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GLOBAL OPTIMIZATION ALGORITHM BASED ON AUGMENTED LAGRANGIAN FOR MULTIOBJECTIVE OPTIMIZATION PROBLEMS

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Abstract. In this article, we propose a method for solving constrained multi-objective optimization problems using an extension of the classical Augmented Lagrangian method. We demonstrate that any sequence generated by the algorithm is feasible, and that any limit point is an optimal Pareto solution. A second algorithm is introduced to solve the subproblem within the main algorithm, using the steepest descent method and a non-monotone Max-type linear search technique. The theoretical and numerical results validate the performance and efficiency of the proposed method.

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1. Introduction

Multiobjective optimization is a critical branch of the optimization field that aims to find solutions that optimize multiple criteria simultaneously. In many real-world problems, decision-making is often confronted with complex and diverse constraints, which necessitates the use of constrained optimization methods to obtain viable solutions [12]. These constraints play a crucial role in ensuring the practical applicability and feasibility of the obtained solutions, making constrained multiobjective optimization an essential tool for addressing real-world decision-making challenges.

Amongst these methods, augmented Lagrangian has emerged as a promising approach for solving constrained multiobjective optimization problems. It effectively handles constraints through an iterative process, enhancing the accuracy of the obtained solutions. The augmented Lagrangian method provides a robust and reliable solution for tackling constrained multi-objective optimization problems.

In the literature, extensive studies have been conducted on the convergence analysis of the augmented Lagrangian method in this context. It all started with the groundbreaking article by Powell (1969) [22], which introduced the concept of augmented Lagrangian to address nonlinear optimization problems with equality constraints. This seminal work paved the way for numerous subsequent research endeavors. Subsequently, Hestenes (1969) [16] and Rockafellar (1973) [23] laid down solid theoretical foundations for the augmented Lagrangian method and established its global convergence properties for convex problems. Building upon this foundation, adaptations of the augmented Lagrangian have been developed to handle non-convex and nondifferentiable optimization problems. Notable contributions, such as those by Birgin and Martínez (2000) [6], proposed variants tailored to address these specific types of problems.

Recently, there has been research on extending the augmented Lagrangian method to solve multi-objective problems. Cocchi et al. [10, 11] introduced a method based on non-scalar augmented Lagrangian, while Undapayer et al. [24] proposed an augmented Lagrangian method based on the cone method, which transforms the multiobjective problem into a scalar single-objective problem.

Augmented Lagrangian methods that employ multiplier safeguarding techniques have gained significant interest in recent years. Notable contributions in this area include the works of Andreani et al. [1], Birgin et al. [7, 8], and Galván et al. [15]. These methods offer advantages over classical approaches, such as penalty and multiplier methods [4, 12, 23], as summarized in [19].

The presence of abstract constraints and the complexity associated with handling approximate stationary points contribute to the intriguing nature of this problem. Augmented Lagrangian methods with multiplier safeguarding techniques provide effective tools for addressing these challenges, enabling efficient and reliable solutions for constrained multi-objective optimization problems.

Computing the global minimum of each Lagrangian subproblem would simplify the convergence analysis without requiring additional assumptions about the admissible domain, except for its closure [3]. However, computing the global minimum is often challenging in the presence of non-convex objective functions. Therefore, our approach focuses on stationary points. Additionally, it is often practical to start with approximate stationary points at the beginning of the algorithm and gradually demand more accurate solutions as the algorithm progresses, which simplifies the computations.

In this context, we propose in this study a method based on Augmented Lagrangian to solve global constrained multi-objective problems. Our objective is to develop an approach that minimizes the number of additional conditions, such as the requirement of convexity.

The rest of this work is structured as follows. In Section 2, we provide an overview of the preliminary concepts related to multi-objective optimization. Section 3 focuses on the augmented Lagrangian method for multi-objective optimization, providing a detailed description of the algorithm used and presenting the results regarding the feasibility and optimality of the generated sequences. In Section 4, we examine the practical application of the algorithm by presenting results that demonstrate its validity. Section 5 is dedicated to the application of the proposed algorithm through test problems. Finally, Section 6 summarizes our conclusions, offers future perspectives, and suggests research directions to explore in this field.

2. Preliminaries

We will consider the multiobjective programming problem defined as follows:

$$\text{(MOP)} \qquad \begin{array}{l} \min\ F(x) = (f_1(x), f_2(x), \cdots, f_q(x)) \\ \text{s.t. :} \left\{ \begin{array}{ll} h_i(x) = 0 & \forall i \in I = \{1, 2, 3, \cdots, p\} \\ g_l(x) \leq 0 & \forall l \in L = \{1, 2, 3, \cdots, m\} \\ x \in \mathbb{R}^n. & \end{array} \right. \end{array}$$

In this formulation, $F : \mathbb{R}^n \to \mathbb{R}^q$ is a vector function with components $f_1(x), f_2(x), \ldots, f_q(x)$. The constraints include equality constraints $h_i(x) = 0$ for all i in the index set $I = \{1, 2, 3, \cdots, p\}$, and inequality constraints $g_l(x) \leq 0$ for all l in the index set $L = \{1, 2, 3, \cdots, m\}$. In the following, we assume that the functions f_j, g_i, h_l are continuous and differentiable functions. Let $\mathcal{X}$ denote the feasible space of problem (MOP), defined as $\mathcal{X} = \{x \in \mathbb{R}^n : h(x) = 0 \text{ and } g(x) \leq 0\}$. In the rest of this work, we assume that the admissible space $\mathcal{X}$ is non-empty.

Throughout our study, we will adhere to the following conventions: the set of positive real numbers is denoted as $\mathbb{R}_{++}$. $\mathbb{R}^n$ represents the set of column vectors with dimension n. The image space of a matrix $A \in \mathbb{R}^{m\times n}$ is referred to as $\mathrm{Im}(A)$. The unit vector of dimension q is represented by e. For any vectors $u = (u_1, u_2, \cdots, u_n)^T$ and $v = (v_1, v_2, \cdots, v_n)^T$, we establish the following conventions regarding equality and inequality:

$$\begin{array}{rl} (i) & u = v \Leftrightarrow u_i = v_i \text{ for every } i = 1, 2, \cdots, n \\ (ii) & u < v \Leftrightarrow u_i < v_i \text{ for every } i = 1, 2, \cdots, n \\ (iii) & u \leqq v \Leftrightarrow u_i \leq v_i \text{ for every } i = 1, 2, \cdots, n \\ (iv) & u \leq v \Leftrightarrow u \leqq v \text{ and } u \neq v. \end{array}$$

Definition 2.1. ([20]) A point $x^* \in \mathcal{X}$ is *Pareto optimal* for problem (MOP) if there does not exist another $x \in \mathcal{X}$ such that:

$$F(x) \leq F(x^*) \text{ and } F(x) \neq F(x^*).$$

Definition 2.1 provides an important property of Pareto optimality. Therefore, we present the following definition, which proposes simpler conditions to obtain in practice. We can express the classical definitions of optimality in the Pareto sense since it is not always possible to find a solution that minimizes all objective functions simultaneously.

Definition 2.2. ([20]) A point $x^* \in \mathcal{X}$ is *weakly Pareto optimal* for problem (MOP) if there does not exist another $x \in \mathcal{X}$ such that:

$$F(x) < F(x^*).$$

Considering these two definitions, we present the following lemma extracted from [10], which provides equivalent relationships between Definition 2.1 and Definition 2.2 of Pareto optimality.

Lemma 2.3. *A point $x^* \in \mathcal{X}$ is*

(a): *a Pareto optimum for (MOP) if and only if for all $y \in \mathcal{X}$ at least one of the following relations holds:*

(i): $\max_{j=1,2,\ldots,q} \{f_j(y) - f_j(x^*)\} > 0,$

(ii): $\min_{j=1,2,\ldots,q} \{f_j(y) - f_j(x^*)\} \geq 0.$

(b): *a weak Pareto optimum for (MOP) if and only if for all* $y \in \mathcal{X}$ *we have* $\max_{j=1,2,\dots,q} \{f_j(y) - f_j(x^*)\} \geq 0$.

We also define $x^* \in \mathbb{R}^n$ as a local Pareto optimal point (or weak local Pareto optimal point) if there exists a neighborhood $V(x^*) \in \mathbb{R}^n$ such that x^* is a Pareto optimal point (or weakly Pareto optimal point) for F restricted to $V(x^*)$.

A necessary but generally not sufficient condition for weak Pareto optimality can be expressed by the following relation:

$$-(\mathbb{R}_{++})^q \cap \operatorname{Im}(J_F(x^*)) = \emptyset, \tag{2.1}$$

where $J_F(.)$ denotes the Jacobian matrix of F. A point $x^* \in \mathcal{X}$ is considered stationary for F if it satisfies relation (2.1). Now, a necessary condition for Pareto optimality is given by the following definition.

Definition 2.4. ([13]) A point $x^* \in \mathcal{X}$ is considered a *Pareto stationary point* for the problem (MOP) if, for any $d \in \mathbb{R}^n$, the following inequality holds:

$$\max_{j=\overline{1;q}} \nabla f_j(x^*)^\top d \geq 0.$$

Note that if x^* is not a Pareto stationary point, there exists an admissible direction d such that $\max_{j=\overline{1;q}} \nabla f_j(x^*)^\top d < 0$.

We can define an ε-Pareto-Stationary solution as follows:

Definition 2.5 ([10])**.** Let $\varepsilon \geq 0$. A point $x^* \in \mathbb{R}^n$ is ε-Pareto-stationary for problem (MOP) if

$$\max_{i=1,\dots,m} \nabla f_j(x^*)^\top d \geq -\varepsilon, \ \forall d \in \{\xi \in \mathbb{R}^n \mid \|\xi\| \leq 1\}.$$

Definition 2.6 presents the concepts of ε-Pareto optimal solution and weakly ε-Pareto optimal solution.

Definition 2.6. Let $\varepsilon \geq 0$. A point $x^* \in \mathcal{X}$ is:

(a): ε-Pareto optimal for (MOP) if for every $y \in \mathcal{X}$ at least one of the following conditions is satisfied:

(i): $\max_{j=1,2,\dots,q} \{f_j(y) - f_j(x^*)\} > -\varepsilon$, or

(ii): $\min_{j=1,2,\dots,q} \{f_j(y) - f_j(x^*)\} \geq -\varepsilon$.

(b): weakly ε-Pareto optimal for (MOP) if $\max_{j=1,2,\dots,q} \{f_j(y) - f_j(x^*)\} \geq -\varepsilon$ for all $y \in \mathcal{X}$.

3. Augmented Lagrangian function for Multiobjective Optimization problems

In this section, we introduce the construction of the Augmented Lagrangian method for the resolution of multiobjective optimization problems.

3.1. Principle.

Let's consider the problem (MOP). Note that (MOP) is equivalent to

$$\begin{aligned} &\min : F(x) = (f_1(x), f_2(x), \cdots, f_q(x)) \\ &\text{s.t.} \begin{cases} h_i(x) = 0 \quad \forall i \in I = \{1, 2, 3, \dots, p\}, \\ g_l(x) + r_l = 0, \ \ r_l \geq 0 \quad \forall l \in L = \{1, 2, 3, \dots, m\} \\ x \in \mathbb{R}^n, \end{cases} \end{aligned}$$

where $r = (r_1, r_2, \dots, r_m)^\top$. Applying the augmented Lagrangian method to problem (3.1) leads to solving the following problem [3, 2]:

$$\min_{x\in\mathbb{R}^n,\, r\geqq 0} \overline{\mathcal{L}_\eta}(x, r, \mu, \lambda), \tag{3.1}$$

where

$$\overline{\mathcal{L}_\tau}(x, r, \mu, \lambda) = F(x) + \left\{ \sum_{l=1}^{m} \left[\mu_l \left(g_l(x) + r_l \right) + \frac{\tau}{2} \left(g_l(x) + r_l \right)^2 \right] + \sum_{i=1}^{p} \left[\lambda_i \left(h_i(x) \right) + \frac{\tau}{2} \left(h_i(x) \right)^2 \right] \right\} \cdot e,$$

and $\tau > 0$, $\mu = (\mu_1, \mu_2, \dots, \mu_m) \in \mathbb{R}^m_+$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p) \in \mathbb{R}^p_+$. The minimum of (3.1) can be obtained [2] by first minimizing $\overline{\mathcal{L}_\tau}(x, r, \mu, \lambda)$ over the slack variables $r \geqq 0$, which implies

$$\mathcal{L}_\tau(x, \mu, \lambda) = \min_{r\geqq 0} \overline{\mathcal{L}_\tau}(x, r, \mu, \lambda),$$

and then minimizing $\mathcal{L}_\tau(x, \mu, \lambda)$ when $x \in \mathbb{R}^n$. Note that $\mu_l g_l(x) + \frac{\tau}{2}(g_l(x) + r_l)^2$ is quadratic in r. Thus, it is easy to obtain a closed-form expression for $\mathcal{L}_\tau(x, \mu, \lambda)$ for each fixed x. For a given x, we have

$$\min_{r\geq 0} \overline{\mathcal{L}_\tau}(x, r, \mu, \lambda) = F(x) + \left\{ \sum_{l=1}^{m} \min_{r\geq 0} \left[\mu_l \left(g_l(x) + r_l \right) + \frac{\tau}{2} \left(g_l(x) + r_l \right)^2 \right] + \sum_{i=1}^{p} \left[\lambda_i \left(h_i(x) \right) + \frac{\tau}{2} \left(h_i(x) \right)^2 \right] \right\} \cdot e. \tag{3.2}$$

However, the minimum of $\left\{ \mu_l \left(g_l(x) + r_l \right) + \frac{\tau}{2} \left(g_l(x) + r_l \right)^2 \right\}$ is obtained for each

$$r_l^* = \max\left\{ 0, -\left(\frac{\mu_l}{\tau} + g_l(x) \right) \right\}.$$

This implies

$$\min_{r\geq 0} \left\{ \mu_l \left(g_l(x) + r_l \right) + \frac{\tau}{2} \left(g_l(x) + r_l \right)^2 \right\} = \left\{ \mu_l \max\left\{ g_l(x), -\frac{\mu_l}{\tau} \right\} + \frac{\tau}{2} \left(\max\left\{ g_l(x), -\frac{\mu_l}{\tau} \right\} \right)^2 \right\}. \tag{3.3}$$

Substituting into (3.2), we obtain

$$\mathcal{L}_\tau(x, r, \mu, \lambda) = F(x) + \left\{ \sum_{l=1}^{m} \left[\mu_l \max\left\{ g_l(x), -\frac{\mu_l}{\tau} \right\} + \frac{\tau}{2} \left(\max\left\{ g_l(x), -\frac{\mu_l}{\tau} \right\} \right)^2 \right] + \sum_{i=1}^{p} \left[\lambda_i \left(h_i(x) \right) + \frac{\tau}{2} \left(h_i(x) \right)^2 \right] \right\} \cdot e \tag{3.4}$$

which represents the Augmented Lagrange function of (MOP), with λ and μ as vectors of Lagrange multipliers and τ as the penalty parameter [5, 7, 8, 15, 17]. These parameters are updated at each iteration according to the following equations:

$$\lambda_i^{k+1} = P_{\mathcal{C}} \left[\lambda_i^k + \tau_k h_i(x^{k+1}) \right] \text{ with } \mathcal{C} = [0, \lambda_{\max}] \text{ for all } i = \overline{1, p}$$

$$\mu_l^{k+1} = P_{\Omega} \left[\mu_l^k + \tau_k g_l(x^{k+1}) \right] \text{ with } \Omega = [0, \mu_{\max}] \text{ for all } l = \overline{1, m},$$

with $P_{\mathcal{K}}(.)$ being the projection operator onto the convex space $\mathcal{K}$.

Now, posing $g_{l,+}(x, \mu_l, \tau) = \max\left\{g_l(x), -\frac{\mu_l}{\tau}\right\}$, and substituting in (3.4), we obtain the following simplified expression (3.5), which we will use throughout the rest of the paper.

$$\mathcal{L}_{\tau_k}(x, \lambda^k, \mu^k) = F(x) + \left[\sum_{i=1}^{p}\left\{\lambda_i^k h_i(x) + \frac{\tau_k}{2}\left(h_i(x)\right)^2\right\} + \sum_{l=1}^{m}\left\{\mu_l^k g_{l,+}(x, \mu_l^k, \tau_k) + \frac{\tau_k}{2}\left(g_{l,+}(x, \mu_l^k, \tau_k)\right)^2\right\}\right] \cdot e. \tag{3.5}$$

3.2. **Algorithm.** In this subsection, we introduce a new algorithm for solving multiobjective optimization problems using the Augmented Lagrangian method. This algorithm is based on the information presented in the preceding sections. The algorithm is outlined as follows:

Algorithm 1: Global Optimization Algorithm based on Augmented Lagrangian for Multiobjective Optimization Problems.

Data: $\mu^0 \in \mathbb{R}_+^m$ $\sigma \in (0, 1)$, $\lambda^0 \in \mathbb{R}_+^p$; $\tau_0 > 0$; $x_0 \in \mathbb{R}^n$; $\alpha \geq 1$

for $k = 0, 1, 2, \cdots$ **do**

  Find $x^{k+1} \in \mathbb{R}^n$ a ϵ_k-Pareto-Point of

$$\mathcal{L}_{\tau_k}(x, \lambda^k, \mu^k) = F(x) + \left[\sum_{i=1}^{p}\left\{\lambda_i^k h_i(x) + \frac{\tau_k}{2}\left(h_i(x)\right)^2\right\} + \sum_{l=1}^{m}\left\{\mu_l^k g_{l,+}(x, \mu_l^k, \tau_k) + \frac{\tau_k}{2}\left(g_{l,+}(x, \mu_l^k, \tau_k)\right)^2\right\}\right] \cdot e,$$

  for $i = 1, 2, 3, \cdots, p$ **do**

    $\lambda_i^{k+1} = P_{[0;\overline{\lambda}]}\left(\lambda_i^k + \tau_k h_i(x^{k+1})\right)$

  for $l = 1, 2, 3, \cdots, m$ **do**

    $\mu_l^{k+1} = P_{[0;\overline{\mu}]}\left(\tau_k g_l(x^{k+1}) + \mu_l^k\right)$

    $\beta_l^{k+1} = \max\left\{g_l(x^{k+1}), -\dfrac{\mu_l^k}{\tau_k}\right\}$

  if $\max\left\{\left\|h(x^{k+1})\right\|, \left\|\beta^{k+1}\right\|\right\} \leq \sigma \max\left\{\left\|h(x^k)\right\|, \left\|\beta^k\right\|\right\}$ **then**

    $\tau_{k+1} = \tau_k$

  else

    $\tau_{k+1} = \alpha\tau_k$

3.3. **Convergence analysis of Algorithm 1.** In this sub-section, we initiate the discussion on convergence analysis within well-defined assumptions. We examine the required conditions to ensure the convergence of the Augmented Lagrangian method in the context of multi-objective problems under constraints. By analyzing these assumptions, our objective is to understand the convergence behaviors and limitations of these methods. The obtained results provide us with information regarding the performance and reliability of these approaches for solving complex multiobjective problems.

Assumption 3.1. *The objective function F has bounded level sets in the multiobjective sense, meaning that the set $\{x \in \mathbb{R}^n, F(x) \leqq F(x_0)\}$ is compact.*

Assumption 3.2. *The sequence $\{\epsilon_k\}$ satisfies $\lim_{k\to\infty} \epsilon_k = 0$.*

Assumption 3.3. *For any $k = 0, 1, \ldots,$ there exists $x^{k+1} \in \mathbb{R}$ an ϵ_k-Pareto point for $\mathcal{L}_{\mathcal{T}_k}(x, \lambda^k, \mu^k)$, that is: for any $x \in \mathbb{R}^n$, there is a $j \in \{1, \ldots, q\}$ such that*

$$\left(\mathcal{L}_{\mathcal{T}_k}(x^{k+1}, \lambda^k, \mu^k)\right)_j < \left(\mathcal{L}_{\mathcal{T}_k}(x, \lambda^k, \mu^k)\right)_j + \epsilon_k,$$

where ϵ_k is a given bounded sequence.

The following theorem presents the admissibility results of the sequence $\{x_k\}$ generated by Algorithm 1 at each iteration.

Theorem 3.4. (Algorithm 1 Feasibility) *Assume that Assumption 3.3 holds. Let $\{x_k\}$ be the sequence generated by Algorithm 1. Suppose that there exists $K \underset{\infty}{\subseteq} \mathbb{N}$ such that $\lim_{\substack{k\to\infty \\ k\in K}} x^k = x^*$. Then, for all $x \in \mathbb{R}^n$, we have*

$$\|h(x^*)\|^2 + \|g_+(x^*)\|^2 \leq \|h(x)\|^2 + \|g_+(x)\|^2, \quad \text{where } g_+(x) = \max\{g(x), 0\} \tag{3.6}$$

Proof. We consider two cases on the sequence of penalty parameters:

- **(a):** the sequence $\{\tau_k\}$ is bounded,
- **(b):** the sequence $\{\tau_k\}$ is unbounded.

Case (a): From the definition of the sequence of penalty parameters in lines 10 to 12 of Algorithm 1, it can be observed that the terms of the sequence τ_k satisfy either $\tau_{k+1} = \tau_k$ or $\tau_{k+1} = \alpha\tau_k$ with $\alpha > 1$. This indicates that τ_k forms a monotonically increasing sequence. In order for $\{\tau_k\}$ to be bounded, the number of times the equality $\tau_{k+1} = \alpha\tau_k$ occurs must be finite. Therefore, there exists a positive integer k_0 such that for any $k > k_0$, we have $\tau_k = \tau_{k_0}$. Hence, for $k_0 \geq 1$,

$$\begin{aligned}
\max\left\{\left\|h(x^{k_0+1})\right\|, \|\beta_{k_0+1}\|\right\} &\leq \sigma \max\left\{\left\|h(x^{k_0})\right\|, \|\beta_{k_0}\|\right\} \\
\max\left\{\left\|h(x^{k_0+2})\right\|, \|\beta_{k_0+2}\|\right\} &\leq \sigma \max\left\{\left\|h(x^{k_0+1})\right\|, \|\beta_{k_0+1}\|\right\} \\
&\leq \sigma^2 \max\left\{\left\|h(x^{k_0})\right\|, \|\beta_{k_0}\|\right\} \\
&\vdots \\
\max\left\|h(x^{k_0+m})\right\|, \|\beta_{k_0+m}\| &\leq \sigma^m \max\left\{\left\|h(x^{k_0})\right\|, \|\beta_{k_0}\|\right\}.
\end{aligned}$$

When $m \to \infty$, since $\sigma \in (0; 1)$, we have :

$$\sigma^m \max\left\{\left\|h(x^{k_0})\right\|, \|\beta_{k_0}\|\right\} \to 0.$$

Hence, $\left\|h(x^{k+1})\right\| \to 0$ and $\max\left\{g_l(x^{k+1}), -\frac{\mu_l^k}{\tau_k}\right\} \to 0$ for all l. Since h and g are continuous, it follows that $h(x^*) = 0$ and $g(x^*) \leq 0$.

Thus, the limit point x^* of x^k is feasible.

Case (b): assume that $\tau_k \to \infty$. Let $K \underset{\infty}{\subseteq} \mathbb{N}$ be such that $x^k \xrightarrow{k\in K} x^*$. Assume by contradiction that there exists $x \in \mathbb{R}^n$ such that

$$\|h(x^*)\|^2 + \|g_+(x^*)\|^2 > \|h(x)\|^2 + \|g_+(x)\|^2 . \tag{3.7}$$

By the continuity of h and g, the boundedness of $\{\mu^k\}$ and $\{\lambda^k\}$ and the fact that $\tau_k \to \infty$, there exist $\zeta > 0$ and $k_0 \in \mathbb{N}$ such that for all $k \in K$, $k \geq k_0$,

$$\left\|h(x^{k+1}) + \frac{\lambda^k}{\tau_k}\right\|^2 + \left\|\left(g(x^{k+1}) + \frac{\mu^k}{\tau_k}\right)_+\right\|^2 - \left\|\frac{\mu^k}{\tau_k}\right\|^2 - \left\|\frac{\lambda^k}{\tau_k}\right\|^2 > \left\|h(x) + \frac{\lambda^k}{\tau_k}\right\|^2 + \left\|\left(g(x) + \frac{\mu^k}{\tau_k}\right)_+\right\|^2 - \left\|\frac{\lambda^k}{\tau_k}\right\|^2 - \left\|\frac{\mu^k}{\tau_k}\right\|^2 + \zeta. \tag{3.8}$$

Therefore, for all $k \in K$, $k \geq k_0$, we have for all $j = \overline{1, q}$

$$\begin{aligned} f_j(x^{k+1}) &+ \sum_{i=1}^{p} \left\{\lambda_i h_i(x^{k+1}) + \frac{\tau_k}{2}\left(h_i(x^{k+1})\right)^2\right\} + \sum_{l=1}^{m} \left\{\mu_l^k g_{l,+}(x^{k+1}, \mu_l^k, \tau)\right. \\ &\left. + \frac{\tau}{2}\left(g_{l,+}(x^{k+1}, \mu_l^k, \tau)\right)^2\right\} > f_j(x) + \sum_{i=1}^{p} \left\{\lambda_i^k h_i(x) + \frac{\tau_k}{2}(h_i(x))^2\right\} \\ &+ \sum_{l=1}^{m} \left\{\mu_l^k g_{l,+}(x, \mu_l^k, \tau) + \frac{\tau}{2}\left(g_{l,+}(x, \mu_l^k, \tau)\right)^2\right\} + \frac{\tau_k \cdot \zeta}{2} + f_j(x^{k+1}) - f_j(x). \end{aligned} \tag{3.9}$$

Since $x^k \xrightarrow{k\in K} x^*$, and $\{\epsilon_k\}$ is bounded there exists $k_0 \leq k_0'$ such that for all $k \geq k_0'$ and $j = 1, 2, \ldots, q$

$$\frac{\tau_k \cdot \zeta}{2} + f_j(x^{k+1}) - f_j(x) > \epsilon_k.$$

Therefore for $k > k_0'$ from equation (3.9), we can write

$$\left(\mathcal{L}_{\tau_k}(x^{k+1}, \lambda^k, \mu^k)\right)_j > \left(\mathcal{L}_{\tau_k}(x, \lambda^k, \mu^k)\right)_j + \epsilon_k, \text{ for all } j,$$

which contradicts assumption (3.3). Hence, (3.6) holds.

□

Theorem 3.5. (Optimality of solution generated by Algorithm 1) *Let x^* be a cluster point for the sequence $\{x^{k+1}\}$ generated by Algorithm 1 under assumptions 3.1 and assumption 3.2. Then x^* is a weak Pareto point for problem (MOP).*

Proof. Since the problem (MOP) is feasible by Theorem 3.4, the point x^* is feasible i.e $h(x^*) = 0$ and $g(x^*) \leq 0$.

Assume now by contradiction that x^* is not a weak Pareto point for problem (MOP). Then, there exists $x \in \mathcal{X}$ such that for all $j = \overline{1, q}$

$$f_j(x) < f_j(x^*). \tag{3.10}$$

From the instruction of the algorithm and from Lemma 2.3 we know that

$$\begin{aligned}(3.11)\quad \min_{j=1,2,\ldots,q} \Bigg\{ & f_j(x^{k+1}) + \sum_{i=1}^{p} \left\{ \lambda_i^k h_i(x^{k+1}) + \frac{\tau_k}{2} \left(h_i(x^{k+1}) \right)^2 \right\} \\ & + \sum_{l=1}^{m} \left\{ \mu_l^k g_{l,+}(x^{k+1}, \mu_l^k, \tau_k) + \frac{\tau_k}{2} \left(g_{l,+}(x^{k+1}, \mu_l^k, \tau_k) \right)^2 \right\} - f_j(x) \\ & - \sum_{l=1}^{m} \left\{ \mu_l^k g_{l,+}(x, \mu_l^k, \tau_k) + \frac{\tau_k}{2} \left(g_{l,+}(x, \mu_l^k, \tau_k) \right)^2 \right\} \Bigg\} \leq \epsilon_k,\end{aligned}$$

which means that

$$\begin{aligned}(3.12)\quad \min_{j=1,2,\ldots,q} \left\{ f_j(x^{k+1}) - f_j(x) \right\} \leq & - \sum_{i=1}^{p} \left\{ \lambda_i^k h_i(x^{k+1}) + \frac{\tau_k}{2} \left(h_i(x^{k+1}) \right)^2 \right\} \\ & - \sum_{l=1}^{m} \left\{ \mu_l^k g_{l,+}(x^{k+1}, \mu_l^k, \tau_k) + \frac{\tau_k}{2} \left(g_{l,+}(x^{k+1}, \mu_l^k, \tau_k) \right)^2 \right\} + \\ & \sum_{l=1}^{m} \left\{ \mu_l^k g_{l,+}(x, \mu_l^k, \tau_k) + \frac{\tau_k}{2} \left(g_{l,+}(x, \mu_l^k, \tau_k) \right)^2 \right\} + \epsilon_k,\end{aligned}$$

By rearranging, we obtain

(3.13)

$$\begin{aligned}\min_{j=1,2,\ldots,q} \left\{ f_j(x^{k+1}) - f_j(x) \right\} \leq & -\frac{\tau_k}{2} \sum_{i=1}^{p} \left\{ \left(h_i(x^{k+1}) + \frac{\lambda_i^k}{\tau_k} \right)^2 \right\} - \frac{\tau_k}{2} \sum_{l=1}^{m} \left[\max \left\{ 0, g_l(x^{k+1}) + \frac{\mu_l^k}{\tau_k} \right\} \right]^2 \\ & + \frac{\tau_k}{2} \sum_{i=1}^{p} \left(\frac{\lambda_i^k}{\tau_k} \right)^2 + \frac{\tau_k}{2} \sum_{l=1}^{m} \left[\max \left\{ 0, g_l(x) + \frac{\mu_l^k}{\tau_k} \right\} \right]^2 + \epsilon_k.\end{aligned}$$

We consider two cases on the sequences of penalty parameters

(i): $\{\tau_k\}$ tend to infinity

(ii): $\{\tau_k\}$ is bounded

Case (i): $\tau_k \to \infty$ now we have

$$\begin{aligned}(3.14)\quad \min_{j=1,2,\ldots,q} \left\{ f_j(x^{k+1}) - f_j(x) \right\} \leq & \frac{\tau_k}{2} \sum_{i=1}^{p} \left(\frac{\lambda_i^k}{\tau_k} \right)^2 \\ & + \frac{\tau_k}{2} \sum_{l=1}^{m} \left[\max \left\{ 0, g_l(x) + \frac{\mu_l^k}{\tau_k} \right\} \right]^2 + \epsilon_k, \text{ for all } k \in \mathbb{N}.\end{aligned}$$

As x is feasible by assumption, i.e $h(x) = 0$ and $g(x) \leq 0$, we have that for all l,

$$\left[\max \left\{ 0, g_l(x) + \frac{\mu_l^k}{\tau_k} \right\} \right]^2 \leq \left\{ \frac{\mu_l^k}{\tau_k} \right\}^2.$$

Therefore, by equation (3.15),

$$\min_{j=1,2,\dots,q}\left\{f_j(x^{k+1})-f_j(x)\right\}\le\frac{\tau_k}{2}\sum_{i=1}^{p}\left(\frac{\lambda_i^k}{\tau_k}\right)^2+\frac{\tau_k}{2}\sum_{l=1}^{m}\left(\frac{\mu_l^k}{\tau_k}\right)^2+\epsilon_k. \tag{3.15}$$

$$\Rightarrow\min_{j=1,2,\dots,q}\left\{f_j(x^{k+1})-f_j(x)\right\}\le\sum_{i=1}^{p}\frac{\left(\lambda_i^k\right)^2}{\tau_k}+\sum_{l=1}^{m}\frac{\left(\mu_l^k\right)^2}{\tau_k}+\epsilon_k. \tag{3.16}$$

Taking limits for $k\in K$ and using that $\lim_{k\to\infty}\frac{\left(\lambda_i^k\right)^2}{\tau_k}=\lim_{k\to\infty}\frac{\left(\mu_l^k\right)^2}{\tau_k}=0$ and $\epsilon_k\to 0$, recalling again $\left\{\lambda^k\right\}$ and $\left\{\mu^k\right\}$ are bounded, we get

$$\min_{j=1,2,\dots,q}\{f_j(x^*)-f_j(x)\}\le 0,$$

which contradicts (3.10).

Case (ii): in this case, there exists $k_0\in\mathbb{N}$ such that $\tau_k=\tau_{k_0}$ for all $k\ge k_0$.

From the instruction of the algorithm, $\max\left\{\left\|h(x^{k+1})\right\|,\left\|\beta^{k+1}\right\|\right\}\to 0$ This implies that $h_i(x^{k+1})\to 0$ and $\beta^{k+1}\to 0\implies\frac{\mu_i^k}{\tau_k}\to 0$ as $k\to\infty$, $k\in K$ for all i such that $g_i(x^*)<0$. In particular, for such indices i it holds $\mu_i^k\to 0$ as $k\to\infty$, $k\in K$. We therefore have from the Lemma 2.3

$$\begin{aligned}\min_{j=1,2,\dots,q}\left\{f_j(x^{k+1})-f_j(x)\right\}\le&-\frac{\tau_{k_0}}{2}\sum_{i=1}^{p}\left\{\left(h_i(x^{k+1})+\frac{\lambda_i^k}{\tau_{k_0}}\right)^2\right\}\\&-\frac{\tau_{k_0}}{2}\sum_{l=1}^{m}\left[\max\left\{0,g_l(x^{k+1})+\frac{\mu_l^k}{\tau_{k_0}}\right\}\right]^2+\frac{\tau_{k_0}}{2}\sum_{i=1}^{p}\left\{\left(\frac{\lambda_i^k}{\tau_{k_0}}\right)^2\right\}\\&+\frac{\tau_{k_0}}{2}\sum_{l=1}^{m}\left[\max\left\{0,g_l(x)+\frac{\mu_l^k}{\tau_{k_0}}\right\}\right]^2+\epsilon_k.\end{aligned} \tag{3.17}$$

Given that $h(x)=0$ and $g(x)\le 0$, it follows from equation (3.17) that

$$\begin{aligned}\min_{j=1,2,\dots,q}\left\{f_j(x^{k+1})-f_j(x)\right\}\le&-\frac{\tau_{k_0}}{2}\sum_{i=1}^{p}\left(h_i(x^{k+1})+\frac{\lambda_i^k}{\tau_{k_0}}\right)^2-\frac{\tau_{k_0}}{2}\sum_{l=1}^{m}\left[\max\left(0,g_l(x^{k+1})+\frac{\mu_l^k}{\tau_{k_0}}\right)\right]^2\\&+\frac{\tau_{k_0}}{2}\sum_{i=1}^{p}\left(\frac{\lambda_i^k}{\tau_{k_0}}\right)^2+\frac{\tau_{k_0}}{2}\sum_{l=1}^{m}\left(\frac{\mu_l^k}{\tau_{k_0}}\right)^2+\epsilon_k\quad\text{for any }k\in\mathbb{N}.\end{aligned} \tag{3.18}$$

Since $x^k\to x^*$, and $h(x^*)=0$, and because the functions h_i for all i are continuous, and the sequences $\{\lambda_i^k\}$ are bounded, we deduce that

$$-\frac{\tau_{k_0}}{2}\sum_{i=1}^{p}\left(h_i(x^{k+1})+\frac{\lambda_i^k}{\tau_{k_0}}\right)^2+\frac{\tau_{k_0}}{2}\sum_{i=1}^{p}\left(\frac{\lambda_i^k}{\tau_{k_0}}\right)^2\longrightarrow 0.$$

For the other two sums, we can write:

$$\sum_{l=1}^{m}\left(\frac{\mu_l^k}{\tau_{k_0}}\right)^2-\sum_{l=1}^{m}\left[\max\left\{0,g_l(x^{k+1})+\frac{\mu_l^k}{\tau_{k_0}}\right\}\right]^2$$
$$\leq \sum_{l:g_l(x^*)<0}\left(\frac{\mu_l^k}{\tau_{k_0}}\right)^2+\sum_{l:g_l(x^*)=0}\left\{\left(\frac{\mu_l^k}{\tau_{k_0}}\right)^2-\left[\max\left\{0,g_l(x^{k+1})+\frac{\mu_l^k}{\tau_{k_0}}\right\}\right]^2\right\}.$$

Since $\mu_l^k \to 0$ for all l such that $g_l(x^*) < 0$, the first sum tends to 0. The second sum also goes to 0 because $x^k \to x^*$, $g_l(x^*) = 0$, the functions g_l are continuous, and the sequences $\{\mu_l^k\}$ are bounded.

Thus, it follows that

$$\min_{j=1,\dots,q}(f_j(x^*)-f_j(x))\leq 0,$$

which contradicts (3.10). □

4. Practical Algorithm

In this section, we present an algorithm specifically designed to solve the sub-problem of Algorithm 1, aiming for an efficient resolution. This algorithm utilizes the Steepest Descent method. In the rest of this paper, we denote

$$T_j=(\mathcal{L}_\tau(x,\lambda,\mu))_j=f_j(x)+\sum_{i=1}^{p}\left\{\lambda_i^k h_i(x)+\frac{\tau_k}{2}(h_i(x))^2\right\}$$
$$+\sum_{l=1}^{m}\left\{\mu_l^k g_{l,+}(x,\mu_l^k,\tau)+\frac{\tau}{2}\left(g_{l,+}(x,\mu_l^k,\tau)\right)^2\right\}\text{ for all } j, \tag{4.1}$$

and

$$\nu_q=\left\{w:\sum_{j\in 1,2,\dots,q}w_j=1,w_j\geq 0,j\in 1,2,\dots,q\right\}.$$

The algorithm is presented as follows:

Algorithm 2: Algorithm to solve subproblem of Algoritm 1

Data: Choose $\rho \in (0,1)$, $\delta \in (0,1)$; tolerance $\epsilon_k \geq 0$; $x^0 \in \mathbb{R}^n$

1 Set $C^0 = T(x^0)$
2 Set a nonnegative integer M
3 **for** $k = 0, 1, 2, \cdots$ **do**
4   Set $w^k = \arg\min_{w \in \nu_q} \frac{1}{2} \left\| \sum_{j=1}^{q} w_j \nabla T_j(x^k) \right\|^2$
5   Set $d_k = -\sum_{j=1}^{q} w_j^k \nabla T_j(x^k)$
6   Set $\theta_k = -\frac{1}{2} \|d_k\|$
7   **if** $\mid \theta_k \mid \leq \epsilon_k$ **then**
8   Stop and **Return** ϵ_k-Pareto-critical point x^k
9   **else**
10   Set $\varphi = 1$
11   **while** $\left(T_j(x^k + \varphi d_k) \nleq C_j^k + \rho\varphi \nabla T_j(x^k)^\top d_k \right)$ *for all* j **do**
12             $\varphi = \delta\varphi$
13   Set $x^{k+1} = x^k + \varphi d_k$
14         $k = k + 1$
15   **for** $j = 1, 2, \ldots, q$ **do**
16   Set $C_j^k = \max_{0 \leq i \leq \min(k,M)} T_j(x^{k-i})$

The description of Algorithm 2 is as follows: first, we evaluate the function $(\mathcal{L}_\tau(x, \lambda, \mu))_j$ using the parameters λ and μ for each j calculated in Algorithm 1, while keeping x as a variable. Then, we initialize the vector C^0 by evaluating the function $T_j(x)$ with the value x^0. In the main loop, we start by solving the problem presented at line 5 to determine the weighting factors. Then, at line 6, these factors are used to determine the descent direction. The stopping condition is checked at line 8, where we compare the computed value θ_k at line 9 with a tolerance threshold ϵ_k defined previously in Algorithm 1. It is important to note that the threshold ϵ_k varies at each iteration of Algorithm 1 and tends to 0. Finally, if the point x_k is not ϵ_k-Pareto-Stationary, we compute the descent step in steps 11 to 16. It should be emphasized that the proper definition of Algorithm 2 relies on the use of the linear search technique to determine the step size φ in these steps . Therefore, we start by presenting Assumption 4.1 regarding the proposed descent direction d_k in [21], and then we present a technical result stated in Lemma 4.2.

Assumption 4.1. *For a sequence of iteration* $\{x^k\}$ *and search direction* $\{d_k\}$ *there exist positive constants* ϱ_1 *and* ϱ_2 *such that*

$$\max_{j=\overline{1;q}} \left\{ \nabla F_j(x^k)^\top d_k \right\} \leq -\varrho_1 \mid \theta_k \mid,$$

$$\|d_k\| \leq \varrho_2 \mid \theta_k \mid .$$

Lemma 4.2. *For every iteration k of Algorithm 2, the following inequality holds:*

$$T(x^k) \leq C^k.$$

Proof. Since for all j , $C_j^k = \max_{0 \leq i \leq \min(k,M)} T_j(x^{k-i})$ we have $T(x^k) \leq C^k$. □

The following theorem demonstrates that the linear search technique presented in algorithm 2 is well-defined, meaning that the step size is determined in a finite number of iterations.

Theorem 4.3. *Let $\{x^k\}$ be an iteration of Algorithm 2. If $J_T(x^k)d_k < 0$, indicating that d_k is a descent direction, then for any $\rho \in (0,1)$, there exists a $\varphi > 0$ such that*

$$T(x^k + \varphi d_k) \leq C^k + \rho\varphi J_T(x^k)d_k.$$

Proof. Assume that $J_T(x)d_k < 0$. Since by the definition T is differentiable, we have

$$T(x^k + \varphi d_k) = T(x^k) + \varphi \left(J_T(x^k)d_k + \Theta(\varphi) \right),$$

with $\lim_{\varphi \to 0} \Theta(\varphi) = 0$.

Observe that $\max \nabla T(x^k)^\top d_k < 0$ and $d_k \neq 0$.

Since $\rho \in (0,1)$, there exist $\delta > 0$ such that for some $\varphi \in [0,\delta]$, $\|\Theta(\varphi)\|$ is small enough, so that

$$\Theta(\varphi) \leq -(1-\rho)J_T(x)d_k \text{ for all } \varphi \in [0,\delta].$$

Therefore

$$\begin{aligned} T(x^k + \varphi d_k) &= T(x^k) + \varphi \left(J_T(x^k)d_k + \Theta(\varphi) \right) \\ &\leq T(x^k) + \varphi \left(J_T(x^k)d_k - (1-\rho)J_T(x)d_k \right) \\ &\leq T(x^k) + \varphi J_T(x^k)d_k \text{ for all } \varphi \in [0,\delta]. \end{aligned}$$

Since $T(x^k) \leq C^k$ from Lemma 4.2. We conclude that

$$T(x^k + \varphi d_k) \leq C^k + \rho\varphi J_T(x^k)d_k \text{ for all } \varphi \in [0,\delta].$$

□

Theorem 4.4. *Assume that the Assumptions 3.1 and 4.1 hold. Let x^* be a cluster point for sequence $\{x^k\}$ generated by Algorithm 2. Then x^* is Pareto critical point.*

Proof. Since the steepest descent direction $d_k(x^k) = -\sum_{j=1}^{q} w_j^k \nabla T_j(x^k)$ with

$$w^k = \arg\min_{w \in \nu_q} \frac{1}{2} \left\| \sum_{j=1}^{q} w_j \nabla T_j(x^k) \right\|^2$$

clearly satisfies Assumption 4.1, the result holds for Algorithm 2 (see Theorem 6 of [21]). □

5. Numerical Experiments

In this section, we present the application of the proposed method to test problems to demonstrate its ability to generate Pareto optimal solutions. The set of test problems used is summarized in Table 1, encompassing both convex and non-convex problems. To evaluate the performance of the method, we utilize specific parameter settings. The values used for the parameters are as follows: $\rho = 10^{-4}$, $\alpha = 10$, $\sigma = 0.9$, $\lambda_0 = 0 \in \mathbb{R}^p$, $\mu_0 = 0 \in \mathbb{R}^m$, $\overline{\mu} = 10^5$, $\overline{\lambda} = 1$, $\varphi_0 = 1$, $\delta = 0.5$. The termination criterion used in all problems is that $\epsilon_k \geq 10^{-6}$. At each iteration, ϵ_k is updated according to the formula

$$\epsilon_{k+1} = \begin{cases} 0.75 * \epsilon_k \text{ if } k = 1, \\ 0.9 * \epsilon_k \text{ if } k > 1. \end{cases}$$

Regarding the problems that have constraints in the form $lb \leq x \leq ub$, we transform them into constraints of the form $lb - x \leq 0$ and $x - ub \leq 0$. This transformation is carried out to adapt the constraints to the formalism used by the method. The total number of constraints generated is equal to $2n$, where n represents the number of variables in the problem. This transformation ensures the feasibility of the obtained solutions and guarantees that they adhere to the specified bounds. we used an HP EliteBook laptop equipped with an Intel Core i7-3687U processor with a base frequency range of 2.10GHz to 2.60 GHz and 4GB of RAM to test our algorithms.

Table 1. List of multiobjective optimization test problems

Problems	**n**	**q**	**Parameters bornes**	**Source**
DGO1	1	2	$[-10, 13]$	[18]
BNH1	2	2	$[0, 5]^2$	[10]
SCH	1	2	$[-4, 4]$	[18]
ZDT1	5,10,15,20,25,30	2	$[0, 1] \times [0, 1/100]^{n-1}$	[21]
ZDT2	5,10,15,20,25,30	2	$[0, 1] \times [0, 1/100]^{n-1}$	[14]
JOS1	$20, 30, 35, 40, 45, 50$	2	$[-2, 2]^n$	[18]
FON	$20, 30, 35, 40, 45, 50$	2	$[-4, 4]^n$	[18]
MLF1	1	2	$[0, 20]$	[18]
LE1	2	2	$[-5, 10]$	[18]
IKK1	2	3	$[-50, 50]^2$	[18]
DD1	5	2	$[-20, 20]$	[21]
Deb	2	2	$[0.1, 1] \times [0, 1]$	[9]
FDS	10	3	$[-2, 2]^{10}$	[21]
TRIDIA	3	3	$[-1, 1]^3$	[21]

Table 2 shows the mathematical formulation of some test problems taken from Table 1.

Table 2. Mathematical formulation of some multiobjective optimization test problems

Problems	Mathematical formulation	Source
DGO1	$\begin{cases} \min f_1(x) = \sin(x) \\ \min f_2(x) = \sin(x+0.7) \\ s.t. \\ g_1(x) = x - 13 \leq 0 \\ g_2(x) = -x - 10 \leq 0 \end{cases}$	[18]
BNH1	$\begin{cases} \min f_1(x) = 4x_1^2 + 4x_2^2 \\ \min f_2(x) = (x_1-5)^2 + (x_2-5)^2 \\ s.t. \\ g_1(x) = (x_1-5)^2 + x_2^2 \leq 25 \\ g_2(x) = (x_1-8)^2 + (x_2+3)^2 \geq 7.7 \end{cases}$	[10]
LE1	$\begin{cases} \min f_1(x) = \left(x_1^2 + x_2^2\right)^{0.125} \\ \min f_2(x) = \left((x_1-0.5)^2 + (x_2-0.5)^2\right)^{0.25} \\ s.t. \\ g_1(x) = -x_1 - 5 \leq 0 \\ g_2(x) = x_1 - 10 \leq 0 \\ g_1(x) = -x_2 - 5 \leq 0 \\ g_2(x) = x_2 - 10 \leq 0 \end{cases}$	[18]
DD1	$\begin{cases} \min f_1(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2, \\ \min f_2(x) = 3x_1 + 2x_2 - \frac{x_3}{3} + 0.01(x_4 - x_5)^3, \\ s.t. \\ g_{1-5}(x) = -x_i - 20 \leq 0, \quad i = 1, 2, \ldots, 5 \\ g_{6-10}(x) = x_i - 20 \leq 0, \quad i = 1, 2, \ldots, 5 \end{cases}$	[21]
Deb	$\begin{cases} \min f_1(x) = x_1 \\ \min f_2(x) = \frac{h(x)}{x_1} \\ h(x) = 2 - \exp\left\{-\left(\frac{x_2-0.2}{0.004}\right)^2\right\} - 0.8\exp\left\{-\left(\frac{x_2-0.6}{0.4}\right)^2\right\} \\ s.t. \\ g_1(x) = -x_1 - 0.1 \leq 0 \\ g_2(x) = x_1 - 1 \leq 0 \\ g_3(x) = -x_2 \leq 0 \\ g_4(x) = x_2 - 1 \leq 0 \end{cases}$	[9]
FDS	$\begin{cases} \min f_1(x) = \frac{1}{n^2}\sum_{i=1}^{n} i(x_i - i)^4, \qquad n = 10 \\ \min f_2(x) = \exp\left(\sum_{i=1}^{n} \frac{x_i}{n}\right) + \Vert x\Vert_2^2, \\ \min f_3(x) = \frac{1}{n(n+1)}\sum_{i=1}^{n} i(n-i+1)e^{-x_i}, \\ s.t. \\ g_{\overline{1,10}}(x) = -x_i - 2 \leq 0, \qquad i = 1, \ldots, 10 \\ g_{\overline{11,20}}(x) = x_i - 2 \leq 0, \qquad i = 1, \ldots, 10 \end{cases}$	[21]

The figures below display the Pareto fronts of the problems mentioned in Table 1. These Pareto fronts illustrate the optimal solutions that offer an optimal compromise between the objectives of the problem, where no improvement in one objective is possible without sacrificing another objective.

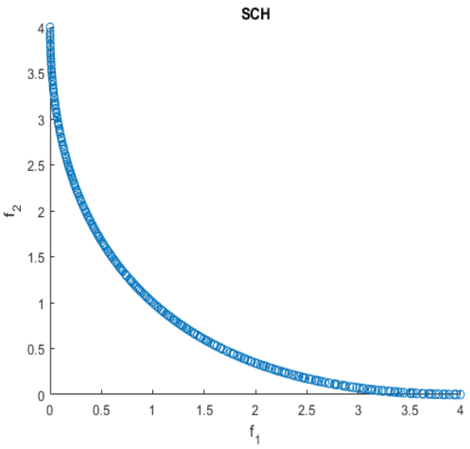

Figure 1. Pareto front SCH

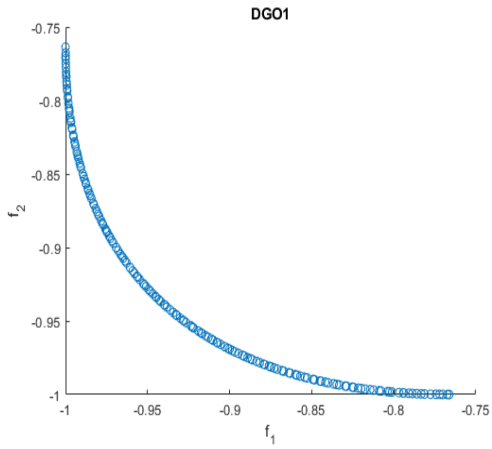

Figure 2. Pareto front of DGO1

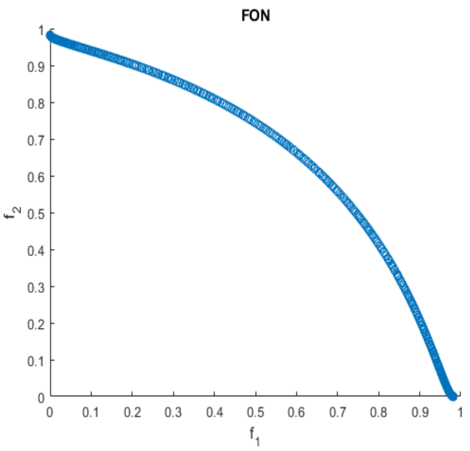

Figure 3. Pareto front FON

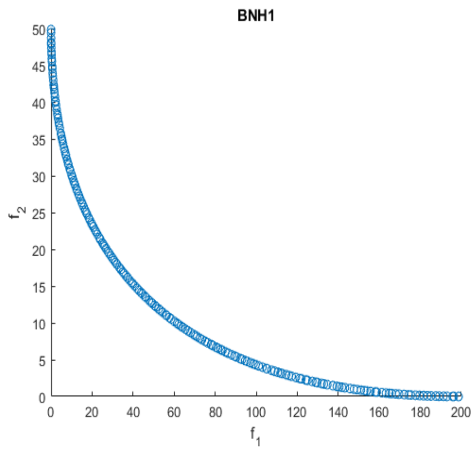

Figure 4. Pareto front of BNH1

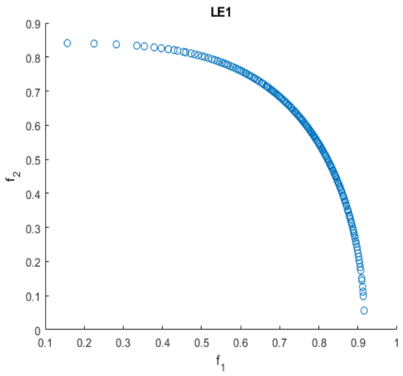

Figure 5. Pareto front of LE1

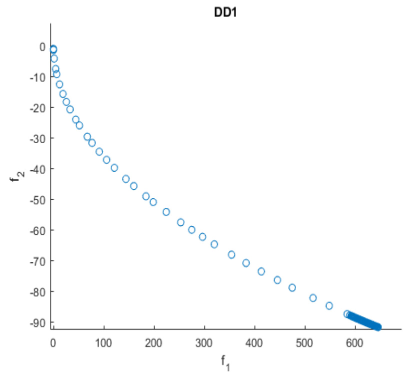

Figure 6. Pareto front of DD1

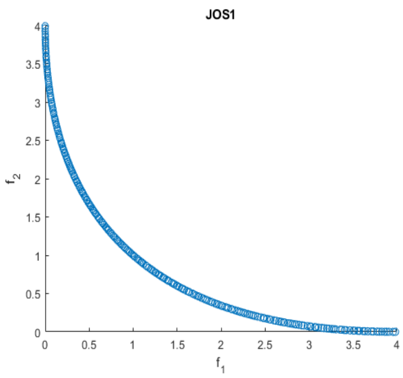

Figure 7. Pareto front of JOS1

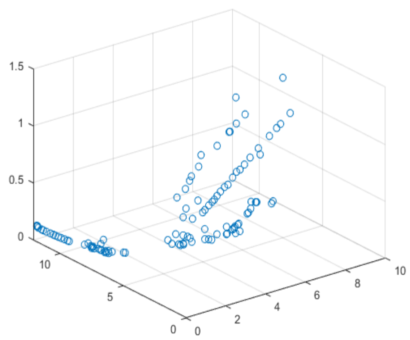

Figure 8. Pareto front of FDS

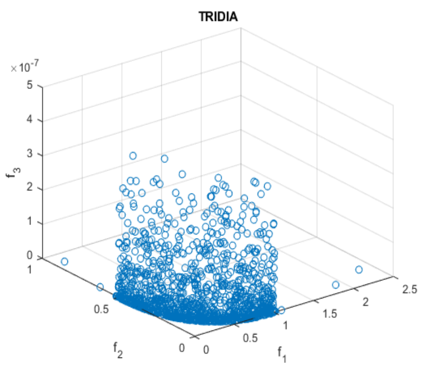

Figure 9. Pareto front of TRIDIA

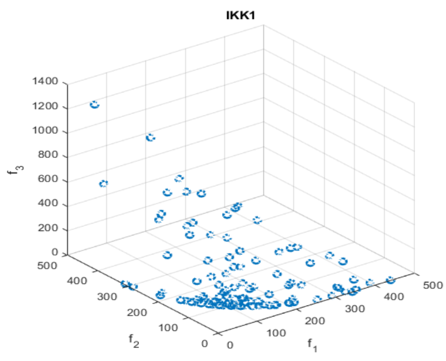

Figure 10. Pareto front of IKK1

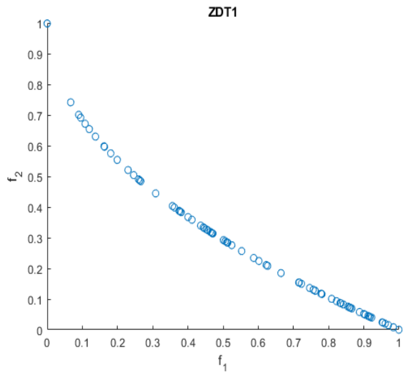

FIGURE 11. Pareto front of ZDT1

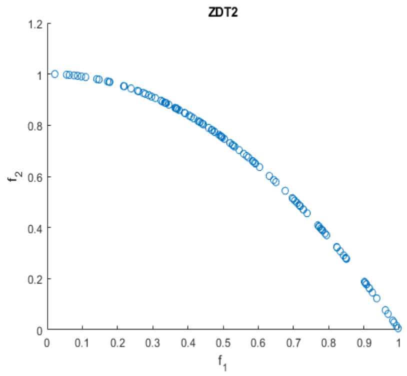

FIGURE 12. Pareto front of ZDT2

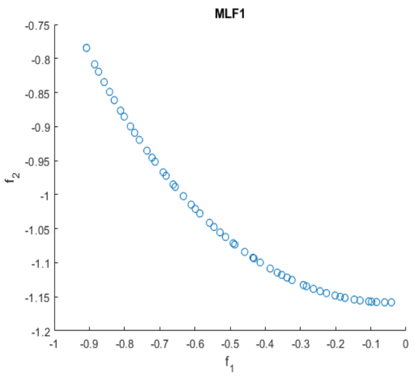

FIGURE 13. Pareto front of MLF1

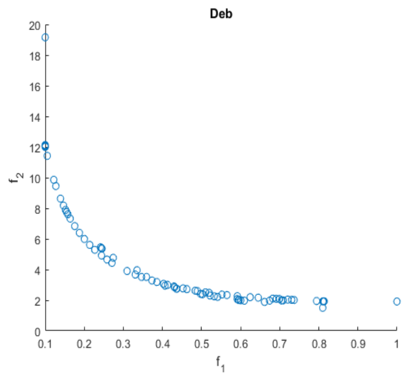

FIGURE 14. Pareto front of Deb

6. CONCLUSION

In this paper, we have presented a method based on Augmented Lagrangian for solving global constrained multi-objective optimization problems. We have demonstrated the feasibility and optimality of the sequences generated by the proposed algorithm. Additionally, we have introduced a second algorithm for its practical application, highlighting the results regarding its validity and emphasizing the numerical outcomes obtained in various test problems.

Future research directions in this field include exploring variants of the proposed algorithm, adapting it to specific problem classes, and extending the method to handle more complex constraints.

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