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A QUATERNIONIC PRODUCT OF PLANES IN $\mathbb{E}^3$

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Abstract. This note introduces a product of Euclidean planes inspired by the product of quaternions. A technical condition is necessary for the existence of this product and some examples (squares of planes, the coordinates planes, involutions) are discussed.

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1. The quaternionic product of distingusihed planes

Fix the set of all planes in the Euclidean 3-dimensional space i.e.:

$$\mathcal{P} := \{\pi : Ax + By + Cz + D = 0; A^2 + B^2 + C^2 > 0\},$$

then a plane π is given by $\pi = \pi(D, \overrightarrow{N})$ with the normal vector $\overrightarrow{N} = (A, B, C) \in \mathbb{R}^3 \setminus \{\overrightarrow{0}\}$ and $D \in \mathbb{R}$. The aim of this work is to introduce a product in $\mathcal{P}$ and hence the starting point of this paper is the identification of the given plane $\pi = \pi(D, \overrightarrow{N}) \in \mathbb{R}^4 \setminus \{(0, 0, 0, D); D \in \mathbb{R}\}$ with the quaternion *in a projective manner*:

$$q(\pi) := D + A\bar{i} + B\bar{j} + C\bar{k} = (D, A, B, C) \in \mathbb{H} = \mathbb{R}^4. \tag{1.1}$$

Here $\bar{i}$, $\bar{j}$ and $\bar{k}$ are the usual complex units of the quaternionic algebra i.e.:

$$\bar{i} = (0, 1, 0, 0), \quad \bar{j} = (0, 0, 1, 0), \quad \bar{k} = (0, 0, 0, 1).$$

Since:

$$D = dist_{\mathbb{E}^3}(O, \pi) \cdot \|\overrightarrow{N}\|_{\mathbb{E}^3}, \quad \|\overrightarrow{N}\|_{\mathbb{E}^3} > 0, \tag{1.2}$$

the quaternion $q(\pi)$ is pure imaginary if and only if the origin $O \in \pi$; such a plane π belongs to the Grassmannian manifold $Gr(2; 3) = O(3)/(O(2) \times O(1))$ of 2-dimensional subspaces of $\mathbb{R}^3$. We point out that although there are alternative ways to associate a quaternion to a given plane we choose the expression (1.1) according to our previous studies, namely (in the chronological order) [4], [2], [3] and [5]; remark that in all these previous studies the coefficient of $\bar{k}$ is 1.

From the real algebra structure of the quaternions ([6, p. 89]) it follows the product of two associated quaternions:

$$q(\pi_1)\cdot q(\pi_2) = (D_1D_2 - A_1A_2 - B_1B_2 - C_1C_2) + (A_1D_2 + A_2D_1 + B_1C_2 - B_2C_1)\bar{\imath}+$$

$$+(B_1D_2+B_2D_1+C_1A_2-C_2A_1)\bar{\jmath}+(C_1D_2+C_2D_1+A_1B_2-A_2B_1)\bar{k} =: \hat{D}+\hat{A}\bar{\imath}+\hat{B}\bar{\jmath}+\hat{C}\bar{k} \quad (1.3)$$

and hence we suggest the following definition:

Definition 1.1. The given pair of planes is called *q-distinguished* if*:*

$$\hat{A}^2 + \hat{B}^2 + \hat{C}^2 = |q(\pi_1)\cdot q(\pi_2)|^2 - \hat{D}^2 = |q(\pi_1)|^2|q(\pi_2)|^2 - \hat{D}^2 > 0. \quad (1.4)$$

Example 1.2. For a fixed plane $\pi(A, B, C, D)$ we have the square:

$$q(\pi)\cdot q(\pi) = (D^2 - A^2 - B^2 - C^2) + (2AD)\bar{\imath} + (2BD)\bar{\jmath} + (2CD)\bar{k}. \quad (1.5)$$

Then the technical condition (1.4) is satisfied if and only if $O \notin \pi$ and then the pair (π, π) is q-distinguished; also $q(\pi)\cdot q(\pi) \in \mathbb{R}^4$ is a purely imaginary quaternion only for $D_\pm = \pm\|\overrightarrow{N}\|_{\mathbb{E}^3}$. $\Box$

Let now $\mathcal{P}^2(q)$ be the set of q-distinguished pairs of planes; it follows a *quaternionic product* in $\mathcal{P}^2(q)$:

$$\pi_1 \odot_q \pi_2 := q^{-1}(q(\pi_1)\cdot q(\pi_2)) = \pi(\hat{A}, \hat{B}, \hat{C}, \hat{D}). \quad (1.6)$$

An important tool of the quaternionic theory is that of *conjugate*, which for our quaternion (1.1) means:

$$\overline{q(\pi)} := D - A\bar{\imath} - B\bar{\jmath} - C\bar{k} = (D, -A, -B, -C) = -q(-D, \overrightarrow{N}) \quad (1.7)$$

and the same projective way of thinking allows the definition of *the conjugate plane*: $\overline{\pi} = \pi(A, B, C, -D)$; obviously the conjugation is an involution map. The pair of parallel planes $(\pi, \overline{\pi})$ is not q-distinguished. The real part D of the quaternion (1.1) is the Euclidean scalar product in $\mathbb{E}^4$ of the vectors $q(\pi_1)$ and $\overline{q(\pi_2)}$.

2. Concrete examples

In the following we study this new product introduced in (1.6) through four large examples.

Example 2.1. Revisiting the example 1.2 (recall that $D \neq 0$) we have immediately the square of a plane π not containing O:

$$\pi^2_{\odot_q} : Ax + By + Cz + \frac{D^2 - A^2 - B^2 - C^2}{2D} = 0 \to \pi^2_{\odot_q} \neq \pi, \quad \pi^2_{\odot_q} \parallel \pi. \quad (2.1)$$

The expression above suggests as remarkable example the case of a cuboid $\Box : D^2 = A^2+B^2+C^2$, which gives the associated planes:

$$\begin{cases} \pi_{(\Box,+)} : Ax + By + Cz + \sqrt{A^2 + B^2 + C^2} = 0, \quad A > 0, \quad B > 0, \quad C > 0, \\ (\pi_{(\Box,+)})^2_{\odot_q} : Ax + By + Cz = 0. \end{cases} \quad (2.3)$$

We note that the quaternion associated to the cuboid can be expressed in the form suggested in [7, p. 138]:

$$q(\Box) = 1 + \mu, \quad \mu \in S^2 \subset \mathbb{R}^3$$

and hence, μ is a solution in $\mathbb{H}$ of the equation $q^2 = -1 \in \mathbb{R}$. Here, S^2 is the unit sphere of the 3-dimensional Euclidean space $\mathbb{E}^3$ and μ is the quaternionic version of the vector $\overrightarrow{N}$:

$$\mu = \frac{1}{\sqrt{A^2+B^2+C^2}}(A\bar{i} + B\bar{j} + C\bar{k}).$$

A particular case of cuboid $\square$ is provided by the case when (A, B, C, D) is a *Pythagorean quadruple* i.e. these are natural (strictly positive) numbers and hence we know its parametrization (provided exactly by the quaternion algebra):

$$\begin{cases} A := m^2 + n^2 - p^2 - q^2, B := 2(mq + np), C := 2(nq - mp), D := m^2 + n^2 + p^2 + q^2, \\ 0 < m, n, p, q \in \mathbb{N}, \quad 1 = gcd(m, n, p, q), \quad m + n + p + q = odd. \end{cases} \tag{2.4}$$

For a concrete example we choose the minimal Pythagorean quadruple ($A = 1, B = C = 2, D = 3, m = n = q = 1, p = 0$) with the associated planes:

$$\begin{cases} \pi(minimal) : x + 2y + 2z + 3 = 0, \quad (\pi(minimal))^2_{\odot_q} : x + 2y + 2z = 0, \\ \overrightarrow{N}(minimal) = \frac{1}{3}(1, 2, 2) \in S^2. \end{cases} \tag{2.5}$$

A trigonometrical generalization of $\pi(minimal)$ uses two angles $(\varphi, \theta) \in [0, 2\pi) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ as usual parameters for S^2:

$$\pi(\varphi, \theta) : (\cos\varphi \cos\theta)x + (\cos\varphi \sin\theta)y + (\sin\varphi)z + 1 = 0. \tag{2.6}$$

Example 2.2. In this example we will perform the quaternionic product of two different q-distinguished planes. The coordinates planes are so since:

$$q(xOy) = \bar{k}, \quad q(yOz) = \bar{i}, \quad q(zOx) = \bar{j} \tag{2.7}$$

and then we have:

$$\begin{cases} xOy \odot_q yOz = yOz \odot_q xOy = zOx, \\ yOz \odot_q zOx = zOx \odot_q yOz = xOy, \\ zOx \odot_q xOy = xOy \odot_q zOx = yOz. \end{cases} \tag{2.8}$$

although, generally speaking, the quaternionic product is not commutative (see the following example); we have only $Re(q(\pi_1) \cdot q(\pi_2)) = Re(q(\pi_2) \cdot q(\pi_1))$.

Example 2.3. The previous example suggests the case of two distinct planes intersecting through a line containing O. The intersection condition:

$$rang \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2 \tag{2.9}$$

means exactly the condition (1.4). Hence, (π_1, π_2) is a q-distinguished pair with:

$$q(\pi_1) \cdot q(\pi_2) = -\langle \overrightarrow{N}_1, \overrightarrow{N}_2 \rangle_{\mathbb{E}^3} + (B_1C_2 - B_2C_1)\bar{i} + (C_1A_2 - C_2A_1)\bar{j} + (A_1B_2 - A_2B_1)\bar{k}. \tag{2.10}$$

In particular, if the planes are orthogonal then $q(\pi_1) \cdot q(\pi_2)$ is a pure quaternion.

As concrete example let us consider the bisectrix line $l : x = y = z$ which is the intersection of the planes $\pi_1 : x - y = 0$, $\pi_2 : x - z = 0$. Since:

$$q(\pi_1) \cdot q(\pi_2) = (\bar{i} - \bar{j})(\bar{i} - \bar{k}) = -1 + \bar{j} + \bar{k} + \bar{i}, \quad q(\pi_2) \cdot q(\pi_1) = (\bar{i} - \bar{k})(\bar{i} - \bar{j}) = -1 - \bar{j} - \bar{k} - \bar{i} \tag{2.11}$$

it results:

$$\pi_1 \odot_q \pi_2 : x + y + z - 1 = 0, \quad \pi_2 \odot_q \pi_1 : x + y + z + 1 = 0, \quad \pi_1 \odot_q \pi_2 \parallel \pi_2 \odot_q \pi_1 = 0. \tag{2.12}$$

Example 2.4. A main result of the paper [7] is that the real algebra $\mathbb{H}$ admits an infinite number of involutions. Apart from the conjugation map, discussed in the first section, there are another three remarkable involutions discussed in the cited paper and which in our setting gives three associated planes to the initial one π:

$$\begin{cases} \alpha(\pi): Ax - By - Cz + D = 0, \\ \beta(\pi): -Ax + By - Cz + D = 0, \\ \gamma(\pi): -Ax - By + Cz + D = 0. \end{cases} \tag{2.13}$$

It follows immediately:

Proposition 2.5. *The conditions of q-distinguished pair are as follows:*
i) for $(\pi, \alpha(\pi))$*:* $A \neq 0$ *and* $B^2 + C^2 + D^2 > 0$*,*
ii) for $(\pi, \beta(\pi))$*:* $B \neq 0$ *and* $C^2 + A^2 + D^2 > 0$*,*
iii) for $(\pi, \gamma(\pi))$*:* $C \neq 0$ *and* $A^2 + B^2 + D^2 > 0$*.*

Proof. We compute the quaternionic product for each pair:
i) $q(\pi) \cdot q(\alpha(\pi)) = (D^2 - A^2 + B^2 + C^2) + (2AD)\bar{i} + (2AC)\bar{j} + (-2AB)\bar{k}$,
ii) $q(\pi) \cdot q(\alpha(\pi)) = (D^2 + A^2 - B^2 + C^2) + (-2BC)\bar{i} + (2BD)\bar{j} + (2AB)\bar{k}$,
iii) $q(\pi) \cdot q(\alpha(\pi)) = (D^2 + A^2 + B^2 - C^2) + (2BC)\bar{i} + (-2AC)\bar{j} + (2CD)\bar{k}$. □

Hence:
a) if the condition *i)* holds then there exists the plane (not containing O):

$$\tilde{\alpha}(\pi): Bx + Cy + Dz + A = 0$$

b) if the condition *ii)* is satisfied then there exists the plane (not containing O):

$$\tilde{\beta}(\pi): Cx + Ay + Dz + B = 0$$

c) if the condition *iii)* holds then there exists the plane (not containing O):

$$\tilde{\gamma}(\pi): Ax + By + Dz + C = 0$$

3. A matrix approach to the quaternionic product

In this section we use the tools of matrices in our study by following the method of [8]. Namely, the quaternion (1.1) is replace with a 2×2 complex matrix:

$$M(\pi) := \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in M_2(\mathbb{C}), \quad a := D + A\bar{i}, \quad b := C + D\bar{i}. \tag{3.1}$$

The motivation for this choice is the expression of $q \in \mathbb{H}$ as $q := a + b\bar{j}$ as well as the equalities:

$$\det M(\pi) = |q(\pi)|^2, \quad M(\overline{\pi}) := \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix}. \tag{3.2}$$

Moreover, the quaternionic product is expressed by the product of associated matrices and the eigenvalues of $M(\pi)$ are:

$$\lambda_{\pm}(\pi) = D \pm \|\overrightarrow{N}\|_{\mathbb{E}^3}\bar{i}. \tag{3.3}$$

For our framework the condition (1.4) says the fact that the eigenvalues of the product $M(\pi_1) \cdot M(\pi_2)$ are not reducible to real numbers; this means also $\lambda_+ \neq \lambda_-$.

Example 3.1. We treat now the case of the minimal plane from the example 2.1:

$$\pi(minimal) : \frac{x}{3} + \frac{2y}{3} + \frac{2z}{3} + 1 = 0. \tag{3.4}$$

Its associated matrix as well as the corresponding diagonal form are:

$$\begin{cases} M(minimal) = \begin{pmatrix} 1+\frac{i}{3} & \frac{2}{3}(1+i) \\ -\frac{2}{3}(1-i) & 1-\frac{i}{3} \end{pmatrix} = S \cdot D \cdot S^{-1}, \quad S = \frac{1+i}{2}\begin{pmatrix} i & -2i \\ 1-i & 1-i \end{pmatrix}, \\ D = (1+i)\begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix}, \quad S^{-1} = \frac{1}{3}\begin{pmatrix} -(1+i) & 2 \\ 1+i & 1 \end{pmatrix}. \end{cases} \tag{3.5}$$

Also, the trace and the determinant are equal: $TrM(minimal) = 2 = \det M(minimal)$. We point out also the eigenvectors:

$$\lambda_1 = 1 - i \to v_- = \left(\frac{1}{2}(-1+i), 1\right), \quad \lambda_+ = 1 + i \to v_+ = (1-i, 1) \tag{3.6}$$

which are chosen in that form in order to be orthogonal with respect to *the Hermitian inner product* of $\mathbb{C}^2$:

$$\langle z, w\rangle_{\mathbb{C}^2} := z^1\overline{w^1} + z^2\overline{w^2}, \quad z = (z^1, z^2) \in \mathbb{C}^2, \quad w = (w^1, w^2) \in \mathbb{C}^2. \tag{3.7}$$

A direct computation yields the square:

$$(M(minimal))^2 = \frac{2}{3}\begin{pmatrix} i & 2(1+i) \\ -2(1-i) & -i \end{pmatrix} \tag{3.8}$$

and the first line of this matrix recast the square $(\pi(minimal))^2_{\odot_q} : x + 2y + 2z = 0$ from the relation (2.5).

Returning to the matrix $M(minimal)$ since its determinant is 2 it results that the following matrix (though as a homothetic transformation of $M(minimal)$) is *special unitary* with the trace $\sqrt{2}$:

$$\begin{cases} \frac{1}{\sqrt{2}}M(minimal) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1+\frac{i}{3} & \frac{2}{3}(1+i) \\ -\frac{2}{3}(1-i) & 1-\frac{i}{3} \end{pmatrix} \in SU(2), \\ \lambda_1 = \frac{1-i}{\sqrt{2}} = e^{-\frac{\pi}{4}i} \to v_- = (-1+i, 2), \quad \lambda_2 = \frac{1+i}{\sqrt{2}} = e^{\frac{\pi}{4}i} \to v_+ = (1-i, 1). \end{cases} \tag{3.9}$$

The element $(1, 1)$ in the matrix above can be written in the exponential-trigonometric form:

$$\frac{1}{\sqrt{2}}\left(1 + \frac{i}{3}\right) = \frac{\sqrt{5}}{3}e^{i\psi}, \quad \cos\psi = \frac{3}{\sqrt{10}}, \quad \sin\psi = \frac{1}{\sqrt{10}}. \tag{3.10}$$

In fact, $\psi \cong 18.43°$.

Recently, we use the adjoint representation of the Lie group $SU(2)$ to produces a special class of conics in [1].

Example 3.2. In the very recent paper [3] we associate to the plane line $l : ax + by + c = 0$ (with obviously $a^2 + b^2 > 0$) the quaternion $q(l) := \bar{k} + a\bar{i} + b\bar{j} + c$. Hence, the minimal plane gives the quaternion $q(minimal) = \bar{k} + \frac{\bar{i}}{2} + \bar{k} + \frac{3}{2}$ which in turn yields the line:

$$l(minimal) : x + 2y + 3 = 0. \tag{3.11}$$

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