ON A FEFFERMAN-PHONG TYPE INEQUALITY, A NEW AND SIMPLIFIED PROOF

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ABSTRACT. We give a simplified proof of an imbedding theorem by C.Fefferman and we study some extention.
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1. Introduction

The purpose of this work is to give an extension and a simplified proof of a difficult result by C.Fefferman [4] concerning the imbedding

$$\int_{\mathbb{R}^n} |f|^2 V \leq C \int_{\mathbb{R}^n} |\nabla f|^2, \ \ \forall f \in C_0^\infty. \tag{1.1}$$

Here the weight V will be always a measurable function with values in $[0, \infty]$.

The estimate 1.1 is proved in [4], assuming $V \in F^r(\mathbb{R}^n),\ 1 < r \leq \frac{n}{2}$, i.e, there is a constant $C > 0$ such that

$$|Q|^{\frac{2}{n}} \left(\frac{1}{|Q|} \int_Q V^r \right)^{1/r} \leq C, \tag{1.2}$$

for all cubes Q in $\mathbb{R}^n$.

In his work, C.Fefferman remarked that it is probably a sharp version of 1.2 in which the $L_{\log}L$ norm is used in place of the L^r-norm. Also, it is well known that the F^1-condition is necessary but not sufficient for 1.1 to hold.

F.Chiarenza and M.Frasca [1] have extended 1.1 to the L_p spaces. They proved that if

$$|Q|^{\frac{p}{n}} \left(\frac{1}{|Q|} \int_Q V^r(x) dx \right)^{1/r} \leq C \tag{1.3}$$

for all cubes Q, with $1 < p < n$ and $1 < r < n/p$, then

$$\int_{\mathbb{R}^n} |f|^p V \leq C \int_{\mathbb{R}^n} |\nabla f|^p, \ \ \forall f \in C_0^\infty. \tag{1.4}$$

Here and bellow C is an unspecified positive constant, possibly different at each occurrence.
In this work we replace 1.2 by a weaker condition involving the dyadic $L_{\log}L$ norm.
Next we recall some definitions and notations.

By a dyadic system $\mathcal{D}$ we mean a collection of cubes with the following properties:

- the side length of each cube in $\mathcal{D}$ is of the form $2^{-j},\ j \in \mathbb{Z}$,
- any two cubes in $\mathcal{D}$ are either disjoint or one is contained in the other,
- the cubes of a given size form a partition of $\mathbb{R}^n$.

Definition 1.1. Let f be a locally integrable function in $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, the *dyadic Hardy-Littlewood maximal function* of f is defined by

$$M^d f(x) = \sup_{Q \ni x, Q \subset \mathcal{D}} \frac{1}{|Q|} \int_Q |f(x)| dx.$$

As in the usual case, see [5, Theorem 2.1], we have

$$|\{x \in \mathbb{R}^n : M^d f(x) > t\}| \le \frac{C}{t} \int_{\{x \in \mathbb{R}^n : f(x) > \frac{t}{2}\}} |f(x)| dx, \tag{1.5}$$

$$|\{x \in \mathbb{R}^n : M^d f(x) > t\}| \ge \frac{C}{t} \int_{\{x \in \mathbb{R}^n : f(x) > t\}} |f(x)| dx, \tag{1.6}$$

and

$$\int_{\mathbb{R}^n} |M^d f(x)|^p dx \le C \int_{\mathbb{R}^n} |f(x)|^p dx, \ \ 1 < p < \infty. \tag{1.7}$$

Definition 1.2. We say that V satisfies the *dyadic Wilson-condition* (see [18]) or $V \in RH_1^d$ if

$$\int_Q M^d(\chi_Q V)(x) dx \le C \int_Q V(x) dx,$$

for all cubes $Q \in \mathcal{D}$.
RH_1^d is refereed as the dyadic maximal reverse Hölder inequality. Our result is the following:

Theorem 1.1. *Let p and q be such that $1 < p < n$ and $p \le q < \infty$. Assume $V \in RH_1^d$ and satisfying for all cubes $Q \in \mathcal{D}$*

$$|Q|^{\frac{1}{n} - \frac{1}{p}} \left(\int_Q V(x) dx \right)^{1/q} \le C.$$

Then

$$\left(\int_{\mathbb{R}^n} |f(x)|^q V(x) dx \right)^{1/q} \le C \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p dx \right)^{1/p}, \quad \forall f \in C_0^\infty(\mathbb{R}^n).$$

Remark 1.1. The non dyadic version of Theorem 1.1 can be found in [15], with $p = q$.

Corollary 1.1. *Let V be a weight satisfying the dyadic Wilson-condition and $1 < p < n$. Assume that there is a constant $C > 0$ such that for all cubes $Q \in \mathcal{D}$*

$$|Q|^{\frac{p}{n} - 1} \int_Q V(x) dx \le C.$$

Then

$$\int_{\mathbb{R}^n} |f(x)|^p V(x) dx \le C \int_{\mathbb{R}^n} |\nabla f(x)|^p dx, \quad \forall f \in C_0^\infty(R^n).$$

2. Some Preliminary Results

Definition 2.1. Let $\varphi(t)$ be a non-negative,increasing function on $(0,\infty)$ and V a weight on $\mathbb{R}^n$. We say that V satisfies a *dyadic φ-reverse Hölder inequality* and write $V \in RH^d_\varphi$ if

$$\int_Q V(x)\varphi\left(\frac{V(x)}{\frac{1}{|Q|}\int_Q V(x)dx}\right)dx \leq C\int_Q V(x)dx \tag{2.1}$$

for all cube $Q \in \mathcal{D}$.
Example 2.1. When $\varphi(t) = t^\epsilon$ and $0 < \epsilon < \infty$, then 2.1 is equivalent to

$$\left(\frac{1}{Q}\int_Q V^{1+\epsilon}(x)\right)^{\frac{1}{1+\epsilon}} \leq \frac{C}{|Q|}\int_Q V(x)dx, \tag{2.2}$$

for all cube $Q \in \mathcal{D}$. Condition 2.2 is called a *reverse Hölder inequality.* To simplify notation, we write $RH^d_\epsilon = RH^d_{t^\epsilon}$.

Lemma 2.1. *If for some $\epsilon > 0$, V belongs to RH^d_ϵ then it must satisfy*

$$\int_Q M^d(\chi_Q V)(x)dx \leq C\int_Q V(x)dx, \tag{2.3}$$

for all cubes $Q \in \mathcal{D}$. Where χ_Q denotes the characteristic function of the cube Q.

Proof. For $Q \in \mathcal{D}$ and $x \in Q$, we have by the RH^d_ϵ condition on V

$$M^d(\chi_Q V^{1+\epsilon})(x) \leq C(M^d(\chi_Q V))^{1+\epsilon}(x).$$

Thus Hölder's inequality and 1.7 lead to

$$\begin{aligned}
\frac{1}{|Q|}\int_Q M^d(\chi_Q V)(x)dx &\leq \frac{1}{|Q|}\int_Q \left(M^d(\chi_Q V^{1+\epsilon})\right)^{\frac{1}{1+\epsilon}}(x)dx \\
&\leq \left(\frac{1}{|Q|}\int_Q M^d(\chi_Q V^{1+\epsilon})(x)dx\right)^{\frac{1}{1+\epsilon}} \\
&\leq \left(\frac{C}{|Q|}\int_Q (M^d(\chi_Q V))^{1+\epsilon}(x)dx\right)^{\frac{1}{1+\epsilon}} \\
&\leq \left(\frac{C}{|Q|}\int_Q V^{1+\epsilon}(x)dx\right)^{\frac{1}{1+\epsilon}} \\
&\leq \frac{C}{|Q|}\int_Q V(x)dx.
\end{aligned}$$

□

Definition 2.2. A *weight V is in A^d_p*, $1 < p < \infty$, if and only if

$$\sup_{Q\in\mathcal{D}}\left(\frac{1}{|Q|}\int_Q V(x)dx\right)\left(\frac{1}{|Q|}\int_Q V^{\frac{-1}{p-1}}(x)dx\right)^{p-1} \leq C.$$

The class A^d_∞ is defined by $A^d_\infty = \cup_{p>1}A^d_p$.
Remark 2.1. As in the continuous case, see [5], if V is in A^d_∞, then it must be dyadic doubling ,i.e, $V(2Q) \leq CV(Q)$, $\forall Q \in \mathcal{D}$, with $V(Q) = \int_Q V(x)dx$. Also if V is in A^d_∞, then it must satisfy the dyadic reverse Hölder's inequality, i.e, RH_ϵ-condition on V holds on all $Q \in \mathcal{D}$, for some $\epsilon > 0$.

The converse is false: the dyadic RH_ϵ-condition on V does not implies $V \in A^d_\infty$. In fact the weight $V = \chi_{\mathbb{R}^n \setminus [0,1]^n}$ is in RH^d_ϵ, for all $\epsilon > 0$, but it is not dyadic doubling. Hence, V can not be in A^d_∞.

Lemma 2.2. *Let V in RH^d_ϵ, $\epsilon > 0$. Then V is in $RH^d_{\log^+}$ with*

$$\log^+ t = \begin{cases} \log t & t > 1 \\ 0 & 0 < t \le 1. \end{cases}$$

Proof. Let Q be a fixed cube in $\mathcal{D}$. Then we have for all $\epsilon \in (0, 1/2)$

$$\int_Q V(x) \log^+ \left(\frac{V}{\frac{1}{|Q|}\int_Q V(x)dx} \right) dx \le \frac{1}{\epsilon} \left(\int_Q V^{1+\epsilon}(x) dx \right) \left(\frac{1}{|Q|} \int_Q V(x)dx \right)^{-\epsilon}. \tag{2.4}$$

The last inequality follows from the estimate

$$\log(e + t) \le \frac{t^\epsilon}{\epsilon}, \quad \forall \epsilon \in (0, 1/2), \forall t \ge 1.$$

Let $\epsilon \in (0, 1/2)$ be such that 2.2 holds. Then by 2.4 we have

$$\begin{aligned} \int_Q V(x) \log^+ \left(\frac{V(x)}{\frac{1}{|Q|}\int_Q V(x)dx} \right) dx &\le C_\epsilon |Q| \left(\frac{1}{|Q|} \int_Q V(x)dx \right)^{1+\epsilon} \left(\frac{1}{|Q|} \int_Q V(x)dx \right)^{-\epsilon} \\ &\le C \int_Q V(x)dx. \end{aligned}$$

□

Lemma 2.3. $V \in RH^d_{\log^+} \Longleftrightarrow V \in RH^d_1$.

Proof. Let $\delta > 0$ to be choose later. The estimate 1.5 implies

$$\begin{aligned} \int_Q M^d(\chi_Q V)(x)dx &= \int_0^\infty |\{x \in Q : M^d(\chi_Q V)(x) > t\}| dt \\ &= \left(\int_0^\delta + \int_\delta^\infty \right) \left(|\{x \in Q : M^d(\chi_Q V(x) > t\}| \right) dt \\ &\le \delta |Q| + C \int_\delta^\infty \frac{1}{t} \left(\int_{\{x \in \mathbb{R}^n : V(x) > t\}} \chi_Q V(x) dx \right) dt \\ &\le C \left(\delta |Q| + \int_{\mathbb{R}^n} \chi_Q V(x) \left(\int_\delta^{V(x)} \frac{1}{t} dt \right) dx \right) \\ &\le C \left(\delta |Q| + \int_Q V(x) \log^+(\delta^{-1} V(x)) dx \right) \end{aligned}$$

Pick $\delta = \frac{1}{|Q|} \int_Q V(x)dx$ and using the $RH^d_{\log^+}$-condition to obtain

$$\int_Q M^d(\chi_Q V)(x)dx \le C \int_Q V(x)dx.$$

To prove the converse, write

$$\int_Q V(x) \log \left(e + \frac{V(x)}{V(Q)} \right) dx = I + II,$$

with

$$I=\frac{1}{V(Q)}\int_0^{V(Q)}\frac{1}{e+\frac{t}{V(Q)}}V(\{x\in Q:V(x)>t)\})dt\leq\int_Q M^d(\chi_Q V)(x)dx.$$

To estimate II, we use 1.6 to get

$$\begin{aligned}II=&\int_{\{x\in Q:V(x)>V(Q)\}}V(x)\log\left(e+\frac{V(x)}{V(Q)}\right))dx\\=&\frac{1}{V(Q)}\int_{V(Q)}^{\infty}\frac{1}{e+\frac{t}{V(Q)}}V(\{x\in Q:V(x)>t)\})dt\\\leq&\frac{C}{V(Q)}\int_0^{\infty}\frac{t}{e+\frac{t}{V(Q)}}|\{x\in Q:M^d(\chi_Q V)(x)>t)\}|dt\\\leq&C\int_0^{\infty}\{x\in Q:M^d(\chi_Q V)(x)>t)\}|dt\\\leq&C\int_Q M^d(\chi_Q V)(x)dx.\end{aligned}$$

□

Corollary 2.1. $A^d_\infty\subset RH^d_{\log^+}$.

Remark 2.2. As mentioned in Remark 2.1, the class $RH^d_{\log^+}$ is more large than the class A^d_∞. However if V is doubling and belongs to $RH^d_{\log^+}$ then it must be in A^d_∞.

3. Strong type inequality for Riesz potentials

Let α to be a real number such that $0<\alpha<n$. By a Riesz potential operator, we mean an operator of the type

$$I_\alpha(f)(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy.$$

The corresponding usual maximal fractional operator is defined by

$$M_\alpha(f)(x)=\sup_{Q\ni x}|Q|^{\frac{\alpha}{n}-1}\int_Q|f(x)|dx,$$

for all cubes Q.

Note. The Riesz operators play a crucial role in the study of partial differential equations. For instance, to study Carleman estimates and unique continuation problem with a singular potential we need some some estimates of this operator, see for instance [2], [16].

Riesz operators are also closely related to various function spaces in harmonic analysis. A rich literature on this can be found in [6, 7, 8, 9, 10, 11, 12, 13] and the references there.

Proposition 3.1. *Let $0<\alpha<n$, $1<p<\frac{n}{\alpha}$ and $p\leq q<\infty$. If σ is a weight and V is in RH^d_1 such that*

$$|Q|^{\frac{\alpha}{n}-1}\left(\int_Q V(x)dx\right)^{1/q}\left(\int_Q\sigma(x)dx\right)^{1/p'}\leq C\tag{3.1}$$

for all cubes $Q\in\mathcal{D}$, then

$$\left(\int_Q(M_\alpha(\chi_Q V))^{p'}(x)\sigma(x)dx\right)^{1/p'}\leq C\left(\int_Q V(x)dx\right)^{1/q'}\tag{3.2}$$

for all cubes $Q \in \mathcal{D}$.

Proof. Let $R > 0$ be large and let $\{Q_{jk}\}$ be the maximal dyadic sub-cubes of Q such that

$$|Q_{jk}|^{\frac{\alpha}{n}-1} \int_{Q_{jk}} V(y)dy > R^k.$$

Then the collection $\{Q_{jk}\}$ satisfies the Carleson nesting condition:

$$\sum_{Q_{jk} \subset Q} |Q_{jk}| \leq C|Q| \tag{3.3}$$

so that

$$\sum_{Q_{jk} \subset Q} \int_{Q_{jk}} V \leq C \int_Q M^d(\chi_Q V). \tag{3.4}$$

See [3]. The estimates 3.1, 3.4 and the dyadic Wilson-condition lead to

$$\begin{aligned}
\int_Q (M_\alpha(\chi_Q V))^{p'}(x)\sigma(x)dx &\leq C \sum_{Q_{jk} \subset Q} \left(|Q_{jk}|^{\frac{\alpha}{n}-1} \int_{Q_{jk}} V\right)^{p'} \int_{Q_{jk}} \sigma \\
&\leq C \sum_{Q_{jk} \subset Q} \left(|Q_{jk}|^{\frac{\alpha}{n}-1} \left(\int_{Q_{jk}} V\right)^{1/q} \left(\int_{Q_{jk}} \sigma(x)dx\right)^{1/p'}\right)^{p'} \left(\int_{Q_{jk}} V\right)^{p'/q'} \\
&\leq C \sum_{Q_{jk} \subset Q} \left(\int_{Q_{jk}} V\right)^{p'/q'} \leq C \left(\int_Q V\right)^{p'/q'-1} \sum_{Q_{jk} \subset Q} \int_{Q_{jk}} V \\
&\leq C \left(\int_Q V\right)^{p'/q'-1} \int_Q M^d(\chi_Q V) \leq C \left(\int_Q V\right)^{p'/q'}.
\end{aligned}$$

□

Theorem 3.1. *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ *and* $p \leq q < \infty$. *If* V *is in* RH_1^d *and satisfies 3.1. Then*

$$\left(\int_{\mathbb{R}^n} |I_\alpha f(x)|^q V(x)dx\right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{1/p}, \quad \forall f \in C_0^\infty(\mathbb{R}^n).$$

Proof. We have by a result of *R.Kerman and E.Sawyer* [14] that for $1 < p \leq q < \infty$, the estimate

$$\left(\int_{\mathbb{R}^n} |I_\alpha f(x)|^q V(x)dx\right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{1/p}, \quad \forall f \in C_0^\infty(\mathbb{R}^n)$$

holds if and only if

$$\left(\int_Q (M_\alpha(\chi_Q V))^{p'}(x)dx\right)^{1/p'} \leq C \left(\int_Q V(x)dx\right)^{1/q'} \tag{3.5}$$

for all cubes $Q \in \mathcal{D}$. Then using Proposition 3.1 to finish the proof. □

Proof of Theorem 1.1. The proof is immediate by using Theorem 3.1 and the following well known estimate

$$|f(x)| \leq C I_1(|\nabla f(x)|) \quad , \quad \forall f \in C_0^\infty(R^n).$$

□

The following corollary is immediate.

Corollary 3.1. *Let* $0 < \alpha < n$, $1 < p < n$, $p \leq q < \infty$ *and* $V \in RH^d_\epsilon$. *Assume that there is a constant* $C > 0$ *such that for all cubes* $Q \in \mathcal{D}$

$$|Q|^{\frac{1}{n}-\frac{1}{p}}\left(\int_Q V(x)dx\right)^{1/q} \leq C.$$

Then

$$\left(\int_{\mathbb{R}^n} |f(x)|^q V(x)dx\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n} |\nabla f(x)|^p dx\right)^{1/p}, \quad \forall f \in C_0^\infty(\mathbb{R}^n).$$

Remark 3.1. Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ and $p \leq q < \infty$. If σ is in RH^d_1 and V is a weight such that

$$|Q|^{\frac{\alpha}{n}-1}\left(\int_Q V(x)dx\right)^{1/q}\left(\int_Q \sigma(x)dx\right)^{1/p'} \leq C \tag{3.6}$$

for all cubes $Q \in \mathcal{D}$, then one can show, by a similar argument given in the proof of Proposition 3.1, that

$$\left(\int_Q \left(M_\alpha(\chi_Q\sigma)\right)^q(x)V(x)dx\right)^{1/q} \leq C\left(\int_Q \sigma(x)dx\right)^{1/p} \tag{3.7}$$

for all cubes $Q \in \mathcal{D}$.
This last estimate is well known to be equivalent to

$$\left(\int_{\mathbb{R}^n} |M_\alpha(f\sigma)(x)|^q V(x)dx\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n} |f(x)|^p \sigma(x)dx\right)^{1/p}, \ \forall f \in C_0^\infty(\mathbb{R}^n).$$

If in addition $V \in A^d_\infty$, then

$$\left(\int_{\mathbb{R}^n} |I_\alpha(f\sigma)(x)|^q V(x)dx\right)^{1/q} \simeq \left(\int_{\mathbb{R}^n} |M_\alpha(f\sigma)(x)|^q V(x)dx\right)^{1/q}$$

$$\leq C\left(\int_{\mathbb{R}^n} |f(x)|^p \sigma(x)dx\right)^{1/p}, \ \forall f \in C_0^\infty(\mathbb{R}^n).$$

This remark includes Pérez's result [17].

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