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HERMITE INTERPOLATING SERIES

MARILENA JIANU

Abstract. The paper introduces a new type of polynomial series (*Hermite interpolating series*). If $f(x)$ represents the sum of such a series, then its full-Hermite interpolating polynomials are partial sums of this series. We state and prove sufficient conditions for a function and its derivatives to be representable as Hermite interpolating series.

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1. Introduction

Interpolating polynomials are a very useful tool in the approximation of functions and, as Philip Davis remarks in [4], "the flame of interest in interpolation and approximation has burned brighter" as the automatic calculation speed highly increased. When these polynomials represent partial sums of some uniformly convergent series, they can be successfully used to approximate the solution of ordinary differential equations. The Taylor polynomial (corresponding to Taylor series) is often used for initial-value problems, when the solution is supposed to be analytic. However, for boundary-value problems, the Newton polynomial (corresponding to Newton series) is a more adequate instrument, either using an infinite sequence of distinct points (see [8, 9]) or a periodic sequence on a finite set of points (see [7]).

Hermite interpolating polynomial generalizes both the Newton (Lagrange) polynomial and the Taylor polynomial, as it makes use of the values of the function and its derivatives at several points. When only the values of the first derivative are considered, we refer to this as simple Hermite (or osculatory) interpolation; when the values of higher derivatives are involved, the interpolating polynomial is known as full Hermite interpolating polynomial. If we want to represent this polynomial as the partial sum of a series, we need to write it in a suitable form, in a similar way that Lagrange polynomial must be written in the Newton form to reveal that it is the partial sum of the Newton series.

Hermite interpolating polynomial was used for solving boundary value problems for differential equations (see [2, 3, 12, 13]) and integral equation (in [11]), for approximating functions in topological fields [6], or even the reliability polynomial of a network (in [5]).

Our work was inspired by [7], where the functions are expanded into Newton interpolating series on p interpolation points and it is proved that some of the derivatives of the polynomial represented by the partial sum, calculated at the interpolation points, are equal to the derivatives of the function. This paper introduces the Hermite series, composed by p independent series and investigates sufficient conditions for their convergence. Particularly, any Hermite interpolating polynomial is obtained by adding partial sums of the p series.

2. Hermite interpolating polynomials

Consider p distinct points, $x_1 < x_2 < \ldots < x_p$ in the interval $[a, b]$ and p nonnegative integers, $\alpha_1, \alpha_2, \ldots, \alpha_p$. Denote by $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)$. Let $f : [a, b] \to \mathbb{R}$ be a function of class $C^k[a, b]$, where $k \geq \alpha_i, i = 1, 2, \ldots, p$. The *full Hermite interpolation problem* ([10], [4]) is the problem of finding a polynomial $H_\alpha(x)$ of degree at most

$$N = \alpha_1 + \alpha_2 + \ldots + \alpha_p + p - 1$$

such that

$$H_\alpha^{(j)}(x_i) = f^{(j)}(x_i)\,,\ \forall j = 0, 1, \ldots, \alpha_i,\ \forall i = 1, 2, \ldots, p. \tag{2.1}$$

Theorem 2.1. [1, 14] *Given the distinct points $x_1 < x_2 < \ldots < x_p$, the nonnegative integers $\alpha_1, \alpha_2, \ldots, \alpha_p$, and the arbitrary real numbers $f_i^{(j)}$, $j = 0, 1, \ldots, \alpha_i$, $i = 1, 2, \ldots, p$, there exists a unique polynomial $H_\alpha(x)$ of degree at most $N = \alpha_1 + \alpha_2 + \ldots + \alpha_p + p - 1$ such that*

$$H_\alpha^{(j)}(x_i) = f_i^{(j)}\,,\ \forall j = 0, 1, \ldots, \alpha_i,\ \forall i = 1, 2, \ldots, p. \tag{2.2}$$

The expression of this polynomial (called ***full Hermite interpolating polynomial****) is:*

$$H_\alpha(x) = \sum_{i=1}^{p} \sum_{j=0}^{\alpha_i} f_i^{(j)} l_{\alpha,i,j}(x)\,, \tag{2.3}$$

where, for any $j = 0, 1, \ldots, \alpha_j$ and $i = 1, 2, \ldots, p$,

$$l_{\alpha,i,j}(x) = u_{\alpha,i}(x) \frac{(x - x_i)^j}{j!} \sum_{k=0}^{\alpha_i - j} \frac{1}{k!} v_{\alpha,i}^{(k)}(x_i)(x - x_i)^k, \tag{2.4}$$

$$u_\alpha(x) = \prod_{i=1}^{p} (x - x_i)^{\alpha_i + 1}, \quad u_{\alpha,i}(x) = \frac{u_\alpha(x)}{(x - x_i)^{\alpha_i + 1}}, \quad v_{\alpha,i}(x) = \frac{1}{u_{\alpha,i}(x)}. \tag{2.5}$$

Theorem 2.2. *Let $x_1 < x_2 < \ldots < x_p$ be distinct points in the interval $[a, b]$, $\alpha_1, \alpha_2, \ldots, \alpha_p$ be nonnegative integers and $f : [a, b] \to \mathbb{R}$ be a function of class $C^k[a, b]$, where $k \geq \alpha_i, i = 1, 2, \ldots, p$. Then the full Hermite interpolating polynomial, $H_\alpha(x)$ of degree at most $N = \alpha_1 + \alpha_2 + \ldots + \alpha_p + p - 1$ verifying (2.1) can be written in the following form:*

$$H_\alpha(x) = \sum_{i=1}^{p} u_{\alpha,i}(x) \sum_{k=0}^{\alpha_i} \frac{(x - x_i)^k}{k!} \left(\frac{f(x)}{u_{\alpha,i}(x)} \right)^{(k)}_{x = x_i}. \tag{2.6}$$

Proof. By Theorem 2.1 we can write:

$$H_\alpha(x) = \sum_{i=1}^{p} u_{\alpha,i}(x) \sum_{j=0}^{\alpha_i} \sum_{l=0}^{\alpha_i - j} f^{(j)}(x_i) \frac{(x-x_i)^{j+l}}{j!\ l!} v_{\alpha,i}^{(l)}(x_i)$$

$$= \sum_{i=1}^{p} u_{\alpha,i}(x) \sum_{j=0}^{\alpha_i} \sum_{k=j}^{\alpha_i} f^{(j)}(x_i) \frac{(x-x_i)^{k}}{k!} \binom{k}{j} v_{\alpha,i}^{(k-j)}(x_i)$$

$$= \sum_{i=1}^{p} u_{\alpha,i}(x) \sum_{k=0}^{\alpha_i} \sum_{j=0}^{k} f^{(j)}(x_i) \frac{(x-x_i)^{k}}{k!} \binom{k}{j} v_{\alpha,i}^{(k-j)}(x_i)$$

$$= \sum_{i=1}^{p} u_{\alpha,i}(x) \sum_{k=0}^{\alpha_i} \frac{(x-x_i)^{k}}{k!} \sum_{j=0}^{k} \binom{k}{j} f^{(j)}(x_i) v_{\alpha,i}^{(k-j)}(x_i)$$

$$= \sum_{i=1}^{p} u_{\alpha,i}(x) \sum_{k=0}^{\alpha_i} \frac{(x-x_i)^{k}}{k!} \left[f(x) v_{\alpha,i}(x)\right]_{x=x_i}^{(k)}$$

and, since $v_{\alpha,i}(x) = u_{\alpha,i}(x)^{-1}$, the formula (2.6) is obtained. □

Theorem 2.3. [1, 4] *Let $x_1 < x_2 < \ldots < x_p$ be distinct points in the interval $[a, b]$, $\alpha_1, \alpha_2, \ldots, \alpha_p$ nonnegative integers and $f : [a, b] \to \mathbb{R}$ a function of class $C^{N+1}[a, b]$, where $N = \alpha_1 + \alpha_2 + \ldots + \alpha_p + p - 1$. If $H_\alpha(x)$ is the corresponding full Hermite interpolating polynomial then, for any $x \in [a, b]$, there exists ξ such that $\min\{x_1, x\} < \xi < \max\{x_p, x\}$ and*

$$f(x) - H_\alpha(x) = \frac{f^{(N+1)}(\xi)}{(N+1)!} u_\alpha(x). \tag{2.7}$$

In the particular case when $\alpha = (n, n, \ldots, n)$, $n = 0, 1, \ldots$, we denote the full Hermite polynomial by $H_n(x)$. Using the notations

$$u(x) = (x - x_1)(x - x_2) \ldots (x - x_p), \quad u_i(x) = \frac{u(x)}{x - x_i} \tag{2.8}$$

the expression (2.6) is written:

$$H_n(x) = \sum_{i=1}^{p} u_i(x)^{n+1} \sum_{k=0}^{n} \frac{(x-x_i)^k}{k!} \left(\frac{f(x)}{u_i(x)^{n+1}}\right)_{x=x_i}^{(k)}. \tag{2.9}$$

Notice that $H_n(x)$ is actually a polynomial of degree at most $np + p - 1$.

Corollary 2.4. *Let $f : [a, b] \to \mathbb{R}$ be an infinitely differentiable function and $a = x_1 < x_2 < \ldots < x_p = b$ be distinct points in the interval $[a, b]$. For each $n = 0, 1, \ldots$, we denote by $M_n = \max_{x\in[a,b]} \left|f^{(n)}(x)\right|$. If there exists a positive constant $\lambda > 0$ such that $M_n < \lambda^{n+1}$, for every $n = 0, 1, \ldots$, then the sequence of full Hermite interpolating polynomials (2.9) uniformly converge to $f(x)$ on $[a, b]$.*

Proof. By applying Theorem 2.3 for $\alpha = (n, n, \ldots, n)$, we can write, for any $x \in [a, b]$:

$$|f(x) - H_n(x)| \le \frac{M_{np+p}}{(np+p)!} |(x - x_1)(x - x_2) \ldots (x - x_p)|^{n+1}$$

$$\leq \lambda \frac{[\lambda(b-a)]^{np+p}}{(np+p)!} \to 0$$

hence the corollary follows. ☐

Remark 2.5. *For $p = 2$, $x_1 = 0$, $x_2 = 1$ and $\alpha_1 = \alpha_2 = n$, the full Hermite interpolating polynomial can be written in the following form, called "Two-point Taylor formula" [2]:*

$$H_n(x) = \sum_{j=0}^{n} \left[f^{(j)}(0) C_{n,j}(x) + (-1)^j f^{(j)}(1) C_{n,j}(1-x) \right], \tag{2.10}$$

where the polynomials $C_{n,j}(x)$, $j = 0, 1, \ldots, n$, $n = 0, 1, \ldots$, are defined by:

$$C_{n,j}(x) = (1-x)^{n+1} \sum_{k=0}^{n-j} \binom{n+k}{k} \frac{x^{k+j}}{j!}.$$

3. Hermite interpolating series

Given p distinct points, $x_1 < x_2 < \ldots < x_p$, we consider the polynomials

$$\mu_{i,n}(x) = (x - x_i)^n u_i(x)^{n+1} = u(x)^n u_i(x), \tag{3.1}$$

for $i = 1, 2, \ldots, p$, $n = 0, 1, \ldots$, and the series of the form

$$\sum_{n \geq 0} a_{i,n} \mu_{i,n}(x), \tag{3.2}$$

where $a_{i,n} \in \mathbb{R}, i = 1, \ldots, p, n = 0, 1, \ldots$. Using the *Weierstrass M-test* we can prove the following sufficient condition of convergence for the series (3.2) and the series of $k-$derivatives,

$$\sum_{n \geq 0} a_{i,n} \mu_{i,n}^{(k)}(x). \tag{3.3}$$

Theorem 3.1. *Let $a = x_1 < x_2 < \ldots < x_p = b$. Denote by M the maximum absolute value of the function $u(x) = (x - x_1)(x - x_2) \ldots (x - x_p)$:*

$$M = \max_{x \in [a,b]} |u(x)| = \max_{x \in [a,b]} |(x - x_1)(x - x_2) \ldots (x - x_p)|.$$

If the sequences $\{a_{i,n}\}_{n \geq 0}$ verify, for any $i = 1, 2, \ldots, p$,

$$\limsup_{n \to \infty} \sqrt[n]{|a_{i,n}|} < \frac{1}{M}, \tag{3.4}$$

then the series (3.3) is absolutely and uniformly convergent on $[a, b]$, for any $k = 0, 1, \ldots$.

Proof. We first study the case when $k = 0$. For any $x \in [a, b]$ we have:

$$|a_{i,n} u(x)^n u_i(x)| \leq |a_{i,n}| M^n (b-a)^{p-1}.$$

By (3.4) the series $\sum_{n \geq 0} |a_{i,n}| M^n (b-a)^{p-1}$ is convergent and the absolute and uniform convergence of the series (3.2) follows by *Weierstrass M-test.*

Now, let us suppose that $k \geq 1$. For any $x \in [a, b]$ we have:

$$\left| a_{i,n} \left[u(x)^n u_i(x) \right]^{(k)} \right| \leq |a_{i,n}| (pn+p-1)(pn+p-2) \ldots (pn+p-k)(b-a)^{pn+p-k-1}$$

if $n = 0, 1, \dots, k$, and

$$\left| a_{i,n} \left[u(x)^n u_i(x) \right]^{(k)} \right| \le |a_{i,n}|(pn + p - 1)(pn + p - 2) \dots (pn + p - k) M^{n-k}(b - a)^{pk+p-k-1}$$

if $n > k$.

Since

$$\limsup_{n\to\infty} \sqrt[n]{|a_{i,n}|(pn + p - 1)(pn + p - 2) \dots (pn + p - k) M^{n-k}(b - a)^{pk+p-k-1}}$$

$$= \limsup_{n\to\infty} \sqrt[n]{|a_{i,n}|} M < 1,$$

it follows that the series $\sum_{n\ge 0} |a_{i,n}|(pn + p - 1)(pn + p - 2) \dots (pn + p - k) M^{n-k}(b - a)^{pk+p-k-1}$ converges, hence the series (3.3) is uniformly and absolutely convergent on $[a, b]$. □

Remark 3.2. If the condition (3.4) is fulfilled for any $i = 1, 2, \dots, p$, then one can write:

$$\sum_{i=1}^{p} \sum_{n\ge 0} a_{i,n} \mu_{i,n}(x) = \sum_{n\ge 0} \sum_{i=1}^{p} a_{i,n} \mu_{i,n}(x).$$

Theorem 3.3. *Let $a = x_1 < x_2 < \dots < x_p = b$ be p distinct points and $f(x)$ be a function infinitely differentiable on $[a, b]$ which can be written as the sum of the uniformly convergent series:*

$$f(x) = \sum_{n\ge 0} \sum_{i=1}^{p} a_{i,n} \mu_{i,n}(x). \tag{3.5}$$

Then, for any $i = 1, 2, \dots, p$ and $n \ge 0$, the coefficients $a_{i,n}$ verify the formulas:

$$a_{i,n} = \frac{1}{u_i(x_i)} \left[\frac{1}{n!} \left(\frac{f(x)}{u_i(x)^n} \right)^{(n)}_{x=x_i} + \sum_{\substack{j=1 \\ j\ne i}}^{p} \frac{1}{(n-1)!} \left(\frac{f(x)}{u_j(x)^n (x - x_i)} \right)^{(n-1)}_{x=x_j} \right], \tag{3.6}$$

for $n = 1, 2, \dots,$ and

$$a_{i,0} = \frac{f(x_i)}{u_i(x_i)}. \tag{3.7}$$

Proof. The formula (3.7) follows by simply observing that $f(x_i) = a_{i,0} u_i(x_i)$.

We can also remark that the partial sum of (3.5),

$$H_n(x) = \sum_{k=0}^{n} \sum_{i=1}^{p} a_{i,k}(x - x_i)^k u_i(x)^{k+1}, \tag{3.8}$$

is the full Hermite interpolating polynomial of the function $f(x)$: it is a polynomial of degree at most $np + p - 1$ such that $H_n^{(k)}(x_i) = f^{(k)}(x_i)$, for any $i = 1, 2, \dots, p$ and $k = 0, 1, \dots, n$. For any $n \ge 1$ and $i = 1, \dots, p$, one can write

$$f(x) - H_{n-1}(x) = a_{i,n}(x - x_i)^n u_i(x)^{n+1} + (x - x_i)^{n+1} g_{i,n}(x),$$

where $g_{i,n}(x)$ is a continuous function, infinitely differentiable in the intervals $(x_i-\varepsilon, x_i)$ and $(x_i, x_i+\varepsilon)$. Then, since there exists the n-th derivative of the function $(x-x_i)^{n+1}g_{i,n}(x)$ at x_i and its value is 0, it follows that

$$a_{i,n}=\frac{f^{(n)}(x_i)-H^{(n)}_{n-1}(x_i)}{n!\ u_i(x_i)^{n+1}}. \tag{3.9}$$

We use the formula (2.9) to calculate $H^{(n)}_{n-1}(x_i)$. Since

$$H_{n-1}(x)=\sum_{i=1}^{p}u_i(x)^n\sum_{k=0}^{n-1}\frac{(x-x_i)^k}{k!}\left(\frac{f(x)}{u_i(x)^n}\right)^{(k)}_{x=x_i},$$

it follows that

$$\begin{aligned}H^{(n)}_{n-1}(x_i)&=\left[u_i(x)^n\sum_{k=0}^{n-1}\frac{(x-x_i)^k}{k!}\left(\frac{f(x)}{u_i(x)^n}\right)^{(k)}_{x=x_i}\right]^{(n)}_{x=x_i}\\&+\sum_{\substack{j=1\\i\neq j}}^{p}\left[u_j(x)^n\sum_{k=0}^{n-1}\frac{(x-x_j)^k}{k!}\left(\frac{f(x)}{u_j(x)^n}\right)^{(k)}_{x=x_j}\right]^{(n)}_{x=x_i}\\&=\sum_{k=0}^{n-1}\binom{n}{k}\left[u_i(x)^n\right]^{(n-k)}_{x=x_i}\left[\frac{f(x)}{u_i(x)^n}\right]^{(k)}_{x=x_i}\\&+\sum_{\substack{j=1\\i\neq j}}^{p}n!\frac{u_i(x_i)^n}{(x_i-x_j)^n}\sum_{k=0}^{n-1}\frac{(x_i-x_j)^k}{k!}\left(\frac{f(x)}{u_j(x)^n}\right)^{(k)}_{x=x_j}\\&=f^{(n)}(x_i)-u_i(x)^n\left(\frac{f(x)}{u_i(x_i)^n}\right)^{(n)}_{x=x_i}\\&+u_i(x_i)^n n\sum_{\substack{j=1\\i\neq j}}^{p}\sum_{k=0}^{n-1}\binom{n-1}{k}(n-k-1)!(x_i-x_j)^{k-n}\left(\frac{f(x)}{u_j(x)^n}\right)^{(k)}_{x=x_j}.\end{aligned}$$

Since $(n-k-1)!(x_i-x_j)^{k-n}=\left[\frac{1}{x_i-x}\right]^{(n-1-k)}_{x=x_j}$, we obtain:

$$H^{(n)}_{n-1}(x_i)=f^{(n)}(x_i)-u_i(x_i)^n\left[\left(\frac{f(x)}{u_i(x_i)^n}\right)^{(n)}_{x=x_i}+n\sum_{\substack{j=1\\i\neq j}}^{p}\left(\frac{f(x)}{(x-x_i)u_j(x)^n}\right)^{(n-1)}_{x=x_j}\right]$$

and by replacing in equation (3.9), the formula (3.6) follows at once. □

Let $a=x_1<x_2<\ldots<x_p=b$ be p distinct points in the interval $[a,b]$. If the series

$$\sum_{n\geq 0}\sum_{i=1}^{p}a_{i,n}\mu_{i,n}(x) \tag{3.10}$$

converges absolutely and uniformly to $f(x)$, then we say that the function $f(x)$ can be represented as a Hermite interpolating series. Based on Corollary 2.4, we present in Theorem 3.5

a sufficient condition such that an infinitely differentiable function and its derivatives can be represented as Hermite interpolating series.

Remark 3.4. There exist functions that can be represented as Hermite interpolating series, but are not differentiable. One such example is the function $f(x) = |x|$ which can be expanded in Hermite series on the interval $[-1, 1]$ taking as interpolation points $x_1 = -1$ and $x_2 = 1$ as follows:

$$f(x) = \sqrt{1-(1-x^2)} = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{2^{2n}(2n-1)} \binom{2n}{n} (x+1)^n (x-1)^n, \tag{3.11}$$

and, denoting $\mu_{1,n}(x) = (x+1)^n(x-1)^{n+1}$ and $\mu_{2,n}(x) = (x-1)^n(x+1)^{n+1}$, one can write

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{2^{2n+1}(2n-1)} \binom{2n}{n} \left(\mu_{2,n}(x) - \mu_{1,n}(x)\right).$$

We notice that the series (3.11) is absolutely and uniformly convergent, by Weierstrass test, because $\left|x^2 - 1\right| \le 1$ for all $x \in [-1, 1]$, and the series $\sum\limits_{n=0}^{\infty} \frac{1}{2^{2n}(2n-1)} \binom{2n}{n}$ converges.

Theorem 3.5. *Let $f : [a, b] \to \mathbb{R}$ be an infinitely differentiable function for which there exists a positive constant $\lambda > 0$ such that*

$$\left|f^{(n)}(x)\right| < \lambda^{n+1}, \text{ for all } x \in [a, b], n = 0, 1, \ldots. \tag{3.12}$$

Then, for any p distinct points $a \le x_1 < x_2 < \ldots < x_p \le b$ in the interval $[a, b]$, the functions $f(x)$ and $f'(x)$ can be represented as full Hermite interpolating series,

$$f(x) = \sum_{n\ge 0} \sum_{i=1}^{p} a_{i,n} \mu_{i,n}(x), \tag{3.13}$$

$$f'(x) = \sum_{n\ge 0} \sum_{i=1}^{p} a_{i,n}^{(1)} \mu_{i,n}(x) \tag{3.14}$$

and the coefficients of the derivative series are given by the formula:

$$a_{i,n}^{(1)} = (2n+1) a_{i,n} \sum_{\substack{j=1 \\ j\ne i}}^{p} \frac{1}{x_i - x_j} + \sum_{\substack{j=1 \\ j\ne i}}^{p} a_{j,n} \frac{1}{x_i - x_j} \left(n + 1 - n\frac{u_j(x_j)}{u_i(x_i)}\right) + (n+1) a_{i,n+1} u_i(x_i), \tag{3.15}$$

for any $i = 1, 2, \ldots, p$ and $n = 0, 1, \ldots$.

Proof. By applying Corollary 2.4 to the functions $f(x)$ and $f'(x)$ we obtain that the corresponding sequences of full Hermite interpolating polynomials (which are partial sums of the Hermite series (3.13), (3.14)) uniformly converge to $f(x)$, respectively $f'(x)$.

For $n = 0$ and $i = 1, 2, \ldots, p$ we have:

$$\mu'_{i,0}(x) = \left[\prod_{\substack{j=1 \\ j\ne i}}^{p} (x - x_j)\right]' = \sum_{\substack{j=1 \\ j\ne i}}^{p} \prod_{\substack{k=1 \\ k\ne i,j}}^{p} (x - x_k) \cdot \frac{(x - x_i) - (x - x_j)}{x_j - x_i}$$

$$= \sum_{\substack{j=1\\ j\neq i}}^{p} \frac{\mu_{j,0}-\mu_{i,0}}{x_j - x_i} = \left(\sum_{\substack{j=1\\ j\neq i}}^{p} \frac{1}{x_i - x_j} \right) \mu_{i,0} + \sum_{\substack{j=1\\ j\neq i}}^{p} \frac{1}{x_j - x_i} \cdot \mu_{j,0}$$

For $n \geq 1$ we apply the formulas (3.6)-(3.7) for the function $f(x) = \mu'_{i,n}(x)$

$$\mu'_{i,n}(x) = n(x-x_i)^{n-1} \prod_{\substack{j=1\\ j\neq i}}^{p} (x-x_j)^{n+1} + (n+1) \sum_{\substack{j=1\\ j\neq i}}^{p} (x-x_i)^n (x-x_j)^n \prod_{\substack{k=1\\ k\neq i,j}}^{p} (x-x_k)^{n+1}$$

$$= \sum_{k=0}^{n} \sum_{j=1}^{p} \alpha_{j,k} \mu_{j,k}(x).$$

We remark that, if $F(x)$ is an infinitely differentiable function in a neighborhood of x_i, then

$$\left[(x-x_i)^n F(x) \right]^{(k)}_{x=x_i} = \begin{cases} 0 & if \;\; k < n \\ n!\binom{k}{n} F^{(k-n)}(x_i) & if \;\; k \geq n \end{cases} \tag{3.16}$$

We can also notice that $\mu'_{i,n}(x)$ can be written:

$$\mu'_{i,n}(x) = n(x-x_i)^{n-1} u_i(x)^{n+1} + (n+1) \sum_{\substack{j=1\\ j\neq i}}^{p} (x-x_i)^n \frac{u_i(x)^{n+1}}{x-x_j} \tag{3.17}$$

as well as

$$\mu'_{i,n}(x) = n \frac{(x-x_j)^{n+1} u_j(x)^{n+1}}{(x-x_i)^2} + (n+1) \sum_{\substack{k=1\\ k\neq i,j}}^{p} \frac{(x-x_j)^{n+1} u_j(x)^{n+1}}{(x-x_i)(x-x_k)} \tag{3.18}$$

$$+(n+1) \frac{(x-x_j)^n u_j(x)^{n+1}}{x-x_i},$$

for $j \neq i$.

It follows that $\alpha_{j,k} = 0$ for any $k < n-1$ and $j = 1, 2, \ldots, p$.

For $k = n-1$, using (3.16)-(3.18), we have $\alpha_{j,n-1} = 0$ for $j \neq i$ and

$$\alpha_{i,n-1} = \frac{1}{u_i(x_i)} \cdot \frac{1}{(n-1)!} \left(\frac{\mu'_{i,n}(x)}{u_i(x)^{n-1}} \right)^{(n-1)}_{x=x_i} = n \cdot u_i(x_i).$$

For $k = n$ we have:

$$\alpha_{i,n} = \frac{1}{u_i(x_i)} \left[\frac{1}{n!} \left(\frac{\mu'_{i,n}(x)}{u_i(x)^n} \right)^{(n)}_{x=x_i} + \sum_{\substack{j=1\\ j\neq i}}^{p} \frac{1}{(n-1)!} \left(\frac{\mu'_{i,n}(x)}{u_j(x)^n (x-x_i)} \right)^{(n-1)}_{x=x_j} \right]$$

$$= \frac{1}{u_i(x_i) n!} \left[n(x-x_i)^{n-1} u_i(x) + (n+1) \sum_{\substack{j=1\\ j\neq i}}^{p} (x-x_i)^n \frac{u_i(x)}{x-x_j} \right]^{(n)}_{x=x_i}$$

$$= \frac{1}{u_i(x_i)n!}\left[n!\binom{n}{1}u_i'(x_i) + (n+1)!\sum_{\substack{j=1\\ j\neq i}}^{p}\frac{u_i(x_i)}{x_i - x_j}\right] = (2n+1)\sum_{\substack{j=1\\ j\neq i}}^{p}\frac{1}{x_i - x_j}$$

and, for $j \neq i$,

$$\alpha_{j,n} = \frac{1}{u_j(x_j)}\left[\frac{1}{n!}\left(\frac{\mu_{i,n}'(x)}{u_j(x)^n}\right)^{(n)}_{x=x_j} + \sum_{\substack{k=1\\ k\neq j}}^{p}\frac{1}{(n-1)!}\left(\frac{\mu_{i,n}'(x)}{u_k(x)^n(x-x_j)}\right)^{(n-1)}_{x=x_k}\right]$$

$$= \frac{1}{u_j(x_j)}\left[\frac{n+1}{n!}\left(\frac{(x-x_j)^n u_j(x)}{x-x_i}\right)^{(n)}_{x=x_j} + \frac{n}{(n-1)!}\left(\frac{(x-x_i)^{n-1}u_i(x)}{x-x_j}\right)^{(n-1)}_{x=x_i}\right]$$

$$= \frac{1}{u_j(x_j)}\left[(n+1)\cdot\frac{u_j(x_j)}{x_j - x_i} + n\cdot\frac{u_i(x_i)}{x_i - x_j}\right]$$

$$= \frac{1}{x_j - x_i}\left(n + 1 - n\cdot\frac{u_i(x_i)}{u_j(x_j)}\right).$$

We denote by

$$S_i = \sum_{\substack{j=1\\ j\neq i}}^{p}\frac{1}{x_i - x_j}.$$

We proved that

$$\mu_{i,n}'(x) = (2n+1)S_i\mu_{i,n}(x) + \sum_{\substack{j=1\\ j\neq i}}^{p}\frac{1}{x_i - x_j}\left(n+1-n\frac{u_i(x_i)}{u_j(x_j)}\right)\mu_{j,n}(x) + nu_i(x_i)\mu_{i,n-1}(x) \tag{3.19}$$

and, since

$$f'(x) = \sum_{n\geq 0}\sum_{i=1}^{p} a_{i,n}\mu_{i,n}'(x) = \sum_{n\geq 0}\sum_{i=1}^{p} a_{i,n}^{(1)}\mu_{i,n}(x),$$

the formula (3.15) follows. □

Corollary 3.6. *Let* $U = \operatorname{diag}\big(u_1(x_1), u_2(x_2), \dots, u_p(x_p)\big)$. *For* $n = 0, 1, \dots,$ *we denote by* $a_n = (a_{1,n}, a_{2,n}, \dots, a_{p,n})^T$, $a_n^{(1)} = (a_{1,n}^{(1)}, a_{2,n}^{(1)}, \dots, a_{p,n}^{(1)})^T$ *and*

$$A_n = \begin{pmatrix} (2n+1)S_1 & \frac{1}{x_1-x_2}\left(n+1-n\frac{u_2(x_2)}{u_1(x_1)}\right) & \dots & \frac{1}{x_1-x_p}\left(n+1-n\frac{u_p(x_p)}{u_1(x_1)}\right) \\ \frac{1}{x_2-x_1}\left(n+1-n\frac{u_1(x_1)}{u_2(x_2)}\right) & (2n+1)S_2 & \dots & \frac{1}{x_2-x_p}\left(n+1-n\frac{u_p(x_p)}{u_2(x_2)}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_p-x_1}\left(n+1-n\frac{u_1(x_1)}{u_p(x_p)}\right) & \frac{1}{x_p-x_2}\left(n+1-n\frac{u_2(x_2)}{u_p(x_p)}\right) & \dots & (2n+1)S_p \end{pmatrix}$$

Then, for any $n = 0, 1, \dots,$ *we have:*

$$a_n^{(1)} = A_n a_n + (n+1)U a_{n+1},$$

for any $n = 0, 1, \dots.$

Moreover, if

$$f^{(k)}(x) = \sum_{n\geq 0}\sum_{i=1}^{p} a_{i,n}^{(k)}\mu_{i,n}(x),$$

then

$$a_n^{(k+1)} = A_n a_n^{(k)} + (n+1)Ua_{n+1}^{(k)},$$

for any $n = 0, 1, \ldots,\ k = 1, 2, \ldots.$

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Department of Mathematics and Computer Science, Technical University of Civil Engineering Bucharest, Blvd. Lacul Tei 122–124, 020396 Bucharest, Romania

Email address: marilena.jianu@utcb.ro