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ON (mI, nJ)-CONTINUOUS MULTIFUNCTIONS

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Abstract. Let $mIO(X)$ be the family of $\star$-open (resp. α-I-open, pre-I-open, semi-I-open, β-I-open, etc.) sets in an ideal topological space (X, τ, I). By using $mIO(X)$, we introduce and investigate the notion of an (mI, nJ)-continuous multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$. As a special case of (mI, nJ)-continuous multifunctions, we obtain the notion of $\star(\alpha)$-continuous multifunctions due to Boonpok [3].

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Key words: minimal structure, ideal topological space, mI-open, $i^\star$-continuous, (mI, nJ)-continuous, weakly (mI, nJ)-continuous, multifunction.

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1. Introduction

Generalizations of open sets in a topological space: semi-open sets, pre-open sets, α-open sets, b-open sets and β-open sets play an important role in the research of generalizations of continuity for functions and multifunctions. By using these sets, various generalizations of continuous multifunctions are introduced and investigated.

The notions of minimal structures, m-spaces, m-continuity, M-continuity are introduced and investigated in [11, 14, 15]. By using these notions, the present authors obtained some unified theory of continuity for multifunctions [13, 16].

The notion of ideal topological spaces was introduced in [10, 19]. As generalizations of open sets in an ideal topological space, several authors introduced the notions of semi-I-open sets, pre-I-open sets, α-I-open sets, b-I-open sets and β-I-open sets. Quite recently, the notions of $i^\star$-continuity [4] and $\star(\alpha)$-continuity [3] for multifunctions have been introduced and some characterizations of the multifunctions have been obtained.

In this paper, let $mIO(X)$ be the family of $\star$-open (resp. α-I-open, pre-I-open, semi-I-open, β-I-open, etc.) sets in an ideal topological space (X, τ, I). By using $mIO(X)$, we introduce the notion of an (mI, nJ)-continuous multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$. As special cases of (mI, nJ)-continuous multifunctions, we obtain the notions of $i^\star$-continuous multifunctions [4] and $\star(\alpha)$-continuous multifunctions [3].

Throughout the present paper, spaces (X, τ) and (Y, σ) always mean topological spaces and $F : (X, \tau) \to (Y, \sigma)$ presents a multivalued function. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverses of a subset B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is,

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and } F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

Let $exp(Y)$ be the collection of all nonempty subsets of Y. For any open set V of Y, we denote $V^+ = \{B \in exp(Y) : B \subset V\}$ and $V^- = \{B \in exp(Y) : B \cap V \neq \emptyset\}$ [18].

2. Preliminaries

Definition 2.1. A subfamily m of the power set $exp(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) on X [14] if $\emptyset \in m$ and $X \in m$. Each member of m is said to be *m-open* and the complement of an m-open set is said to be *m-closed*. A set X with an m-structure m is called an m-space and is denoted by (X, m)

Definition 2.2. Let X be a nonempty set and m an m-structure on X. For a subset A of X, the *m-closure* of A and the *m-interior* of A are defined in [11] as follows:
(1) $\mathrm{mCl}(A) = \cap\{F : A \subset F, X - F \in m\}$,
(2) $\mathrm{mInt}(A) = \cup\{U : U \subset A, U \in m\}$.

Lemma 2.3. [11] *Let (X, m) be an m-space. For subsets A and B of X, the following properties hold:*
(1) $\mathrm{mCl}(X \setminus A) = X \setminus \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X \setminus A) = X \setminus \mathrm{mCl}(A)$,
(2) If $(X \setminus A) \in m$, *then* $\mathrm{mCl}(A) = A$ *and if* $A \in m$, *then* $\mathrm{mInt}(A) = A$,
(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,
(4) If $A \subset B$, *then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,
(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,
(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

Definition 2.4. An m-structure m on a nonempty set X is said to have *property* **B** [11] if the union of any family of subsets belonging to m belongs to m.

Lemma 2.5. [17] *Let X be a nonempty set and m an m-structure with property* **B**. *Then, the following properties are hold:*
(1) $\mathrm{mInt}(A) = A$ *if and only if* $A \in m$,
(2) $\mathrm{mCl}(A) = A$ *if and only if A is m-closed,*
(3) $\mathrm{mInt}(A) \in m$ *and* $\mathrm{mCl}(A)$ *is m-closed.*

3. (m, n)-continuous multifunctions

Definition 3.1. A multifunction $F : (X, m) \to (Y, n)$ is said to be *(m, n)-continuous* at $x \in X$ if for each n-open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists $U \in m$ containing x such that $F(u) \in V_1^+ \cap V_2^-$ for every $u \in U$. $F : (X, m) \to (Y, n)$ is said to be *(m, n)-continuous* if it has the property at each point of X.

Theorem 3.2. *A multifunction $F : (X, m) \to (Y, n)$ is (m, n)-continuous at $x \in X$ if and only if for every n-open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, $x \in \mathrm{mInt}(F^+(V_1) \cap F^-(V_2))$.*

Proof. Necessity. Let V_1, V_2 be any n-open sets of (Y, n) such that $F(x) \in V_1^+ \cap V_2^-$. Then there exists $U \in m$ containing x such that $F(U) \subset V_1^+ \cap V_2^-$. Since U is an m-open set, $x \in U \subset \mathrm{mInt}(F^+(V_1) \cap F^-(V_2))$.

Sufficiency. Let V_1, V_2 be any n-open sets such that $F(x) \in V_1^+ \cap V_2^-$. Then we have $x \in \mathrm{mInt}(F^+(V_1) \cap F^-(V_2))$.Therefore, there exists $U \in m$ such that $x \in U \subset F^+(V_1) \cap F^-(V_2)$. Therefore, $F(u) \in V_1^+ \cap V_2^-$ for every $u \in U$. Hence F is (m, n)-continuous at x. □

Theorem 3.3. *For a multifunction $F : (X, m) \to (Y, n)$, where n has property* **B**, *the following properties are equivalent:*

(1) F is (m, n)-continuous;

(2) $F^+(V_1) \cap F^-(V_2) = \mathrm{mInt}(F^+(V_1) \cap F^-(V_2))$ *for every n-open sets V_1, V_2 of Y;*

(3) $F^-(K_1) \cup F^+(K_2) = \mathrm{mCl}(F^-(K_1) \cup F^+(K_2))$ *for every n-closed sets K_1, K_2 of Y;*

(4) $\mathrm{mCl}(F^-(B_1) \cup F^+(B_2)) \subset F^-(\mathrm{nCl}(B_1)) \cup F^+(\mathrm{nCl}(B_2))$ *for every subsets B_1, B_2 of Y;*

(5) $F^-(\mathrm{nInt}(B_1)) \cap F^+(\mathrm{nInt}(B_2)) \subset \mathrm{mInt}(F^-(B_1) \cap F^+(B_2))$ *for every subsets B_1, B_2 of Y.*

Proof. (1) ⇒ (2): Let V_1, V_2 be any n-open sets in Y and $x \in F^+(V_1) \cap F^-(V_2)$. Then $F(x) \in V_1^+ \cap V_2^-$. By Theorem 3.2, $x \in \mathrm{mInt}(F^+(V_1) \cap F^-(V_2))$. Therefore, we have $F^+(V_1) \cap F^-(V_2) \subset \mathrm{mInt}(F^+(V_1) \cap F^-(V_2))$. By Definition 2.2, $\mathrm{mInt}(F^+(V_1) \cap F^-(V_2)) \subset F^+(V_1) \cap F^-(V_2)$ and hence $F^+(V_1) \cap F^-(V_2) = \mathrm{mInt}(F^+(V_1) \cap F^-(V_2))$.

(2) ⇒ (3): This easily follows from Lemma 2.3 and the fact that $F^-(Y \setminus B) = X \setminus F^+(B)$ and $F^+(Y \setminus B) = X \setminus F^-(B)$.

(3) ⇒ (4): Let B_1, B_2 be any subsets of Y. Since n has property **B**, $\mathrm{nCl}(B_1)$ and $\mathrm{nCl}(B_2)$ are n-closed sets in Y. Thus, by (3) and Lemma 2.3 we obtain $\mathrm{mCl}(F^-(B_1) \cup F^+(B_2)) \subset \mathrm{mCl}(F^-(\mathrm{nCl}(B_1)) \cup F^+(\mathrm{nCl}(B_2))) = F^-(\mathrm{nCl}(B_1)) \cup F^+(\mathrm{nCl}(B_2))$.

(4) ⇒ (5): Let B_1, B_2 be any subsets of Y. By (4) and Lemma 2.3, we have

$$X \setminus \mathrm{mInt}(F^-(B_1) \cap F^+(B_2)) = \mathrm{mCl}(X \setminus (F^-(B_1) \cap F^+(B_2))) = \mathrm{mCl}((X \setminus F^-(B_1)) \cup (X \setminus F^+(B_2)) = \mathrm{mCl}(F^+(Y \setminus B_1) \cup F^-(Y \setminus B_2)) \subset F^+(\mathrm{nCl}(Y \setminus B_1) \cup F^-(\mathrm{nCl}(Y \setminus B_2)) = (X \setminus F^-(\mathrm{nInt}(B_1)))) \cup (X \setminus F^+(\mathrm{nInt}(B_2))) = X \setminus (F^-(\mathrm{nInt}(B_1)) \cap F^+(\mathrm{nInt}(B_2))).$$

Therefore, we obtain $F^-(\mathrm{nInt}(B_1)) \cap F^+(\mathrm{nInt}(B_2)) \subset \mathrm{mInt}(F^-(B_1) \cap F^+(B_2))$.

(5) ⇒ (1): Let $x \in X$ and V_1, V_2 be any n-open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. Then $x \in F^+(V_1) \cap F^-(V_2) = F^+(\mathrm{nInt}(V_1) \cap F^-(\mathrm{nInt}(V_2) \subset \mathrm{mInt}(F^+(V_1) \cap F^-(V_2))$. By Theorem 3.2, F is (m, n)-continuous at x. □

Definition 3.4. A multifunction $F : (X, m) \to (Y, n)$ is said to be *weakly (m, n)-continuous* at a point $x \in X$ if for each n-open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists $U \in m$ containing x such that $F(u) \in (\mathrm{nCl}(V_1))^+ \cap (\mathrm{nCl}(V_2))^-$ for every $u \in U$. If F is weakly (m, n)-continuous at every point of $x \in X$, then F is said to be *weakly (m, n)-continuous.*

Theorem 3.5. *For a multifunction $F : (X, m) \to (Y, n)$, where n has property* **B**, *the following properties are equivalent:*

(1) F is weakly (m, n)-continuous;

(2) $F^+(G_1) \cap F^-(G_2) \subset \mathrm{mInt}(F^+(\mathrm{nCl}(G_1)) \cap F^-(\mathrm{nCl}(G_2)))$ *for every n-open sets G_1, G_2 of Y;*

(3) $\mathrm{mCl}(F^-(\mathrm{nInt}(K_1)) \cup F^+(\mathrm{nInt}(K_2))) \subset F^-(K_1) \cup F^+(K_2)$ *for every n-closed sets K_1, K_2 of Y;*

(4) $\mathrm{mCl}(F^-(\mathrm{nInt}(\mathrm{nCl}(B_1))) \cup F^+(\mathrm{nInt}(\mathrm{nCl}(B_2)))) \subset F^-(\mathrm{nCl}(B_1)) \cup F^+(\mathrm{nCl}(B_2))$ *for every subsets B_1, B_2 of Y;*

(5) $F^+(\mathrm{nInt}(B_1)) \cap F^-(\mathrm{nInt}(B_2)) \subset \mathrm{mInt}(F^+(\mathrm{nCl}(B_1)) \cap F^-(\mathrm{nCl}(B_2))))$ *for every subsets B_1, B_2 of Y;*

(6) $\mathrm{mCl}(F^-(G_1) \cup F^+(G_2)) \subset F^-(\mathrm{nCl}(G_1)) \cup F^+(\mathrm{nCl}(G_2))$ *for every n-open sets G_1, G_2 of Y.*

Proof. (1) $\Rightarrow$ (2): Let G_1, G_2 be any n-open sets in Y such that $x \in F^+(G_1) \cap F^-(G_2)$. Then $F(x) \in G_1^+ \cap G_2^-$ and hence there exists $U \in m$ such that $x \in U \subset F^+(\mathrm{nCl}(G_1)) \cap F^-(\mathrm{nCl}(G_2))$. Since $U \in m$, we have $x \in \mathrm{mInt}(F^+(\mathrm{nCl}(G_1)) \cap F^-(\mathrm{nCl}(G_2)))$.

(2) $\Rightarrow$ (3): Let K_1, K_2 be any n-closed sets in Y. Then, $Y \setminus K_1$ and $Y \setminus K_2$ are n-open sets in Y and by (2) and Lemma 2.3, we have

$$X \setminus (F^-(K_1) \cup F^-(K_2)) = (X \setminus F^-(K_1)) \cap (X \setminus F^+(K_2)) = F^+(Y \setminus K_1) \cap F^-(Y \setminus K_2) \subset \mathrm{mInt}(F^+(\mathrm{nCl}(Y \setminus K_1)) \cap F^-(\mathrm{nCl}(Y \setminus K_2))) = \mathrm{mInt}[(X \setminus F^-(\mathrm{nInt}(K_1)) \cap (X \setminus F^+(\mathrm{nInt}(K_2)))] = \mathrm{mInt}(X \setminus [F^-(\mathrm{nInt}(K_1)) \cup F^+(\mathrm{nInt}(K_2))]).$$

Therefore, we obtain $\mathrm{mCl}(F^-(\mathrm{nInt}(K_1)) \cup F^+(\mathrm{nInt}(K_2))) \subset F^-(K_1) \cup F^+(K_2)$.

(3) $\Rightarrow$ (4): Let B_1, B_2 be any subsets of Y. Then, since n has property **B**, by Lemma 2.5 $\mathrm{nCl}(B_1)$ and $\mathrm{nCl}(B_2)$ are n-closed sets of Y and by (3) we obtain $\mathrm{mCl}(F^-(\mathrm{nInt}(\mathrm{nCl}(B_1))) \cup F^+(\mathrm{nInt}(\mathrm{nCl}(B_2)))) \subset F^-(\mathrm{nCl}(B_1)) \cup F^+(\mathrm{nCl}(B_2))$.

(4) $\Rightarrow$ (5): Let B_1, B_2 be any subsets in Y. Then by (4) and Lemma 2.3 we have

$$F^+(\mathrm{nInt}(B_1)) \cap F^-(\mathrm{nInt}(B_2)) = X \setminus [F^-(\mathrm{nCl}(Y \setminus B_1)) \cup F^+(\mathrm{nCl}(Y \setminus B_2))] \subset X \setminus \mathrm{mCl}(F^-(\mathrm{nInt}(\mathrm{nCl}(Y \setminus B_1))) \cup F^+(\mathrm{nInt}(\mathrm{nCl}(Y \setminus B_2)))) = X \setminus \mathrm{mCl}(F^-(Y \setminus \mathrm{nCl}(\mathrm{nInt}(B_1))) \cup F^+(Y \setminus \mathrm{nCl}(\mathrm{nInt}(B_2)))) = X \setminus \mathrm{mCl}[(X \setminus F^+(\mathrm{nCl}(\mathrm{nInt}(B_1)))) \cup (X \setminus F^-(\mathrm{nCl}(\mathrm{nInt}(B_2))))] = X \setminus \mathrm{mCl}(X \setminus [F^+(\mathrm{nCl}(\mathrm{nInt}(B_1)))) \cap F^-(\mathrm{nCl}(\mathrm{nInt}(B_2)))]) = \mathrm{mInt}(F^+(\mathrm{nCl}(\mathrm{nInt}(B_1))) \cap F^-(\mathrm{nCl}(\mathrm{nInt}(B_2)))).$$

Thus, we obtain $F^+(\mathrm{nInt}(B_1)) \cap F^-(\mathrm{nInt}(B_2)) \subset \mathrm{mInt}(F^+(\mathrm{nCl}(B_1)) \cap F^-(\mathrm{nCl}(B_2))))$.

(5) $\Rightarrow$ (2): This is obvious.

(2) $\Rightarrow$ (1): Let G_1, G_2 be any n-open sets of Y such that $F(x) \in G_1^+ \cap G_2^-$. Then $x \in F^+(G_1) \cap F^-(G_2) \subset \mathrm{mInt}(F^+(\mathrm{nCl}(G_1)) \cap F^-(\mathrm{nCl}(G_2)))$. Then there exists $U \in m$ such that $x \in U \subset F^+(\mathrm{nCl}(G_1)) \cap F^-(\mathrm{nCl}(G_2))$. Therefore, $F(u) \subset \mathrm{nCl}(G_1)$ and $F(u) \cap \mathrm{nCl}(G_2) \neq \emptyset$ for every $u \in U$. Hence F is weakly (m, n)-continuous.

(4) $\Rightarrow$ (6): Let G_1, G_2 be any n-open sets of Y. Then we obtain $\mathrm{mCl}(F^-(G_1) \cup F^+(G_2)) \subset \mathrm{mCl}(F^-(\mathrm{nInt}(\mathrm{Cl}(G_1))) \cup F^+(\mathrm{nInt}(\mathrm{Cl}(G_2)))) \subset F^-(\mathrm{nCl}(G_1)) \cup F^+(\mathrm{nCl}(G_2))$.

(6) $\Rightarrow$ (2): Let G_1, G_2 be any n-open sets of Y. Then we have

$$F^+(G_1) \cap F^-(G_2) \subset F^+(\mathrm{nInt}(\mathrm{nCl}(G_1))) \cap F^-(\mathrm{nInt}(\mathrm{nCl}(G_2))) = X \setminus [F^-(\mathrm{nCl}(Y \setminus \mathrm{nCl}(G_1))) \cup F^+(\mathrm{nCl}(Y \setminus \mathrm{nCl}(G_2)))] \subset X \setminus \mathrm{mCl}[F^-(Y \setminus \mathrm{nCl}(G_1)) \cup F^+(Y \setminus \mathrm{nCl}(G_2))] = \mathrm{mInt}(F^+(\mathrm{nCl}(G_1)) \cap F^-(\mathrm{nCl}(G_2))).$$

Therefore, we obtain $F^+(G_1) \cap F^-(G_2) \subset \mathrm{mInt}(F^+(\mathrm{nCl}(G_1)) \cap F^-(\mathrm{nCl}(G_2)))$.

□

4. Ideal topological spaces

Let (X, τ) be a topological space. The notion of ideals on (X, τ) has been introduced in [10] and [19] and further investigated in [9]

Definition 4.1. A nonempty collection I of subsets of a set X is called an *ideal on* X if it satisfies the following two conditions:

(1) $A \in I$ and $B \subset A$ implies $B \in I$,

(2) $A \in I$ and $B \in I$ implies $A \cup B \in I$.

A topological space (X, τ) with an ideal I on X is called an *ideal topological space* and is denoted by (X, τ, I). Let (X, τ, I) be an ideal topological space. For any subset A of X, $A^\star(I, \tau) = \{x \in X : U \cap A \notin I \text{ for every } U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$, is called the *local function* of A with respect to τ and I [9]. Hereafter $A^\star(I,\tau)$ is simply denoted by $A^\star$. It is well known that $\mathrm{Cl}^\star(A) = A \cup A^\star$ defines a Kuratowski closure operator on X and the topology generated by $\mathrm{Cl}^\star$ is denoted by $\tau^\star$.

Lemma 4.2. [9] *Let (X,τ,I) be an ideal topological space and A, B be subsets of X. Then the following properties hold:*
(1) $A \subset B$ implies $\mathrm{Cl}^\star(A) \subset \mathrm{Cl}^\star(B)$,
(2) $\mathrm{Cl}^\star(X) = X$ and $\mathrm{Cl}^\star(\emptyset) = \emptyset$,
(3) $\mathrm{Cl}^\star(A) \cup \mathrm{Cl}^\star(B) \subset \mathrm{Cl}^\star(A \cup B)$.

Definition 4.3. Let (X,τ,I) be an ideal topological space. A subset A of X is said to be
(1) *α-I-open* [8] if $A \subset \mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}(A)))$,
(2) *semi-I-open* [8] if $A \subset \mathrm{Cl}^\star(\mathrm{Int}(A))$,
(3) *pre-I-open* [5] if $A \subset \mathrm{Int}(\mathrm{Cl}^\star(A))$,
(4) *b-I-open* [1] if $A \subset \mathrm{Int}(\mathrm{Cl}^\star(A)) \cup \mathrm{Cl}^\star(\mathrm{Int}(A))$,
(5) *β-I-open* [8] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^\star(A)))$,
(6) *weakly semi-I-open* [6] if $A \subset \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}(A)))$,
(7) *weakly b-I-open* [12] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^\star(A))) \cup \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}(A)))$,
(8) *strongly β-I-open* [7] if $A \subset \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(A)))$,
(9) *$\tau^\star$-α-open* (= *α-$I^\star$*-open [3]) if $A \subset \mathrm{Int}^\star(\mathrm{Cl}^\star(\mathrm{Int}^\star(A)))$,
(10) *$\tau^\star$-semi-open* (= *semi-$I^\star$*-open [2]) if $A \subset \mathrm{Cl}^\star(\mathrm{Int}^\star(A))$,
(11) *$\tau^\star$-pre-open* (= $I^\star$-preopen) if $A \subset \mathrm{Int}^\star(\mathrm{Cl}^\star(A))$,
(12) *$\tau^\star$-b-open* if $A \subset \mathrm{Int}^\star(\mathrm{Cl}^\star(A)) \cup \mathrm{Cl}^\star(\mathrm{Int}^\star(A))$,
(13) *$\tau^\star$-β-open* (= *semi-$I^\star$*-preopen [2]) if $A \subset \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(A)))$.

The family of all α-I-open (resp. semi-I-open, pre-I-open, b-I-open, β-I-open, weakly semi-I-open, weakly b-I-open, strongly β-I-open, $\tau^\star$-α-open, $\tau^\star$-semi-open, $\tau^\star$-pre-open, $\tau^\star$-b-open, $\tau^\star$-β-open) sets in an ideal topological space (X,τ,I) is denoted by $\alpha\mathrm{IO}(X)$ (resp. $\mathrm{SIO}(X)$, $\mathrm{PIO}(X)$, $\mathrm{BIO}(X)$, $\beta\mathrm{IO}(X)$, $\mathrm{WSIO}(X)$, $\mathrm{WBIO}(X)$, $\mathrm{S}\beta\mathrm{IO}(X)$, $\tau^\star\alpha\mathrm{O}(X)$, $\tau^\star\mathrm{SO}(X)$, $\tau^\star\mathrm{PO}(X)$, $\tau^\star\mathrm{BO}(X)$, $\tau^\star\beta\mathrm{O}(X)$).

Definition 4.4. By $\mathrm{mIO}(X)$, we denote each one of the families $\tau^\star$, $\alpha\mathrm{IO}(X)$, $\mathrm{SIO}(X)$, $\mathrm{PIO}(X)$, $\mathrm{BIO}(X)$, $\beta\mathrm{IO}(X)$, $\mathrm{WSIO}(X)$, $\mathrm{WBIO}(X)$, $\mathrm{S}\beta\mathrm{IO}(X)$, $\tau^\star\alpha\mathrm{O}(X)$, $\tau^\star\mathrm{SO}(X)$, $\tau^\star\mathrm{PO}(X)$, $\tau^\star\mathrm{BO}(X)$, and $\tau^\star\beta\mathrm{O}(X)$.

Lemma 4.5. *Let (X,τ,I) be an ideal topological space. Then $\mathrm{mIO}(X)$ is an m-structure on X and has property* **B**.

Definition 4.6. Let (X,τ,I) be an ideal topological space. For a subset A of X, $\mathrm{mCl_I}(A)$ and $\mathrm{mInt_I}(A)$ are defined as follows:
(1) $\mathrm{mCl_I}(A) = \cap\{F : A \subset F, X \setminus F \in \mathrm{mIO}(X)\}$,
(2) $\mathrm{mInt_I}(A) = \cup\{U : U \subset A, U \in \mathrm{mIO}(X)\}$.

Let (X,τ,I) be an ideal topological space and $\mathrm{mIO}(X)$ the m-structure on X. If $\mathrm{mIO}(X) = \tau^\star$ (resp. $\alpha\mathrm{IO}(X)$, $\mathrm{SIO}(X)$, $\mathrm{PIO}(X)$, $\mathrm{BIO}(X)$, $\beta\mathrm{IO}(X)$, $\mathrm{WSIO}(X)$, $\mathrm{WBIO}(X)$, $\mathrm{S}\beta\mathrm{IO}(X)$, $\tau^\star\alpha\mathrm{O}(X)$, $\tau^\star\mathrm{SO}(X)$, $\tau^\star\mathrm{PO}(X)$, $\tau^\star\mathrm{BO}(X)$, $\tau^\star\beta\mathrm{O}(X)$), then we have the following:

(1) $\mathrm{mCl_I}(A) = \mathrm{Cl}^\star(A)$ (resp. $\alpha\mathrm{Cl_I}(A)$, $\mathrm{sCl_I}(A)$, $\mathrm{pCl_I}(A)$, $\mathrm{bCl_I}(A)$, $\beta\mathrm{Cl_I}(A)$, $\mathrm{wsCl_I}(A)$, $\mathrm{wbCl_I}(A)$, $\mathrm{s}\beta\mathrm{Cl_I}(A)$, $\alpha^\star\mathrm{Cl}(A)$, $\mathrm{s}^\star\mathrm{Cl}(A)$, $\mathrm{p}^\star\mathrm{Cl}(A)$, $\mathrm{b}^\star\mathrm{Cl}(A)$, $\beta^\star\mathrm{Cl}(A)$),

(2) $\mathrm{mInt_I}(A) = \mathrm{Int}^\star(A)$ (resp. $\alpha\mathrm{Int_I}(A)$, $\mathrm{sInt_I}(A)$, $\mathrm{pInt_I}(A)$, $\mathrm{bInt_I}(A)$, $\beta\mathrm{Int_I}(A)$, $\mathrm{wsInt_I}(A)$, $\mathrm{wbInt_I}(A)$, $\mathrm{s}\beta\mathrm{Int_I}(A)$, $\alpha^\star\mathrm{Int}(A)$, $\mathrm{s}^\star\mathrm{Int}(A)$, $\mathrm{p}^\star\mathrm{Int}(A)$, $\mathrm{b}^\star\mathrm{Int}(A)$, $\beta^\star\mathrm{Int}(A)$).

5. (mI, nJ)-CONTINUOUS MULTIFUNCTIONS

Let n be an m-structure on a set Y, J be an ideal on Y and (Y, n, J) be an ideal n-space.

Definition 5.1. A multifunction $F : (X, m, I) \to (Y, n, J)$ is said to be (mI, nJ)-*continuous* at $x \in X$ if for each nJ-open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists $U \in mIO(X)$ containing x such that $F(U) \subset V_1$ and $F(u) \cap V_2 \neq \emptyset$ for every $u \in U$. A multifunction $F : (X, m, I) \to (Y, n, J)$ is said to be (mI, nJ)-*continuous* if F has this property at each point of X.

Lemma 5.2. *For a multifunction $F : (X, m, I) \to (Y, n, J)$, the following properties are equivalent:*

(1) $F : (X, m, I) \to (Y, n, J)$ is (mI, nJ)-continuous;

(2) $F : (X, mIO(X)) \to (Y, nJO(Y))$ is (m, n)-continuous.

Proof. By Definitions 3.1 and 5.1, the proof is obvious.

Theorem 5.3. *A multifunction $F : (X, m, I) \to (Y, n, J)$ is (mI, nJ)-continuous at $x \in X$ if and only if for every nJ-open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, $x \in \mathrm{mInt_I}(F^+(V_1) \cap F^-(V_2))$.*

Proof. This follows from Theorem 3.2.

Theorem 5.4. *For a multifunction $F : (X, m, I) \to (Y, n, J)$, the following properties are equivalent:*

(1) F is (mI, nJ)-continuous;

(2) $F^+(V_1) \cap F^-(V_2)$ is mI-open in X for every nJ-open sets V_1, V_2 of Y;

(3) $F^-(K_1) \cup F^+(K_2)$ is mI-closed in X for every nJ-closed sets K_1, K_2 of Y;

(4) $\mathrm{mCl_I}(F^-(B_1) \cup F^+(B_2)) \subset F^-(\mathrm{nCl_J}(B_1)) \cup F^+(\mathrm{nCl_J}(B_2))$ for every subsets B_1, B_2 of Y;

(5) $F^-(\mathrm{nInt_J}(B_1)) \cap F^+(\mathrm{nInt_J}(B_2)) \subset \mathrm{mInt_I}(F^-(B_1) \cap F^+(B_2))$ for every subsets B_1, B_2 of Y.

Proof. This follows from Theorem 3.3 and Lemma 4.5.

Let $m = \tau$, $\mathrm{mIO}(X) = \tau^\star$ and $n = \sigma$, $\mathrm{nJO}(Y) = \sigma^\star$. Then by Definition 5.1 we obtain the following definition:

Definition 5.5. A multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$ is said to be $i^\star$-*continuous* [4] if for each $x \in X$ and each $\sigma^\star$-open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists a $\tau^\star$-open set U of X containing x such that $F(U) \subset V_1$ and $F(u) \cap V_2 \neq \emptyset$ for every $u \in U$.

Let $\mathrm{mIO}(X) = \tau^\star$ and $\mathrm{nJO}(Y) = \sigma^\star$, then by Theorem 5.4, we obtain the following corollary:

Corollary 5.6. [4] *For a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$, the following properties are equivalent:*

(1) F is $i^\star$-continuous;

(2) $F^+(G_1) \cap F^-(G_2)$ is $\tau^\star$-open for every $\sigma^\star$-open sets G_1, G_2 of Y;

(3) $F^-(K_1) \cup F^+(K_2)$ is $\tau^\star$-closed for every $\sigma^\star$-closed sets K_1, K_2 of Y;

(4) $\mathrm{Cl}^\star(F^-(B_1) \cup F^+(B_2)) \subset F^-(\mathrm{Cl}^\star(B_1)) \cup F^+(\mathrm{Cl}^\star(B_2))$ for every subsets B_1, B_2 of Y;

(5) $F^-(\mathrm{Int}^\star(B_1)) \cap F^+(\mathrm{Int}^\star(B_2)) \subset \mathrm{Int}^\star(F^-(B_1) \cap F^+(B_2))$ for every subsets B_1, B_2 of Y.

Definition 5.7. A multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$ is said to be $\star(\alpha)$-*continuous* [3] at $x \in X$ if for each $\star$-open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists an α-$I^\star$-open set U of X containing x such that $F(U) \subset V_1$ and $F(u) \cap V_2 \neq \emptyset$ for every $u \in U$. A multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$ is said to be $\star(\alpha)$-*continuous* if F has this property at each point of X.

Let $\mathrm{mIO}(X) = \tau^\star\alpha O(X)$ and $\mathrm{nJO}(Y) = \sigma^\star$, then by Theorem 5.4, we obtain the following corollary:

Corollary 5.8. [3] *For a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$, the following properties are equivalent:*
(1) F is $\star(\alpha)$-continuous;
(2) $F^+(G_1) \cap F^-(G_2)$ is $\tau^\star$-α-open for every $\sigma^\star$-open sets G_1, G_2 of Y;
(3) $F^-(K_1) \cup F^+(K_2)$ is $\tau^\star$-α-closed for every $\sigma^\star$-closed sets K_1, K_2 of Y;
(4) $\alpha^\star\mathrm{Cl}(F^-(B_1) \cup F^+(B_2)) \subset F^-(\mathrm{Cl}^\star(B_1)) \cup F^+(\mathrm{Cl}^\star(B_2))$ for every subsets B_1, B_2 of Y;
(5) $F^-(\mathrm{Int}^\star(B_1)) \cap F^+(\mathrm{Int}^\star(B_2)) \subset \alpha^\star\mathrm{Int}(F^-(B_1) \cap F^+(B_2))$ for every subsets B_1, B_2 of Y.

Definition 5.9. A multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$ is said to be *weakly (mI, nJ)-continuous* at a point $x \in X$ if for each nJ-open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists $U \in mIO(X)$ containing x such that $F(u) \in (\mathrm{nCl_J}(V_1))^+ \cap (\mathrm{nCl_J}(V_2))^-$ for every $u \in U$. If F is weakly (mI, nJ)-continuous at every point of $x \in X$, then F is said to be *weakly (mI, nJ)-continuous.*

Lemma 5.10. *For a multifunction $F : (X, m, I) \to (Y, n, J)$, the following properties are equivalent:*
(1) $F : (X, m, I) \to (Y, n, J)$ is weakly (mI, nJ)-continuous;
(2) $F : (X, mIO(X)) \to (Y, nJO(Y))$ is weakly (m, n)-continuous.

Proof. By Definitions 3.4 and 5.9, the proof is obvious.

Theorem 5.11. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$, the following properties are equivalent:*
(1) F is weakly (mI, nJ)-continuous;
(2) $F^+(G_1) \cap F^-(G_2) \subset \mathrm{mInt_I}(F^+(\mathrm{nCl_J}(G_1)) \cap F^-(\mathrm{nCl_J}(G_2)))$ for every nJ-open sets G_1, G_2 of Y;
(3) $\mathrm{mCl_I}(F^-(\mathrm{nInt_J}(K_1)) \cup F^+(\mathrm{nInt_J}(K_2))) \subset F^-(K_1) \cup F^+(K_2)$ for every nJ-closed sets K_1, K_2 of Y;
(4) $\mathrm{mCl_I}(F^-(\mathrm{nInt_J}(\mathrm{nCl_J}(B_1))) \cup F^+(\mathrm{nInt_J}(\mathrm{nCl_J}(B_2)))) \subset F^-(\mathrm{nCl_J}(B_1)) \cup F^+(\mathrm{nCl_J}(B_2))$ for every subsets B_1, B_2 of Y;
(5) $F^+(\mathrm{nInt_J}(B_1)) \cap F^-(\mathrm{nInt_J}(B_2)) \subset \mathrm{mInt_I}(F^+(\mathrm{nCl_J}(B_1)) \cap F^-(\mathrm{nCl_J}(B_2))))$ for every subsets B_1, B_2 of Y;
(6) $\mathrm{mCl_I}(F^-(G_1) \cup F^+(G_2)) \subset F^-(\mathrm{nCl_J}(G_1)) \cup F^+(\mathrm{nCl_J}(G_2))$ for every nJ-open sets G_1, G_2 of Y.

Definition 5.12. A multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$ is said to be *weakly $i^\star$-continuous* [4] if for each $x \in X$ and each $\sigma^\star$-open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists a $\tau^\star$-open set U of X containing x such that $F(U) \subset \mathrm{Cl}^\star(V_1)$ and $F(u) \cap \mathrm{Cl}^\star(V_2) \neq \emptyset$ for every $u \in U$.

Let $\mathrm{mIO}(X) = \tau^\star$ and $\mathrm{nJO}(Y) = \sigma^\star$, then by Theorem 5.11, we obtain the following corollary:

Corollary 5.13. [4] *For a multifunction $F : (X, \tau, J) \to (Y, \sigma, J)$, the following properties are equivalent:*
(1) F is weakly $i^\star$-continuous;
(2) $F^+(G_1) \cap F^-(G_2) \subset \mathrm{Int}^\star(F^+(\mathrm{Cl}^\star(G_1)) \cap F^-(\mathrm{Cl}^\star(G_2)))$ for every $\star$-open sets G_1, G_2 of Y;
(3) $\mathrm{Cl}^\star(F^-(\mathrm{Int}^\star(K_1)) \cup F^+(\mathrm{Int}^\star(K_2))) \subset F^-(K_1) \cup F^+(K_2)$ for every $\star$-closed sets K_1, K_2

of Y;

(4) $\mathrm{Cl}^\star(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(B_1)))\cup F^+(\mathrm{Int}^\star(\mathrm{Cl}^\star(B_2)))) \subset F^-(\mathrm{Cl}^\star(B_1))\cup F^+(\mathrm{Cl}^\star(B_2))$ *for every subsets* B_1, B_2 *of Y;*

(5) $F^+(\mathrm{Int}^\star(B_1)) \cap F^-(\mathrm{Int}^\star(B_2)) \subset \mathrm{Int}^\star(F^+(\mathrm{Cl}^\star(B_1)) \cap F^-(\mathrm{Cl}^\star(B_2))))$ *for every subsets* B_1, B_2 *of* Y*;*

(6) $\mathrm{Cl}^\star(F^-(G_1) \cup F^+(G_2)) \subset F^-(\mathrm{Cl}^\star(G_1)) \cup F^+(\mathrm{Cl}^\star(G_2))$ *for every* $\star$*-open sets* G_1, G_2 *of Y.*

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