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ZEROS OF POLYNOMIALS OVER LOCALLY COMPACT FIELDS

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In memory of my Teacher and Master, Dr. doc. Nicolae Popescu

Abstract. Let K be a non-discrete locally compact (commutative) field and let $P \in K[X]$ be a non-constant polynomial. In this note we give two criteria for the existence of a zero for P in the initial field K. Then, we apply these criteria to simply prove that the complex number field is algebraically closed and, under the truth of a conjecture, that any polynomial equation $P(X) = 0$, with $P \in K[X]$, is solvable by radicals.

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1. Introduction

We started from a brilliant idea of C. Fefferman [1] and we tried to use it in the much more general context of a non-discrete locally compact (commutative) field K. From [6], Chapter 1, 2 we know that K is a rank one valued field (K, φ), where $\varphi : K \to \mathbb{R}_{\geq 0}$ is its corresponding multiplicative absolute value. The structure of such fields are well known (see [6], Chapter 1, 2). Namely, if φ is Archimedean, then $(K, \varphi) \simeq (\mathbb{R}, |\cdot|^s)$ or $(K, \varphi) \simeq (\mathbb{C}, |\cdot|^s)$, where $|\cdot|$ is the usual absolute value and $0 < s \leq 1$. If φ is not Archimedean, then $(K, \varphi) \simeq (L, \varphi_L)$ or $(K, \varphi) \simeq (S, \psi_X)$, where L is a finite extension of $\mathbb{Q}_p$, the p-adic number field for a prime number p, φ_L is the unique extension to L of the p-adic absolute value φ_p of $\mathbb{Q}_p$, and S is a finite extension of the power series field $\mathbb{F}_p((X))$, with ψ_X the unique extension to S of the X-adic valuation of $\mathbb{F}_p((X))$. For a prime number p, as usual, $\mathbb{F}_p$ is the finite field with p elements. A more elementary treatment of this subject one can find in [2], Chapter 2, or in [4], Chapter 3.

In Section 2 we give the basic results (Theorem 2.2 and Theorem 2.4) and two applications of them, a simple proof for the Fundamental Theorem of Algebra (Corollary 2.5) and, under the truth of a conjecture, an elementary proof for the solvability by radicals of a polynomial equation $P(X) = 0$, $P \in K[X]$ (Corollary 2.6). In particular, there exists a unique non-discrete locally compact field that contains the radicals of all its elements, the complex number field. We also prove (under the truth of the same conjecture) that $Gal(L_P/K)$ is a solvable group, where $P \in K[X]$ and L_P is a decomposition field of P (Corollary 2.7).

2. The main results

In the following we fix a non-discrete locally compact field (K, φ), where φ is a canonical multiplicative absolute value (see Section 1), we also fix an algebraic closure $\overline{K}$ of K, and a non-constant polynomial,

$$P(x) = c_n x^n + c_{n-1}x^{n-1} + ... + c_0 \in K[x], c_n \neq 0.$$

Lemma 2.1. *With the above notation and assumptions, the function $x \to \varphi(P(x))$, defined on K with values in $\mathbb{R}_+$, has an absolute minimum point x_0 in K, that is there exists x_0 in K such that*

$$\inf_{x\in K} \varphi(P(x)) = \varphi(P(x_0)).$$

Proof. Since

$$\varphi(P(x)) = \varphi(x)^n \varphi\left(c_n + \frac{c_{n-1}}{x} + ... + \frac{c_0}{x^n}\right),$$

we can find $M > 0$ such that $\varphi(P(x)) \geq \varphi(c_0)$ for any $x \in K$ with $\varphi(x) > M$. Since $x \to \varphi(P(x))$ is a continuous function, there exists at least an element $x_0 \in B[0, M] = \{x \in K : \varphi(x) \leq M\}$, so that

$$\inf_{x\in B[0,M]} \varphi(P(x)) = \varphi(P(x_0)),$$

because $B[0, M]$ is a compact subspace in K. Let x be in $K \setminus B[0, M]$. Then,

$$\varphi(P(x)) \geq \varphi(c_0) = \varphi(P(0)) \geq \varphi(P(x_0)).$$

Thus $\varphi(P(x)) \geq \varphi(P(x_0))$ for any $x \in K$. □

Theorem 2.2. *Let (K, φ) be $(\mathbb{R}, |\cdot|^s)$ or $(\mathbb{C}, |\cdot|^s)$, $0 < s \leq 1$, where $|\cdot|$ is the usual absolute value on $\mathbb{R}$ and $\mathbb{C}$ respectively, and let P be a non-constant polynomial of degree n in $K[x]$. Let x_0 be an absolute minimum in K for the continuous function $x \to \varphi(P(x))$, $x \in K$ (see Lemma 2.1), and let*

$$P(x) = a_0 + a_j(x - x_0)^j + \cdots + a_n(x - x_0)^n,\ a_j \neq 0, \tag{2.1}$$

be the Taylor expansion of P at x_0. Then, x_0 is a zero for the equation $P(x) = 0$ if and only if the equation

$$x^j + \frac{a_0}{a_j} = 0 \tag{2.2}$$

has a root in K.

Proof. We see that if $P(x_0) = a_0 = 0$, then the equation (2.2) has the root $x = 0$ in K. Conversely, let us assume now that $a_0 \neq 0$, that is we assume that x_0 is not a root for P, and let $y_0 \in K$ be a root of the equation (2.2). We take a small $\varepsilon > 0$ and we compute

$$P(x_0 + \varepsilon y_0) = a_0 + a_j\varepsilon^j y_0^j + a_{j+1}\varepsilon^{j+1} y_0^{j+1} + \cdots + a_n \varepsilon^n y_0^n.$$

Since $y_0^j = -\frac{a_0}{a_j}$, we can also write:

$$P(x_0 + \varepsilon y_0) = a_0 - a_0\varepsilon^j - a_0\frac{a_{j+1}}{a_j}\varepsilon^{j+1} y_0 - \cdots - a_0\frac{a_n}{a_j}\varepsilon^n y_0^{n-j}.$$

Hence,

$$\varphi(P(x_0+\varepsilon y_0)) = \varphi(a_0)\varphi\left(1-\varepsilon^j-\varepsilon^{j+1}R(\varepsilon)\right),$$

where $R(x)$ is a polynomial in $K[x]$. Since $\varphi(x) = |x|^s$, $0 < s \leq 1$, we see that $\varphi\left(1-\varepsilon^j-\varepsilon^{j+1}R(\varepsilon)\right) < 1$ for ε small enough. Thus $\varphi(P(x_0+\varepsilon y_0)) < \varphi(a_0)$ for such an ε. But this is a contradiction, because $\varphi(a_0) = \varphi(P(x_0))$ and x_0 is an absolute minimum in K for $x \to \varphi(P(x))$, $x \in K$. □

Definition 2.3. *Let (K, φ) be a non-Archimedean locally compact non-discrete field. With the above notation, let x_0 be an absolute minimum point in K for the function $x \to \varphi(P(x))$, $x \in K$. We say that x_0 satisfies property (P) if the coefficients $a_0, a_j, a_{j+1}, \cdots, a_n$ of the polynomial $P(x)$ in its Taylor expansion (2.1) verify the following condition:*

$$\varphi(a_{j+k}) < \frac{\varphi(a_j)^{\frac{j+k}{j}}}{\varphi(a_0)^{\frac{k}{j}}} \tag{2.3}$$

for any $k = 1, 2, \cdots, n-j$. If $a_0 = 0$, we write $\frac{1}{+0} = +\infty$ and, in this case, property (P) is obviously satisfied.

Conjecture C. *If (K, φ) is a non-Archimedean locally compact non-discrete field, then, for any polynomial $P(x) \in K[x]$, the continuous function $x \to \varphi(P(x))$ has at least one absolute minimum point x_0 that satisfies the property (P).*

Theorem 2.4. *Let (K, φ) be a non-Archimedean locally compact non-discrete field, and let P be a non-constant polynomial of degree n in $K[x]$. Let x_0 be an absolute minimum point for the continuous function $x \to \varphi(P(x))$, $x \in K$, that satisfies property (P). Then, x_0 is a root of the equation $P(x) = 0$ if and only if the equation*

$$x^j + \frac{a_0}{a_j} = 0 \tag{2.4}$$

has at least one root in K. Here a_0, a_j are coefficients of P in its Taylor expansion from (2.1)

Proof. If x_0 is a root of P in K, then $a_0 = 0$ and $x = 0$ is a root in K for the equation (2.4). Conversely, we assume that $a_0 = P(x_0) \neq 0$ and $q \in K$ is a root of the equation (2.4), that is

$$q^j = -\frac{a_0}{a_j}. \tag{2.5}$$

Let us come back to (2.1) and compute

$$P(x_0+q) = -a_0\left(\frac{a_{j+1}}{a_j}q + \cdots + \frac{a_n}{a_j}q^{n-j}\right).$$

Thus,

$$\varphi(P(x_0+q)) = \varphi(a_0)\varphi\left(\sum_{k=1}^{n-j}\frac{a_{j+k}}{a_j}q^k\right). \tag{2.6}$$

Since

$$\varphi\left(\sum_{k=1}^{n-j}\frac{a_{j+k}}{a_j}q^k\right) \leq \max_{1\leq k\leq n-j}\left\{\frac{\varphi(a_{j+k})}{\varphi(a_j)}\varphi(q)^k\right\},$$

and since

$$\varphi(q) = \left[\frac{\varphi(a_0)}{\varphi(a_j)}\right]^{\frac{1}{j}}$$

(see (2.5)), we conclude that

$$\frac{\varphi(a_{j+k})}{\varphi(a_j)}\varphi(q)^k = \frac{\varphi(a_{j+k})\varphi(a_0)^{\frac{k}{j}}}{\varphi(a_j)^{\frac{j+k}{j}}} < 1$$

for any $k = 1, 2, \cdots, n-j$ (see (2.3)). Hence,

$$\varphi(P(x_0+q)) < \varphi(a_0) = \varphi(P(x_0)),$$

a contradiction, because x_0 is an absolute minimum point in K for $x \to \varphi(P(x))$, $x \in K$. Therefore, if the equation (2.5) has a root q in K, then $a_0 = P(x_0) = 0$. □

Corollary 2.5. (*The Fundamental Theorem of Algebra*) *The complex number field* $\mathbb{C}$ *is algebraically closed.*

Proof. (See also [1] for a similar idea.) We directly apply Theorem 2.2 and the fact that the equation $X^j + a_0/a_j = 0$ always has all its solutions in $\mathbb{C}$, because $\exp(z) \in \mathbb{C}$ if $z \in \mathbb{C}$ and radicals of positive real numbers are also real numbers. □

Let K be an arbitrary field. We say that R/K is a *finite radical extension* of K if $R = L_s$, so that

$$K = L_0 \subset L_1 \subset \ldots \subset L_s$$

is a tower of fields, such that $L_i = L_{i-1}(\theta_i)$, with θ_i a root of an equation of the following type,

$$X^{n_i} + b_{i-1} = 0,\ b_{i-1} \in L_{i-1},\ i = 1, 2, \ldots, s.$$

We say that a polynomial $P \in K[X]$ is *solvable* (*by radicals*) if its decomposition field L_P is contained in a finite radical extension R of K.

Corollary 2.6. *Now we assume that Conjecture C is true. Let* (K, φ) *be a non-Archimedean, non-discrete locally compact field, and let* $P(X) = 0$ *be a polynomial equation with* $P \in K[X]$*. Then* $P(X) = 0$ *has all its solutions in a finite radical extension* $R \subset \overline{K}$*, where* $\overline{K}$ *is a fixed algebraic closure of* K*. Thus,* P *is solvable by radicals.*

Proof. We can assume that $n = \deg_K P \geq 2$. Since (K, φ) is a complete field, φ can be uniquely extended to $\overline{K}$ by an absolute value $\overline{\varphi}$. In what follows we will denote $\overline{\varphi}$ also by φ.

Let us write again formula (2.1) in which we highlight an arbitrary finite extension T of K in which $P(X)$ can be written as

$$P(X) = a_0^{(T)} + a_{j_T}^{(T)}(X - x_0^{(T)})^{j_T} + a_{j_T+1}^{(T)}(X - x_0^{(T)})^{j_T+1} + \ldots + a_n^{(T)}(X - x_0^{(T)})^n, \tag{2.7}$$

where $x_0^{(T)}$ is an arbitrary element in T and $a_{j_T}^{(T)} \neq 0$. We choose now $x_0^{(T)} \in T$ with property (P), such that

$$\inf_{x \in T} \varphi(P(x)) = \varphi(P(x_0^{(T)})) \tag{2.8}$$

(see Lemma 2.1 and Conjecture C).

If $P(x_0^{(K)}) = 0$ (here $T = K$), we are done. Let us assume that $P(x_0^{(K)}) \neq 0$ and so, from Theorem 2.4 (we use here the fact that Conjecture C is true), we see that any root α_{j_K} of the equation $X^{j_K} + a_0^{(K)}/a_{j_K}^{(K)} = 0$ is not in $K =: K_0$. Let us take such a root $\alpha_{j_{K_0}} \in \overline{K}$, and let us consider $K_1 = K_0[\alpha_{j_{K_0}}]$, and

$$\inf_{x \in K_1} \varphi(P(x)) = \varphi(P(x_0^{(K_1)}))$$

for an element $x_0^{(K_1)} \in K_1$, which satisfies property (P). Thus,

$$\varphi(P(x_0^{(K_1)})) \leq \varphi(P(x_0^{(K_0)})).$$

If $P(x_0^{(K_1)}) = 0$, we are done. If not, we construct $K_2 = K_1[\alpha_{j_{K_1}}]$, where $\alpha_{j_{K_1}}$ is a fixed root of the equation $X^{j_{K_1}} + a_0^{(K_1)}/a_{j_{K_1}}^{(K_1)} = 0$, and so on. We see that

$$K = K_0 \subset K_1 \subset \ldots \subset K_m \subset \ldots \subset \overline{K} \tag{2.9}$$

and

$$\varphi(P(x_0^{(K_0)})) \geq \varphi(P(x_0^{(K_1)})) \geq \ldots \geq \varphi(P(x_0^{(K_m)})) \geq \ldots, \tag{2.10}$$

where $x_0^{(K_j)} \in K_j$ is an absolute minimum point of $x \to \varphi(P(x))$, $x \in K_j$, which satisfies property (P) for $j = 0, 1, \ldots$ Let $\alpha_{j_{K_s}} \in \overline{K}$ be the above fixed solution of the equation

$$X^{j_{K_s}} + a_0^{(K_s)}/\alpha_{j_{K_s}}^{(K_s)} = 0, \;\; s \in \{0, 1, \ldots\}. \tag{2.11}$$

Since $P'(x_0^{(K_h)}) = 0$ for any $h \in \{0, 1, \ldots\}$ where "we are not done" ($j_{K_h} > 1$, otherwise the equation (2.11) has a solution in K_h, and so, "we are done"), we see that there exists a smallest k such that

$$\varphi(P(x_0^{(K_k)})) = \varphi(P(x_0^{(K_{k+1})})) = \ldots = \varphi(P(x_0^{(K_m)})) = \ldots$$

Since $\varphi(P(x_0^{(K_{k+1})})) = \varphi(P(x_0^{(K_k)}))$, the absolute minimum point $x_0^{(K_{k+1})} \in K_{k+1}$, which satisfies property (P), can be taken to be $x_0^{(K_k)} \in K_k$, so we can write, instead of the equation,

$$P(X) = a_0^{(K_{k+1})} + a_{j_{K_{k+1}}}^{(K_{k+1})}(X - x_0^{(K_{k+1})})^{j_{K_{k+1}}} + \ldots,$$

the previous equation:

$$P(X) = a_0^{(K_k)} + a_{j_{K_k}}^{(K_k)}(X - x_0^{(K_k)})^{j_{K_k}} + \ldots,$$

so $\alpha_{j_{K_{k+1}}}^{(K_{k+1})}$ can be taken to be $\alpha_{j_{K_k}}^{(K_k)} \in K_k\left[\alpha_{j_{K_k}}^{(K_k)}\right] = K_{k+1}$ and again, applying Theorem 2.4 (we use here the fact that Conjecture C is true), we get that $P(x_0^{(K_k)}) = P(x_0^{(K_{k+1})}) = 0$. Thus, the equation $P(X) = 0$ has a solution in the radical extension K_k/K. Hence, we can write $P(X) = (X - x_0^{(K_k)})Q(X)$ with $Q \in K_k[X]$, and we continue in the same way as above with Q instead of P and K_k instead of K, etc. Therefore, the equation $P(X) = 0$ has all its solutions in a finite radical extension R/K. □

Corollary 2.7. (*see also* [5], *IV, Corollary 5 with an alternative proof*) *We also assume here that Conjecture C is true. Let* (K, φ) *be a non-discrete locally compact field and let* P *be a non-constant polynomial in* $K[X]$*. Let* $L_P \subset \overline{K}$ *be the decomposition field of* P*. Then* $Gal(L_P/K)$ *is a solvable group.*

Proof. We reconsider the tower of subfields (2.9),

$$K = K_0 \subset K_1 \subset \ldots \subset K_k \subset \overline{K}$$

such that $K_{j+1} = K_j[\alpha_{j_{K_j}}]$, where $\alpha_{j_{K_j}}$ is a root of the equation

$$X^{j_{K_j}} + a_0^{(K_j)}/a_{j_{K_j}} = 0$$

and $a_0^{(K_j)}$, $a_{j_{K_j}} \neq 0$, with $j_{K_j} > 1$, are the first two coefficients in the expansion (2.7) of $P(x)$ for $T = K_j$. We also assume that K_k contains all the roots of P (see the proof of Corollary 2.6). Let us denote by U_m the set of all the m-th roots of unity in $\overline{K}$ and denote by $L_{-1} = K_0 = K$, $L_0 = K_0[U_{j_{K_0}}]$, $L_1 = L_0[\alpha_{j_{K_0}}]$, $L_2 = L_1[U_{j_{K_1}}]$, $L_3 = L_2[\alpha_{j_{K_1}}], \ldots$ We see that

$$L_{-1} \subset L_0 \subset L_1 \subset \ldots \subset L_{2k+1},$$

where L_j/K is a normal extension and, from [3], Theorem 6.2, we see that

$Gal(L_j/L_{j-1})$ is a cyclic group for any $j = 0, 1, ..., 2k+1$. Thus $Gal(L_{2k+1}/K)$ is a solvable group. Moreover, the decomposition field L_P of P is contained in L_{2k+1}. Since

$$Gal(L_P/K) \simeq Gal(L_{2k+1}/K)/Gal(L_{2k+1}/L_P)$$

and since $Gal(L_{2k+1}/K)$ is a solvable group, we see that $Gal(L_P/K)$ is also a solvable group. □

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