the form $A - \lambda B$ is defined by

$$\sigma(A, B) = \{\lambda \in \mathbb{K} : \det(A - \lambda B) = 0\}.$$

The *resolvent set* $\rho(A, B)$ *of a matrix pencil* (A, B) is the complement of $\sigma(A, B)$ in $\mathbb{K}$ given by

$$\rho(A, B) = \{\lambda \in \mathbb{K} : \ R_\lambda(A, B) = (A - \lambda B)^{-1} \text{ exists in } \mathcal{M}_n(\mathbb{K})\}.$$

$R_\lambda(A, B)$ is called the resolvent of the matrix pencil (A, B).

Definition 1.10. [7] Let $A, B \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$. The *ε-pseudospectrum of the matrix pencil* (A, B) of the form $A - \lambda B$ is defined by

$$\sigma_\varepsilon(A, B) = \sigma(A, B) \cup \{\lambda \in \mathbb{K} : \|(A - \lambda B)^{-1}\| > \varepsilon^{-1}\}.$$

The *ε-pseudoresolvent of the matrix pencil* (A, B) is denoted by

$$\rho_\varepsilon(A, B) = \rho(A, B) \cap \{\lambda \in \mathbb{K} : \ \|(A - \lambda B)^{-1}\| \leq \varepsilon^{-1}\},$$

by convention $\|(A - \lambda B)^{-1}\| = \infty$ if, and only if, $\lambda \in \sigma(A, B)$.

2. Main Results

As a generalization of ε-condition pseudospectra given in the paper [2] for the matrix case, we introduce the following definition.

Definition 2.1. Let $A, B, C \in \mathcal{M}_n(\mathbb{K})$ such that $0 \in \rho(B) \cap \rho(C)$ and $\varepsilon > 0$, the *structured condition pseudospectrum of a matrix A*, relative to the pair (B, C) is defined by

$$\Lambda_\varepsilon(A, B, C) = \sigma(A) \cup \left\{\lambda \in \mathbb{K} : \|C^{-1}(A - \lambda I)B^{-1}\|\|B(A - \lambda I)^{-1}C\| > \frac{1}{\varepsilon}\right\}.$$

The *structured condition pseudoresolvent of a matrix A* is given by

$$\rho(A) \cap \left\{\lambda \in \mathbb{K} : \|C^{-1}(A - \lambda I)B^{-1}\|\|B(A - \lambda I)^{-1}C\| \leq \frac{1}{\varepsilon}\right\}.$$

By convention $\|C^{-1}(A - \lambda I)B^{-1}\|\|B(A - \lambda I)^{-1}C\| = \infty$ if and only if $\lambda \in \sigma(A)$.

From Definition 2.1, we conclude the following remark.

Remark 2.2. Let $A, B, C \in \mathcal{M}_n(\mathbb{K})$ such that $0 \in \rho(B) \cap \rho(C)$, we have:
(i) If $C = B = I$, hence for each $\varepsilon > 0, \Lambda_\varepsilon(A, I, I) = \Lambda_\varepsilon(A)$ is the condition pseudospectrum of the matrix A.
(ii) One can see that for any $\varepsilon > 0, \Lambda_\varepsilon(A, B, C) \subset \Lambda_{\varepsilon k}(A)$ in which $k = \|B\|\|C\|\|B^{-1}\|\|C^{-1}\|$.

We have the following results.

Proposition 2.3. *Let $A, B, C \in \mathcal{M}_n(\mathbb{K})$ such that $0 \in \rho(B) \cap \rho(C)$, then*
(i) $\sigma(A) = \bigcap_{\varepsilon>0} \Lambda_\varepsilon(A, B, C)$.
(ii) *If $0 < \varepsilon_1 < \varepsilon_2$, then $\sigma(A) \subset \Lambda_{\varepsilon_1}(A, B, C) \subset \Lambda_{\varepsilon_2}(A, B, C)$.*

Proof. (i) By Definition 2.1, for all $\varepsilon > 0$, $\sigma(A) \subset \Lambda_\varepsilon(A, B, C)$. Conversely, if $\lambda \in \bigcap_{\varepsilon>0} \Lambda_\varepsilon(A, B, C)$, then for all $\varepsilon > 0$, $\lambda \in \Lambda_\varepsilon(A, B, C)$. If $\lambda \notin \sigma(A)$, then $\lambda \in \{\lambda \in \mathbb{K} : \ \|C^{-1}(A - \lambda I)B^{-1}\|\|B(A - \lambda I)^{-1}C\| > \varepsilon^{-1}\}$, taking limits as $\varepsilon \to 0^+$, we get $\|C^{-1}(A - \lambda I)B^{-1}\|\|B(A - \lambda I)^{-1}C\| = \infty$. Then $\lambda \in \sigma(A)$.
(ii) For $0 < \varepsilon_1 < \varepsilon_2$. Let $\lambda \in \Lambda_{\varepsilon_1}(A, B, C)$, then $\|C^{-1}(A - \lambda I)B^{-1}\|\|B(A - \lambda I)^{-1}C\| > \varepsilon_1^{-1} > \varepsilon_2^{-1}$. Hence $\lambda \in \Lambda_{\varepsilon_2}(A, B, C)$.

□

The following lemma is true for any Banach space over a complete field with a nontrivial valuation, in particular see Remark 2.17.

Lemma 2.4. *Let X be a non-Archimedean finite dimensional Banach space over $\mathbb{K}$, let $A, B, C \in \mathcal{L}(X)$ such that $0 \in \rho(B) \cap \rho(C)$ and $\varepsilon > 0$. Then $\lambda \in \Lambda_\varepsilon(A, B, C) \setminus \sigma(A)$ if, and only if, there is $x \in X \backslash \{0\}$ such that*

$$\|C^{-1}(A - \lambda I)B^{-1}x\| < \varepsilon \|C^{-1}(A - \lambda I)B^{-1}\| \|x\|.$$

Proof. If $\lambda \in \Lambda_\varepsilon(A, B, C) \setminus \sigma(A)$, then

$$\|C^{-1}(A - \lambda I)B^{-1}\| \|B(A - \lambda I)^{-1}C\| > \varepsilon^{-1}.$$

Thus

$$\|B(A - \lambda I)^{-1}C\| > \frac{1}{\varepsilon \|C^{-1}(A - \lambda I)B^{-1}\|}.$$

Hence

$$\sup_{y \in X \backslash \{0\}} \frac{\|B(A - \lambda I)^{-1}Cy\|}{\|y\|} > \frac{1}{\varepsilon \|C^{-1}(A - \lambda I)B^{-1}\|}.$$

Then, there is $y \in X \backslash \{0\}$ such that $\|B(A - \lambda I)^{-1}Cy\| > \frac{\|y\|}{\varepsilon \|C^{-1}(A - \lambda I)B^{-1}\|}$. Setting $x = B(A - \lambda I)^{-1}Cy$, hence $y = C^{-1}(A - \lambda I)B^{-1}x$. Consequently

$$\|x\| > \frac{\|C^{-1}(A - \lambda I)B^{-1}x\|}{\varepsilon \|C^{-1}(A - \lambda I)B^{-1}\|}.$$

Hence,

$$\|C^{-1}(A - \lambda I)B^{-1}x\| < \varepsilon \|C^{-1}(A - \lambda I)B^{-1}\| \|x\|.$$

Conversely, suppose that there is $x \in X \backslash \{0\}$ such that

$$\|C^{-1}(A - \lambda I)B^{-1}x\| < \varepsilon \|C^{-1}(A - \lambda I)B^{-1}\| \|x\|. \tag{2.1}$$

If $\lambda \notin \sigma(A)$ and $x = B(A - \lambda I)^{-1}Cy$, then $\|x\| \leq \|B(A - \lambda I)^{-1}C\| \|y\|$. From (2.1), we have $\|x\| < \varepsilon \|B(A - \lambda I)^{-1}C\| \|C^{-1}(A - \lambda I)B^{-1}\| \|x\|$. Then

$$\|B(A - \lambda I)^{-1}C\| \|C^{-1}(A - \lambda I)B^{-1}\| > \frac{1}{\varepsilon}.$$

Consequently, $\lambda \in \Lambda_\varepsilon(A, B, C) \backslash \sigma(A)$. □

Theorem 2.5. *Let $\mathbb{K}$ be a non-Archimedean complete field. Let $A, B, C \in \mathcal{M}_n(\mathbb{K})$ such that $0 \in \rho(A) \cap \rho(B) \cap \rho(C)$, $AC = CA$, $k = \|A^{-1}\| \|A\|$ and $\varepsilon > 0$. We have*

(i) *If $\lambda \in \Lambda_\varepsilon(A^{-1}, B, C) \backslash \{0\}$, then $\frac{1}{\lambda} \in \Lambda_{\varepsilon k}(A, B, C) \backslash \{0\}$.*
(ii) *If $\frac{1}{\lambda} \in \Lambda_{\varepsilon k}(A, B, C) \backslash \{0\}$, then $\lambda \in \Lambda_{\varepsilon k^2}(A^{-1}, B, C) \backslash \{0\}$.*

Proof.
(i) If $\lambda \in \Lambda_\varepsilon(A^{-1}, B, C) \backslash \{0\}$, then

$$\begin{aligned} \frac{1}{\varepsilon} < \|C^{-1}(A^{-1} - \lambda I)B^{-1}\| \|B(A^{-1} - \lambda I)^{-1}C\| &= \|\lambda A^{-1}C^{-1}\Big(\frac{I}{\lambda} - A\Big)B^{-1}\| \times \\ &\quad \|\lambda^{-1}B\Big(\frac{I}{\lambda} - A\Big)^{-1}CA\| \\ &\leq \|A^{-1}\| \|A\| \|C^{-1}\Big(\frac{I}{\lambda} - A\Big)B^{-1}\| \times \\ &\quad \|B\Big(\frac{I}{\lambda} - A\Big)^{-1}C\|. \end{aligned}$$

Hence, $\frac{1}{\lambda} \in \Lambda_{\varepsilon k}(A, B, C) \backslash \{0\}$.

(ii) If $\frac{1}{\lambda} \in \Lambda_{\varepsilon k}(A,B,C)\backslash\{0\}$, hence

$$\begin{aligned}\frac{1}{\varepsilon k} < \|C^{-1}(A-\lambda^{-1}I)B^{-1}\|\|B(A-\lambda^{-1}I)^{-1}C\| &= \|\lambda^{-1}AC^{-1}\Big(\lambda I - A^{-1}\Big)B^{-1}\| \times \\ &\quad \|\lambda B\Big(\lambda I - A^{-1}\Big)^{-1}CA^{-1}\| \\ &\leq \|A^{-1}\|\|A\|\|C^{-1}(\lambda I - A^{-1})B^{-1}\| \times \\ &\quad \|B\Big(\lambda I - A^{-1}\Big)^{-1}C\|.\end{aligned}$$

Then

$$\frac{1}{\varepsilon k^2} < \|C^{-1}(\lambda I - A^{-1})B^{-1}\|\|B\Big(\lambda I - A^{-1}\Big)^{-1}C\|.$$

Consequently, $\lambda \in \Lambda_{\varepsilon k^2}(A^{-1},B,C)\backslash\{0\}$. □

Theorem 2.6. *Let $\mathbb{K}$ be a non-Archimedean complete field. Let $A,B,C \in \mathcal{M}_n(\mathbb{K})$ such that $0 \in \rho(B)\cap\rho(C)$, $\lambda \in \mathbb{K}$ and $\varepsilon > 0$. If there is $D \in \mathcal{M}_n(\mathbb{K})$ such that $\|D\| < \varepsilon\|C^{-1}(A-\lambda I)B^{-1}\|$ and $\lambda \in \sigma(A+CDB)$. Then, $\lambda \in \Lambda_\varepsilon(A,B,C)$.*

Proof. Suppose that there exists $D \in \mathcal{M}_n(\mathbb{K})$ such that $\|D\| < \varepsilon\|C^{-1}(A-\lambda I)B^{-1}\|$. Let $\lambda \notin \Lambda_\varepsilon(A,B,C)$, thus $\lambda \in \rho(A)$ and $\|C^{-1}(A-\lambda I)B^{-1}\|\|B(A-\lambda I)^{-1}C\| \leq \varepsilon^{-1}$.
Consider F defined on $\mathcal{M}_n(\mathbb{K})$ by

$$F = \sum_{n=0}^{\infty}(A-\lambda I)^{-1}C\Big(-DB(A-\lambda I)^{-1}C\Big)^n C^{-1}.$$

From Lemma 1.4, we get $F = (A-\lambda I)^{-1}C\Big(C + CDB(A-\lambda I)^{-1}C\Big)^{-1}$. Then $A + CDB - \lambda I$ is invertible which is a contradiction (with $\lambda \in \sigma(A+CDB)$). Thus $\lambda \in \Lambda_\varepsilon(A,B,C)$. □

Remark 2.7. If X is non-Archimedean finite dimensional Banach space over $\mathbb{K}$, Theorem 2.6 remains valid.

We put $\mathcal{D}_\varepsilon(X) = \{D \in \mathcal{L}(X) : \|D\| < \varepsilon\|C^{-1}(A-\lambda I)B^{-1}\|\}$.

Theorem 2.8. *Let X be a non-Archimedean finite dimensional Banach space over a spherically complete field $\mathbb{K}$ such that $\|X\| \subseteq |\mathbb{K}|$, let $A,B,C \in \mathcal{L}(X)$ such that $0 \in \rho(B)\cap\rho(C)$ and $\varepsilon > 0$. Then*

$$\Lambda_\varepsilon(A,B,C) = \bigcup_{D\in\mathcal{D}_\varepsilon(X)} \sigma(A+CDB).$$

Proof. From Theorem 2.6, we get $\bigcup_{D\in\mathcal{D}_\varepsilon(X)} \sigma(A+CDB) \subseteq \Lambda_\varepsilon(A,B,C)$. Conversely, assume that $\lambda \in \Lambda_\varepsilon(A,B,C)$. If $\lambda \in \sigma(A)$, we may put $D = 0$. If $\lambda \in \Lambda_\varepsilon(A,B,C)$ and $\lambda \notin \sigma(A)$. By Lemma 2.4 and $\|X\| \subseteq |\mathbb{K}|$, there is $x \in X$ such that $\|x\| = 1$ and $\|C^{-1}(A-\lambda I)B^{-1}x\| < \varepsilon\|C^{-1}(A-\lambda I)B^{-1}\|$.
By Theorem 1.6, there is $\varphi \in X^*$ such that $\varphi(x) = 1$ and $\|\varphi\| = \|x\|^{-1} = 1$. Consider D on X defined by for all $y \in X$, $Dy = -\varphi(y)C^{-1}(A-\lambda I)B^{-1}x$. Then $\|D\| < \varepsilon\|C^{-1}(A-\lambda I)B^{-1}\|$. For $x \in X\backslash\{0\}$, $(A-\lambda I)B^{-1}x + CDx = 0$. Set $z = B^{-1}x \in X\backslash\{0\}$, we have $(A+CDB-\lambda I)z = 0$, hence $A+CDB-\lambda I$ is not injective, then $A+CDB-\lambda I$ is not invertible. Consequently,

$$\lambda \in \bigcup_{D\in\mathcal{D}_\varepsilon(X)} \sigma(A+CDB).$$

□

Proposition 2.9. *Let $A, B, C \in \mathcal{M}_n(\mathbb{K}), \varepsilon > 0$ such that $0 \in \rho(B) \cap \rho(C)$ and $\|C^{-1}(A - \lambda I)B^{-1}\| \neq 0$. We have*

(i) $\lambda \in \Lambda_\varepsilon(A, B, C)$ *if and only if* $\lambda \in \sigma_{\varepsilon\|C^{-1}(A-\lambda I)B^{-1}\|}(A, B, C)$.

(ii) $\lambda \in \sigma_\varepsilon(A, B, C)$ *if and only if* $\lambda \in \Lambda_{\frac{\varepsilon}{\|C^{-1}(A-\lambda I)B^{-1}\|}}(A, B, C)$.

Proof. (i) Let $\lambda \in \Lambda_\varepsilon(A, B, C)$, then $\lambda \in \sigma(A)$ and $\|C^{-1}(A - \lambda I)B^{-1}\|\|B(A - \lambda I)^{-1}C\| > \varepsilon^{-1}$. Hence $\lambda \in \sigma(A)$ and $\|B(A - \lambda I)^{-1}C\| > \frac{1}{\varepsilon\|C^{-1}(A-\lambda I)B^{-1}\|}$. Consequently, $\lambda \in \sigma_{\varepsilon\|C^{-1}(A-\lambda I)B^{-1}\|}(A)$. The converse is similar.

(ii) Let $\lambda \in \sigma_\varepsilon(A, B, C)$, then $\lambda \in \sigma(A)$ and $\|B(A - \lambda I)^{-1}C\| > \varepsilon^{-1}$. Thus

$$\lambda \in \sigma(A) \text{ and } \|C^{-1}(A - \lambda I)B^{-1}\|B(A - \lambda I)^{-1}C\| > \frac{\|C^{-1}(A - \lambda I)B^{-1}\|}{\varepsilon}.$$

Then, $\lambda \in \Lambda_{\frac{\varepsilon}{\|C^{-1}(A-\lambda I)B^{-1}\|}}(A, B, C)$. The converse is similar.

□

Theorem 2.10. *Let $A, B, C, U \in \mathcal{M}_n(\mathbb{K})$ such that $0 \in \rho(B) \cap \rho(C) \cap \rho(U)$, $UC = CU$ and $BU = UB$. If $V = U^{-1}AU$ and $k = \|U^{-1}\|\|U\|$, then for all $\varepsilon > 0$, we have*

$$\Lambda_{\frac{\varepsilon}{k^2}}(A, B, C) \subseteq \Lambda_\varepsilon(V, B, C) \subseteq \Lambda_{k^2\varepsilon}(A, B, C).$$

Proof. Let $\lambda \in \Lambda_{\frac{\varepsilon}{k^2}}(A, B, C)$, hence $\lambda \in \sigma(A)\Big(= \sigma(V)\Big)$ and

$$\begin{aligned}
\frac{k^2}{\varepsilon} < \|C^{-1}(A - \lambda I)B^{-1}\|\|B(A - \lambda I)^{-1}C\| &= \|C^{-1}U\Big(V - \lambda I\Big)U^{-1}B^{-1}\| \times \\
&\quad \|BU\Big(V - \lambda I\Big)^{-1}U^{-1}C\| \\
&\leq \Big(\|U\|\|U^{-1}\|\Big)^2\|C^{-1}\Big(V - \lambda I\Big)B^{-1}\| \times \\
&\quad \|B\Big(V - \lambda I\Big)^{-1}C\| \\
&\leq k^2\|C^{-1}\Big(V - \lambda I\Big)B^{-1}\|\|B\Big(V - \lambda I\Big)^{-1}C\|.
\end{aligned}$$

Or, $k^2 > 0$, then $\lambda \in \Lambda_\varepsilon(V, B, C)$. Hence, $\Lambda_{\frac{\varepsilon}{k^2}}(A, B, C) \subseteq \Lambda_\varepsilon(V, B, C)$.
Let $\lambda \in \Lambda_\varepsilon(V, B, C)$. Then

$$\begin{aligned}
\frac{1}{\varepsilon} < \|C^{-1}(V - \lambda I)B^{-1}\|\|B(V - \lambda I)^{-1}C\| &= \|C^{-1}U^{-1}\Big(A - \lambda I\Big)UB^{-1}\| \times \\
&\quad \|BU^{-1}\Big(A - \lambda I\Big)^{-1}UC\| \\
&\leq \Big(\|U\|\|U^{-1}\|\Big)^2\|C^{-1}\Big(A - \lambda I\Big)B^{-1}\| \times \\
&\quad \|B\Big(A - \lambda I\Big)^{-1}C\| \\
&\leq k^2\|B\Big(A - \lambda I\Big)C\|\|C^{-1}\Big(A - \lambda I\Big)^{-1}B^{-1}\|.
\end{aligned}$$

Hence, $\lambda \in \Lambda_{k^2\varepsilon}(A, B, C)$. Consequently $\Lambda_\varepsilon(V, B, C) \subseteq \Lambda_{k^2\varepsilon}(A, B, C)$. □

As a generalization of pseudospectra of a matrix pencil introduced in the paper [9], we introduce the following definition.

Definition 2.11. Let $A, B, C, M \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$, the (B, C)-*structured pseudospectrum of a matrix pencil* (A, M) is defined by

$$\sigma_\varepsilon(A, M, B, C) = \sigma(A, M) \cup \left\{\lambda \in \mathbb{K} : \|B(A - \lambda M)^{-1}C\| > \frac{1}{\varepsilon}\right\}.$$

The (B, C)-*structured pseudoresolvent of a matrix pencil* (A, M) is given by

$$\rho(A, M) \cap \left\{\lambda \in \mathbb{K} : \|B(A - \lambda M)^{-1}C\| \leq \frac{1}{\varepsilon}\right\}.$$

By convention $\|B(A - \lambda M)^{-1}C\| = \infty$ if and only if $\lambda \in \sigma(A, M)$.

By Definition 2.11, we get the following remark.

Remark 2.12.
(i) If $C = B = I$, then $\sigma_\varepsilon(A, M, I, I) = \sigma_\varepsilon(A, M)$ is the pseudospectrum of the matrix pencil (A, M).
(ii) One can see that for any $\varepsilon > 0, \sigma_\varepsilon(A, M, B, C) \subset \sigma_{\varepsilon k}(A, M)$ in which $k = \|B\|\|C\|$.

Theorem 2.13. *Let* $A, B, C, M \in \mathcal{M}_n(\mathbb{K})$. *Then,*
(i) *For all* $\varepsilon_1, \varepsilon_2 > 0$ *such that* $\varepsilon_1 \leq \varepsilon_2$, $\sigma_{\varepsilon_1}(A, M, B, C) \subset \sigma_{\varepsilon_2}(A, M, B, C)$.
(ii) $\sigma(A, M) = \bigcap_{\varepsilon>0} \sigma_\varepsilon(A, M, B, C)$.

Proof.
(i) If $\lambda \in \sigma_{\varepsilon_1}(A, M, B, C)$, then $\|B(A - \lambda M)^{-1}C\| > \varepsilon_1^{-1} \geq \varepsilon_2^{-1}$. Hence $\lambda \in \sigma_{\varepsilon_2}(A, M, B, C)$.
(ii) Since for all $\varepsilon > 0$, $\sigma(A, M) \subseteq \sigma_\varepsilon(A, M, B, C)$, then $\sigma(A, M) \subseteq \bigcap_{\varepsilon>0} \sigma_\varepsilon(A, M, B, C)$. Conversely, if $\lambda \in \bigcap_{\varepsilon>0} \sigma_\varepsilon(A, M, B, C)$, then for each $\varepsilon > 0$, $\lambda \in \sigma_\varepsilon(A, M, B, C)$, if $\lambda \notin \sigma(A, M)$, hence $\lambda \in \{\lambda \in \mathbb{K} : \|B(A - \lambda M)^{-1}C\| > \varepsilon^{-1}\}$, for $\varepsilon \to 0^+$, we obtain $\|B(A - \lambda M)^{-1}C\| = \infty$. Consequently $\lambda \in \sigma(A, M)$. □

Since the Hahn-Banach theorem does not hold in a general field only if it is spherically complete, see [10], we obtain:

Theorem 2.14. *Let X be a non-Archimedean finite dimensional Banach space over a spherically complete field $\mathbb{K}$ such that $\|X\| \subseteq |\mathbb{K}|$, let $A, B, C, M \in \mathcal{L}(X)$ such that $0 \in \rho(B) \cap \rho(C)$ and $\varepsilon > 0$. Then*

$$\sigma_\varepsilon(A, M, B, C) = \bigcup_{D \in \mathcal{L}(X) : \|D\| < \varepsilon} \sigma(A + CDB, M).$$

Proof. Firstly, we show that $\bigcup_{D \in \mathcal{L}(X) : \|D\| < \varepsilon} \sigma(A + CDB, M) \subseteq \sigma_\varepsilon(A, M, B, C)$.

Let $\lambda \in \bigcup_{D \in \mathcal{L}(X) : \|D\| < \varepsilon} \sigma(A + CDB, M)$. If $D = 0$, hence

$$\sigma(A, M) \subseteq \sigma_\varepsilon(A, M, B, C).$$

If $D \neq 0$. We argue by contradiction, if $\lambda \in \rho(A, M)$ and $\|B(A-\lambda M)^{-1}C\| \leq \varepsilon^{-1}$. Then for each $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, we have $\|DB(A-\lambda M)^{-1}C\| < 1$. Thus, $DB(A-\lambda M)^{-1}C + I$ is invertible. From Proposition 1.8, for any $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, $-1 \notin \sigma(DB(A-\lambda M)^{-1}C)$ if and only if $-1 \notin \sigma(CDB(A-\lambda M)^{-1})$. Thus

$$A + CDB - \lambda M = (I + CDB(A-\lambda M)^{-1})(A - \lambda M).$$

Hence $(A + CDB - \lambda M)^{-1} \in \mathcal{L}(X)$ which is a contradiction. Then

$$\bigcup_{D \in \mathcal{L}(X):\|D\|<\varepsilon} \sigma(A + CDB, M) \subseteq \sigma_\varepsilon(A, M, B, C).$$

For the converse inclusion, if $\lambda \notin \sigma(A, M)$, then $\|B(A-\lambda M)^{-1}C\| > \varepsilon^{-1}$. Hence

$$\|B(A-\lambda M)^{-1}C\| > \frac{1}{\varepsilon}.$$

Then

$$\sup_{x \in X\backslash\{0\}} \frac{\|B(A-\lambda M)^{-1}Cx\|}{\|x\|} > \frac{1}{\varepsilon}.$$

Consequently, there exists $x \in X\backslash\{0\}$ such that

$$\|B(A-\lambda M)^{-1}Cx\| > \frac{\|x\|}{\varepsilon}. \tag{2.2}$$

Set $y = B(A-\lambda M)^{-1}Cx$, then $C^{-1}(A-\lambda M)B^{-1}y = x$. From (2.2), we have

$$\|C^{-1}(\lambda M - A)B^{-1}y\| < \varepsilon\|y\|. \tag{2.3}$$

Since $\|X\| \subseteq |\mathbb{K}|$, there is $c \in \mathbb{K}\backslash\{0\}$ such that $\|y\| = |c|$, set $z = c^{-1}y$, thus $\|z\| = 1$. By (2.3),

$$\|(C^{-1}(\lambda M - A)B^{-1})z\| < \varepsilon.$$

By Theorem 1.6, there is $\varphi \in X^*$ such that $\varphi(z) = 1$ and $\|\varphi\| = \|z\|^{-1} = 1$. Set for each $x \in X$, $Dx = \varphi(x)(C^{-1}(\lambda M - A)B^{-1})z$. Then for all $x \in X$,

$$\begin{array}{rcl} \|Dx\| & = & |\varphi(x)|\|(C^{-1}(A-\lambda M)B^{-1})z\| \\ & \leq & \|\varphi\|\|x\|\|(C^{-1}(A-\lambda M)B^{-1})z\| \\ & < & \varepsilon\|x\|. \end{array}$$

Hence $D \in \mathcal{L}(X)$ and $\|D\| < \varepsilon$. Moreover for $z \neq 0$, $Dz + (C^{-1}(A-\lambda M)B^{-1})z = 0$. Set $v = B^{-1}z \in X\backslash\{0\}$. One can see that for $v \neq 0$, $(CDB + A - \lambda M)v = 0$. Thus

$$\lambda \in \bigcup_{D \in \mathcal{L}(X):\|D\|<\varepsilon} \sigma(A + CDB, M).$$

Consequently,

$$\sigma_\varepsilon(A, M, B, C) = \bigcup_{D \in \mathcal{L}(X):\|D\|<\varepsilon} \sigma(A + CDB, M).$$

□

Theorem 2.15. *Let X be a non-Archimedean finite dimensional Banach space over a spherically complete field $\mathbb{K}$ such that $\|X\| \subseteq |\mathbb{K}|$, let $A, B, C \in \mathcal{M}_n(\mathbb{K})$ such that $0 \in \rho(B) \cap \rho(C)$ and $\varepsilon > 0$. Then,*

$$\sigma_\varepsilon(A, M, B, C) = \sigma(A, M) \cup \{\lambda \in \mathbb{K} : \exists\, x \in X, \|x\| = 1,\ \|C^{-1}(A-\lambda M)B^{-1}x\| < \varepsilon\}.$$

Proof. If $\lambda \in \sigma_\varepsilon(A,M,B,C)\backslash\sigma(A,M)$, then $\|B(A-\lambda M)^{-1}C\| > \varepsilon^{-1}$. Thus

$$\sup_{x\in X\backslash\{0\}} \frac{\|B(A-\lambda M)^{-1}Cx\|}{\|x\|} > \frac{1}{\varepsilon}.$$

Hence there exists $x \in X\backslash\{0\}$ such that

$$\|B(A-\lambda M)^{-1}Cx\| > \frac{\|x\|}{\varepsilon}. \tag{2.4}$$

Set $y = B(A-\lambda M)^{-1}Cx \in X\backslash\{0\}$, then $C^{-1}(A-\lambda M)B^{-1}y = x$. By (2.4), we get

$$\|C^{-1}(A-\lambda M)B^{-1}y\| < \varepsilon\|y\|. \tag{2.5}$$

Since $\|X\| \subseteq |\mathbb{K}|$, there exists $c \in \mathbb{K}\backslash\{0\}$ such that $\|y\| = |c|$, put $z = c^{-1}y$, hence $\|z\| = 1$. By (2.5), we have

$$\|C^{-1}(A-\lambda M)B^{-1}z\| < \varepsilon.$$

Let $\lambda \in \mathbb{K}$ such that there exists $z \in X$ with $\|z\| = 1$ and

$$\|C^{-1}(A-\lambda M)B^{-1}z\| < \varepsilon.$$

From Theorem 1.6, there exists $\varphi \in X^*$ such that $\varphi(z) = 1$ and $\|\varphi\| = \|z\|^{-1} = 1$. Set for each $y \in X$, $Dy = \varphi(y)(C^{-1}(\lambda M - A)B^{-1})z$. Hence for any $y \in X$,

$$\begin{aligned}
\|Dy\| &= |\varphi(y)|\|(C^{-1}(A-\lambda M)B^{-1})z\| \\
&\leq \|\varphi\|\|y\|\|(C^{-1}(A-\lambda M)B^{-1})z\| \\
&< \varepsilon\|y\|.
\end{aligned}$$

Thus $D \in \mathcal{L}(X)$ and $\|D\| < \varepsilon$. Moreover for $z \neq 0$, $Dz + (C^{-1}(A-\lambda M)B^{-1})z = 0$. Set $v = B^{-1}z \in X\backslash\{0\}$. One can see that for $v \neq 0$, $(CDB + A - \lambda M)v = 0$. Thus

$$\lambda \in \bigcup_{D\in\mathcal{L}(X):\|D\|<\varepsilon} \sigma(A+CDB, M).$$

By Theorem 2.14, $\lambda \in \sigma_\varepsilon(A,M,B,C)$. □

We have the following example.

Example 2.16. Suppose that $\mathbb{K} = \mathbb{Q}_p$.

(i) If

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p).$$

Then, for all $\lambda \in \mathbb{Q}_p$, $\det(A-\lambda M) = (\lambda-1)(2\lambda-1)$, hence $\sigma(A,M) = \{\frac{1}{2}, 1\}$. Moreover

$$B(A-\lambda M)^{-1}C = \begin{pmatrix} \frac{-1}{1-2\lambda} & \frac{1}{(1-\lambda)(1-2\lambda)} \\ 0 & \frac{-1}{1-\lambda} \end{pmatrix}.$$

Hence

$$\|B(A-\lambda M)^{-1}C\| = \max\left\{\frac{1}{|(1-2\lambda)(1-\lambda)|}, \frac{1}{|1-2\lambda|}, \frac{1}{|\lambda-1|}\right\}.$$

Consequently, for each $\varepsilon > 0$,

$$\sigma_\varepsilon(A,M,B,C) = \sigma(A,M) = \left\{\frac{1}{2}, 1\right\} \cup \left\{\lambda \in \mathbb{Q}_p : \|B(A-\lambda M)^{-1}C\| > \frac{1}{\varepsilon}\right\}.$$

(ii) If

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \textit{ and } M = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p).$$

Then $\sigma(A, M) = \{0, \frac{1}{2}\}$ and

$$\|B(A - \lambda M)^{-1}C\| = \frac{1}{|\lambda(2\lambda - 1)|}.$$

Thus, the structured pseudospectrum of the matrix pencil (A, M) is

$$\sigma_\varepsilon(A, M, B, C) = \left\{0, \frac{1}{2}\right\} \cup \left\{\lambda \in \mathbb{Q}_p : |\lambda(1 - 2\lambda)| < \varepsilon\right\}.$$

(iii) If

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix},\ M = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \textit{ and } C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p).$$

Then $\sigma(A, M) = \{1\}$ and

$$\|B(A - \lambda M)^{-1}C\| = \frac{1}{|\lambda - 1|}.$$

Thus, the structured pseudospectrum of the matrix pencil (A, M) is

$$\sigma_\varepsilon(A, M, B, C) = \{1\} \cup \left\{\lambda \in \mathbb{Q}_p : |\lambda - 1| < \varepsilon\right\}.$$

As a generalization of condition pseudospectra of operators in complex Banach spaces [3], we finish with the following remark.

Remark 2.17. If X is an Archimedean Banach space over $\mathbb{C}$ and $0 < \varepsilon < 1$, the notion of the structured condition pseudospectrum introduced above and the results about it remain valid.

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