BOUNDEDNESS OF HIGHER ORDER COMMUTATORS OF SUBLINEAR OPERATORS ON GRAND VARIABLE HERZ-MORREY SPACES

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Abstract. In harmonic analysis, studies of inequalities of sublinear operators in various function spaces have a very important place. Morrey type spaces with variable exponents and the demonstration of the boundedness of such operators on these spaces have an important place in harmonic analysis and have become an interesting field. Our aim in this paper is to prove the boundedness of higher order commutators of sublinear operators on grand variable Herz-Morrey spaces by applying some properties of variable exponent.

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1. Introduction

In recent years, function spaces with variable exponents have gained increasing attention in research due to their applications in a variety of fields for instance see [1]. Over time, there has been significant theoretical work on variable Lebesgue, Orclicz, Sobolev and Lorentz spaces. This work has helped to establish the properties and characteristics of these spaces, and has contributed to our understanding of their potential applications. Some important references in this area include articles published by researchers such as [2, 3, 4, 5, 6]. In harmonic analysis, Herz space are notable for their norm which incorporates both local and global information. In [7] the first generalization of variable exponents Herz spaces was presented. The most general conclusions were found in [8], allowing for the exponent α to vary.

More recently, in [9] variable parameters were used to define continual Herz spaces. Additionally, the boundedness of sublinear operators on these continual Herz spaces was demonstrated. Additionally, boundedness of other operators the Marcinkiewicz integral and Riesz potential operator can be confirmed in [10, 11]. Morrey spaces are a type of function space that were originally developed by Charles Morrey in 1938 as a tool for studying regularity properties of solutions to certain types of partial differential equations, particularly those arising in the calculus of variations. Morrey spaces provide a more refined notion of local regularity than Lebesgue spaces, which can be useful in many different areas of mathematics, including harmonic analysis and PDE theory, ref [12]. In their paper [13], Meskhi introduced the concept of grand Morrey spaces, denoted by $L^{p,\theta,\lambda}$, which are a generalization of Morrey spaces. Then author went on to show how to obtain the boundedness of a class of integral operators, including the Hardy-Littlewood maximal function, Calderón-Zygmund singular integrals, and potentials, on these spaces. Boundedness results for some integral operators were then studied in subsequent works such as

[14]. Moreover the grand variable Herz-Morrey spaces is the generalization of original grand variable Herz spaces and it was introduced in [15, 16, 17].

The study of higher-order commutators of fractional integral operator can provide important insights into the behavior of these operators in function spaces. Grand weighted Herz spaces and grand weighted Herz-Morrey spaces was introduced by Sultan et al. in [18, 19] respectively. For more references see [20, 21, 22].

In fact in this article are aiming to prove the boundedness of these commutators of sublinear operators in grand variable Herz-Morrey spaces. There are different sections of this paper. We have different sections in this research paper to organize and present the information clearly. Apart from introduction, the preliminaries section is important to establish the necessary background knowledge and definitions that will be used throughout the paper. In the last section we will focus on boundedness of commutators of sublinear operators on grand variable Herz-Morrey spaces.

2. Preliminaries

Consider a measurable set $\mathcal{A}$ in $\mathbb{R}^n$ and a measurable function $p(\cdot) : \mathcal{A} \to [1, \infty)$. Then

(a) Now Lebesgue space with variable exponent $L^{p(\cdot)}(\mathcal{A})$ is defined by

$$L^{p(\cdot)}(\mathcal{A}) = \left\{ f \text{ is measurable} : \int_{\mathcal{A}} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty \text{ where } \lambda \text{ is a constant} \right\}.$$

Norm in $L^{p(\cdot)}(\mathcal{A})$ can be defined as,

$$\|f\|_{L^{p(\cdot)}(\mathcal{A})} = \inf \left\{ \lambda > 0 : \int_{\mathcal{A}} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

(b) The space $L^{p(\cdot)}_{\text{loc}}(\mathcal{A})$ can be defined as

$$L^{p(\cdot)}_{\text{loc}}(\mathcal{A}) := \left\{ f : f \in L^{p(\cdot)}(H) \text{ for all compact subsets } H \subset \mathcal{A} \right\}.$$

We use these notations in this paper:

(i) Suppose that $f \in L^1_{\text{loc}}(\mathcal{A})$ now the Hardy-Littlewood maximal operator M can be given as

$$Mf(x) := \sup_{0<r} \frac{1}{r^n} \int_{B(x,r)} |f(x)| dx \quad (x \in \mathcal{A}),$$

where $B(x, r) := \{y \in \mathcal{A} : |x - y| < r\}$.

(ii) Let $p(\cdot)$ be a measurable function then the set $\mathfrak{B}(\mathcal{A})$ is consists of those measurable functions satisfying the following properties:

$$p_- := \operatorname*{ess\,inf}_{x\in\mathcal{A}} p(x) > 1, \;\; p_+ := \operatorname*{ess\,sup}_{x\in\mathcal{A}} p(x) < \infty. \tag{2.1}$$

(iii) Let $p \in \mathfrak{B}(\mathcal{A})$ then $\mathfrak{B}^{\log} = \mathfrak{B}^{\log}(\mathcal{A})$ is the class of functions satisfying (2.1) and log-condition given by,

$$|p(x) - p(y)| \leq \frac{C(p)}{-\ln|x - y|}, \;\; |x - y| \leq \frac{1}{2}, \;\; x, y \in \mathcal{A}. \tag{2.2}$$

(iv) For an unbounded set $\mathcal{A}$ in $\mathbb{R}^n$, $\mathfrak{B}_\infty(\mathcal{A})$ is the subset of $\mathfrak{B}(\mathcal{A})$ and values are in $[1, \infty)$ satisfying following condition

$$|p(x) - p_\infty| \leq \frac{C}{\ln(e + |x|)}, \tag{2.3}$$

where $p_\infty \in (1,\infty)$. $\mathfrak{B}_{0,\infty}(\mathcal{A})$ is the subset of $\mathfrak{B}(\mathcal{A})$ satisfying the following condition

$$|p(x)-p_0| \le \frac{C}{|\ln|x||}, |x| \le \frac{1}{2}, \tag{2.4}$$

in the case of homogeneous Herz spaces.

(v) For an unbounded set $\mathcal{A}$ in $\mathbb{R}^n$, then $\mathfrak{B}_\infty(\mathcal{A})$ and $\mathfrak{B}_{0,\infty}(\mathcal{A})$ are the subsets of $\mathfrak{B}(\mathcal{A})$.

(vi) Consider an unbounded set $\mathcal{A}$ in $\mathbb{R}^n$, then $\mathfrak{B}^{\log}_\infty(\mathcal{A})$ is the set of exponent $p \in \mathfrak{B}_\infty(\mathcal{A})$ satisfying condition (2.1). $\mathfrak{B}_\infty(\mathcal{A})$ is the subsets of exponents of $L^\infty(\mathcal{A})$ and its values are in $[1,\infty]$ satisfying both conditions (2.2) and (2.3),

(vii) $p(\cdot) \in \mathcal{A}$ then $\mathcal{B}(\mathcal{A})$ is the collection of $p(\cdot)$ where M is bounded on $L^{p(\cdot)}(\mathcal{A})$.

(viii) $B_t = B(0,2^t) = \{z_1 \in \mathbb{R}^n : |z_1| < 2^t\}$ for all $l \in \mathbb{Z}$. $F_t = B_t \setminus B_{t-1}$, $\chi_{F_t} = \chi_t$.

(ix) $F_{t,\tau} = B(0,\tau) \setminus B(0,t) = \{x : t < |x| < \tau\}$.

C is a positive constant, its value can be changes from line to line.

Lemma 2.1. [9] Let $D > 1$ and $p \in \mathfrak{B}_{0,\infty}(\mathbb{R}^n)$. Then

$$\frac{1}{t_0} s^{\frac{n}{p(0)}} \le \|\chi_{R_{s,sD}}\|_{p(\cdot)} \le t_0 s^{\frac{n}{p(0)}}, \; for\; 0 < s \le 1 \tag{2.5}$$

and

$$\frac{1}{t_\infty} s^{\frac{n}{p_\infty}} \le \|\chi_{R_{s,sD}}\|_{p(\cdot)} \le t_\infty s^{\frac{n}{p_\infty}}, \; for\; s \ge 1, \tag{2.6}$$

respectively, where $t_0 \ge 1$ and $t_\infty \ge 1$ and depending on D but independent of s.

Lemma 2.2. [3][*Hölder's inequality*] Consider a measurable subset $\mathcal{A}$ such that $\mathcal{A} \subseteq \mathbb{R}^n$, and $1 \le p_-(\mathcal{A}) \le p_+(\mathcal{A}) \le \infty$. Then

$$\|fg\|_{L^{r(\cdot)}(\mathcal{A})} \le C\|f\|_{L^{p(\cdot)}(\mathcal{A})}\|f\|_{L^{q(\cdot)}(\mathcal{A})}$$

holds, where $f \in L^{p(\cdot)}(\mathcal{A})$, $g \in L^{q(\cdot)}(\mathcal{A})$ and $\frac{1}{r(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}$ for every $x \in \mathcal{A}$.

Definition 2.3 (BMO space)**.** Let u is a locally integrable function then a BMO function is consists of those functions whose mean oscillation given by $\frac{1}{|B|}\int_B |u(x)-u_B|dx$ is bounded. A Mathematically,

$$\|u\|_{BMO} = \sup_B \frac{1}{|B|}\int_B |u(x)-u_B|dx < \infty.$$

Lemma 2.4. [23] For a positive integer k, let $b \in BMO(\mathbb{R}^n)$ and choose $w, l \in \mathbb{Z}$ with $l < w$,

$$\frac{1}{C}\|b\|^k_{BMO} \le \sup_{B:ball} \frac{1}{\|\chi_B\|_{p(\cdot)}} \|(b-b_B)^k\chi_B\|_{p(\cdot)} \tag{2.7}$$

$$\le C\|b\|^k_{BMO}, \tag{2.8}$$

$$||(b-b_{B_t})^k\chi_{B_w}||_{p(\cdot)} \le C(w-t)^k\|b\|^k_{BMO}||\chi_{B_w}||_{p(\cdot)}. \tag{2.9}$$

Lemma 2.5. [23] Consider a ball B in $\mathbb{R}^n$ and $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$; then we have

$$|B|^{-1}\|\chi_B\|_{p(\cdot)}\|\chi_B\|_{p'(\cdot)} \le C. \tag{2.10}$$

3. Grand variable Herz-Morrey spaces

Definition 3.1. Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $p \in [1,\infty)$, $q : \mathbb{R}^n \to [1,\infty)$, $\theta > 0$, $0 \le \lambda < \infty$. We define the homogeneous grand variable Herz-Morrey spaces by the norm:

$$M\dot{K}^{\alpha(\cdot),p),\theta}_{\lambda,q(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}^{\alpha(\cdot),p),\theta}_{\lambda,q(\cdot)}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),p),\theta}_{\lambda,q(\cdot)}(\mathbb{R}^n)} = \sup_{\epsilon>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta \sum_{k=-\infty}^{k_0} \|2^{k\alpha(\cdot)} f\chi_k\|^{p(1+\epsilon)}_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{p(1+\epsilon)}}.$$

For $\lambda = 0$, grand variable Herz-Morrey spaces becomes grand variable Herz spaces.

Non-homogeneous grand variable Herz-Morrey spaces can be defined in the similar way. For more details we refer to [5, 6, 15, 18, 19]

4. Boundedness of higher order commutators of sublinear operators

4.1. **Higher order commutators of sublinear operators.** Let b be a locally integrable function, and $m \in \mathbb{N}$; the higher order commutators of sublinear operators T_b are defined by

$$T_b f(x) = \int_{\mathbb{R}^n} \frac{[b(x)-b(y)]^m}{|x-y|^n} f(y)dy. \tag{4.1}$$

We will prove the boundedness of higher order commutators of sublinear operators on grand variable Herz-Morrey spaces.

Theorem 4.1. Let $1 < p < \infty$, $0 \le \lambda < \infty$, $b \in BMO(\mathbb{R}^n)$, $\alpha, q \in \mathfrak{P}_{0,\infty}(\mathbb{R}^n)$, such that α is satisfying $\frac{-n}{q_\infty} < \alpha_\infty < \frac{n}{q'_\infty}$ and $\frac{-n}{q(0)} < \alpha(0) < \frac{n}{q'(0)}$. Then the commutator of sublinear operators T_b^m is bounded on $M\dot{K}^{\alpha(\cdot),p),\theta}_{\lambda,q(\cdot)}(\mathbb{R}^n)$.

Proof. Let $f \in M\dot{K}^{\alpha(\cdot),p),\theta}_{\lambda,q(\cdot)}(\mathbb{R}^n)$, and $f(x) = \sum_{t=-\infty}^{\infty} f(x)\chi_t(x) = \sum_{t=-\infty}^{\infty} f_t(x)$, we have

$$\|T_b^m(f)\|_{M\dot{K}^{\alpha(\cdot),p),\theta}_{\lambda,q(\cdot)}(\mathbb{R}^n)} = \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta \sum_{w=-\infty}^{k_0} \|2^{w\alpha(\cdot)}\chi_w T_b^m f\|^{p(1+\epsilon)}_{q(\cdot)}\right)^{\frac{1}{p(1+\epsilon)}}$$

$$\le \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta \sum_{w=-\infty}^{k_0}\left(\sum_{t=-\infty}^{\infty} \|2^{w\alpha(\cdot)}\chi_w T_b^m f(\chi_t)\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}$$

$$\le \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta \sum_{w=-\infty}^{k_0}\left(\sum_{t=-\infty}^{w} \|2^{w\alpha(\cdot)}\chi_w T_b^m (f\chi_t)\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}$$

$$+ \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta \sum_{w=-\infty}^{k_0}\left(\sum_{t=w+1}^{\infty} \|2^{k\alpha(\cdot)}\chi_w T_b^m (f\chi_t)\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}$$

$$=: E_1 + E_2.$$

For E_1, we use the facts that, if $x \in F_w$, $y \in F_t$ and $t \le w$, then $|x-y| \sim |x| \sim 2^w$. By virtue of Hölder's inequality and size condition

$$|T_b^m(f\chi_t)(x).\chi_w(x)| \le C \int_{F_t} \frac{|f_t(y)|[b(x)-b(y)]^m}{|x-y|^n} dy.\chi_w(x)$$

$$\le C2^{-wn} \int_{F_t} |f_t(y)||b(x)-b(y)|^m dy.\chi_w(x)$$

$$\le C2^{-wn}\left(|b(x)-b_{B_t}|^m \int_{F_t} |f_t(y)|dy + \int_{F_t} |f_t(y)||b(y)-b_{B_t}|^m dy\right).\chi_w(x)$$

$$\leq C2^{-wn}\|f_t\|_{q(\cdot)}\big(|b(x)-b_{B_t}|^m\|\chi_t\|_{q'(\cdot)}+\|((b-b_{B_t})^m\chi_t\|_{q'(\cdot)}\big)\cdot\chi_w(x).$$

To find the estimate of above expressions apply Lemma (2.4), and the fact that $\|\chi_t\|_{q(\cdot)}\leq\|\chi_{B_t}\|_{q(\cdot)}$, we have

$$\begin{aligned}
&\||T_b^m(f\chi_t)(x).\chi_w(x)\|_{q(\cdot)}\\
&\quad\leq C2^{-wn}\|f_t\|_{q(\cdot)}\big(\|(b-b_{B_t})^m\chi_w\|_{q(\cdot)}\|\chi_t\|_{q'(\cdot)}\\
&\quad+\|((b-b_{B_t})^m\chi_t\|_{q'(\cdot)}\|\chi_t\|_{q(\cdot)}\big)\\
&\quad\leq C2^{-wn}\|f_t\|_{q(\cdot)}\big((w-t)^m\|b\|_{BMO}^m\|\chi_{B_w}\|_{q(\cdot)}\|\chi_t\|_{q'(\cdot)}\\
&\quad+\|b\|_{BMO}^m\|\chi_{B_t}\|_{q'(\cdot)}\|\chi_w\|_{q(\cdot)}\big)\\
&\quad\leq C2^{-wn}(w-t)^m\|b\|_{BMO}^m\|f_t\|_{q(\cdot)}\|\chi_{B_t}\|_{q'(\cdot)}\|\chi_{B_w}\|_{q(\cdot)}.
\end{aligned}$$

Next we use the following inequality:

$$2^{-wn}\|\chi_{B_t}\|_{q'(\cdot)}\|\chi_{B_w}\|_{q(\cdot)}\leq C2^{-wn}2^{\frac{wn}{q(0)}}2^{\frac{tn}{q'(0)}}\leq C2^{\frac{(t-w)n}{q'(0)}}. \tag{4.2}$$

For splitting E_1 we use Minkowski's inequality to obtain

$$\begin{aligned}
E_1\leq&\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^\theta\sum_{w=-\infty}^{-1}\left(\sum_{t=-\infty}^{w}\|2^{w\alpha(\cdot)}\chi_wT_b^m(f\chi_t)\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^\theta\sum_{w=0}^{k_0}\left(\sum_{t=-\infty}^{w}\|2^{w\alpha(\cdot)}\chi_wT_b^m(f\chi_t)\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
:=&E_{11}+E_{12}.
\end{aligned}$$

Applying above results to E_{11},

$$E_{11}\leq\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^\theta\sum_{w=-\infty}^{-1}\left(\sum_{t=-\infty}^{w}\|2^{w\alpha(\cdot)}\chi_wT_b^m(f\chi_t)\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}$$

$$\leq C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^\theta\sum_{w=-\infty}^{-1}2^{w\alpha(0)p(1+\epsilon)}\left(\sum_{t=-\infty}^{w}2^{\frac{(t-w)n}{q'(0)}}(w-t)^m\|b\|_{BMO}^m\|f\chi_l\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}$$

$$\leq C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^\theta\sum_{w=-\infty}^{-1}2^{w\alpha(0)p(1+\epsilon)}\left(\sum_{t=-\infty}^{w}2^{\frac{(t-w)n}{q'(0)}}(w-t)^m\|b\|_{BMO}^m\|f\chi_t\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}.$$

Let $b:=\frac{n}{q'(0)}-\alpha(0)$, again by applying Hölder's inequality, $2^{-p(1+\epsilon)}<2^{-p}$ and Fubini's theorem for series to find the estimate we get,

$$\leq C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^\theta\sum_{w=-\infty}^{-1}\left(\sum_{t=-\infty}^{w}2^{\alpha(0)t}(w-t)^m\|b\|_{BMO}^m\|f\chi_t\|_{q(\cdot)}2^{b(t-w)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}$$

$$\leq C\|b\|_{BMO}^m\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^\theta\sum_{w=-\infty}^{-1}\left(\sum_{t=-\infty}^{w}2^{\alpha(0)t}\|f\chi_t\|_{q(\cdot)}(w-t)^m2^{b(t-w)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}$$

$$\leq C\|b\|_{BMO}^m\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\Bigg[\epsilon^\theta\sum_{w=-\infty}^{-1}\left(\sum_{t=-\infty}^{w}2^{\alpha(0)p(1+\epsilon)t}\|f\chi_t\|_{q(\cdot)}^{p(1+\epsilon)}(w-t)^{mp(1+\epsilon)/2}\right)$$

$$
\begin{aligned}
&\times 2^{bp(1+\epsilon)(t-w)/2}\left(\sum_{t=-\infty}^{w}(w-t)^{m(p(1+\epsilon))'/2}2^{b(p(1+\epsilon))'(t-w)/2}\right)^{\frac{p(1+\epsilon)}{(p(1+\epsilon))'}}\Bigg]^{\frac{1}{p(1+\epsilon)}}\\
=&C\|b\|_{BMO}^{m}\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\Bigg[\epsilon^{\theta}\sum_{w=-\infty}^{-1}\sum_{t=-\infty}^{w}2^{\alpha(0)p(1+\epsilon)t}\|f\chi_t\|_{q(\cdot)}^{p(1+\epsilon)}(w-t)^{mp(1+\epsilon)/2}\\
&\qquad\times 2^{bp(1+\epsilon)(t-w)/2}\Bigg]^{\frac{1}{p(1+\epsilon)}}\\
=&C\|b\|_{BMO}^{m}\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\Bigg[\epsilon^{\theta}\sum_{t=-\infty}^{-1}2^{\alpha(\cdot)p(1+\epsilon)t}\|f\chi_t\|_{q(\cdot)}^{p(1+\epsilon)}\\
&\qquad\times\sum_{k=l}^{-1}(w-t)^{mp(1+\epsilon)/2}2^{bp(1+\epsilon)(t-w)/2}\Bigg]^{\frac{1}{p(1+\epsilon)}}\\
<&C\|b\|_{BMO}^{m}\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{t=-\infty}^{-1}2^{\alpha(0)p(1+\epsilon)t}\|f\chi_t\|_{q(\cdot)}^{p(1+\epsilon)}\sum_{k=l}^{-1}(w-t)^{mp/2}2^{bp(t-w)/2}\right)^{\frac{1}{p(1+\epsilon)}}\\
\le&C\|b\|_{BMO}^{m}\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{t=-\infty}^{-1}2^{\alpha(0)p(1+\epsilon)t}\|f\chi_t\|_{q(\cdot)}^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
=&C\|b\|_{BMO}^{m}\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{t=-\infty}^{k_0}2^{\alpha(\cdot)p(1+\epsilon)t}\|f\chi_t\|_{q(\cdot)}^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
\le&C\|b\|_{BMO}^{m}\|f\|_{M\dot{K}^{\alpha(\cdot),p),\theta}_{\lambda,q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Now for E_{12} we will again use Minkowski's inequality, consequently we have

$$
\begin{aligned}
E_{12}\le&\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{w=0}^{k_0}\left(\sum_{t=-\infty}^{-1}\|2^{w\alpha(\cdot)}\chi_w T_b^m(f\chi_t)\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
&+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{w=0}^{k_0}\left(\sum_{t=0}^{w}\|2^{w\alpha(\cdot)}\chi_w T_b^m(f\chi_t)\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
:=&A_1+A_2.
\end{aligned}
$$

As it can be easily seen that the estimate for A_2 can be checked similarly to E_{11}. For that estimate we will simply replace $q'(0)$ with q'_∞ and using the fact $b:=\frac{n}{q'_\infty}-\alpha_\infty>0$. For A_2 we have

$$
2^{-wn}\|\chi_{B_w}\|_{q(\cdot)}\|\chi_{B_t}\|_{q'(\cdot)}\le C2^{-wn}2^{\frac{wn}{q_\infty}}2^{\frac{tn}{q'(0)}}\le C2^{\frac{-wn}{q_\infty}}2^{\frac{tn}{q'(0)}},\tag{4.3}
$$

as $a_\infty-\frac{n}{q'_\infty}<0$ we have

$$
\begin{aligned}
A_1\le&C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{w=0}^{k_0}2^{w\alpha_\infty p(1+\epsilon)}\left(\sum_{t=-\infty}^{-1}\|\chi_w T_b^m(f\chi_t)\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
\le&C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\Bigg[\epsilon^{\theta}\sum_{w=0}^{k_0}2^{w\alpha_\infty p(1+\epsilon)}
\end{aligned}
$$

$$\times \left(\sum_{t=-\infty}^{-1} 2^{-wn} 2^{\frac{wn}{q_\infty}} 2^{\frac{tn}{q'(0)}} (w-t)^m \|b\|_{BMO}^m \|f\chi_t\|_{q(\cdot)} \right)^{p(1+\epsilon)} \Bigg]^{\frac{1}{p(1+\epsilon)}}$$

$$\leq C\|b\|_{BMO}^m \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \Bigg[\epsilon^\theta \sum_{w=0}^{\infty} 2^{\frac{w\alpha - wn}{q'_\infty} p(1+\epsilon)}$$

$$\times \left(\sum_{t=-\infty}^{-1} 2^{\frac{tn}{q'(0)}} \|f\chi_t\|_{q(\cdot)} (w-t)^m \right)^{p(1+\epsilon)} \Bigg]^{\frac{1}{p(1+\epsilon)}}$$

$$\leq C\|b\|_{BMO}^m \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^\theta \left(\sum_{t=-\infty}^{-1} 2^{\frac{tn}{q'(0)}} \|f\chi_t\|_{q(\cdot)} (w-t)^m \right)^{p(1+\epsilon)} \right)^{\frac{1}{p(1+\epsilon)}}$$

$$\leq C\|b\|_{BMO}^m \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^\theta \left(\sum_{t=-\infty}^{-1} 2^{\alpha(0)t} \|f\chi_t\|_{q(\cdot)} 2^{\frac{tn}{q'(0)} - \alpha(0)t} (w-t)^m \right)^{p(1+\epsilon)} \right)^{\frac{1}{p(1+\epsilon)}}.$$

By using the fact $\frac{n}{q'(0)} - \alpha(0) > 0$ and Hölder's inequality we have

$$A_1 \leq C\|b\|_{BMO}^m \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \Bigg[\epsilon^\theta \left(\sum_{t=-\infty}^{-1} 2^{\alpha(0)tp(1+\epsilon)} \|f\chi_t\|_{q(\cdot)}^{p(1+\epsilon)} \right)$$

$$\times \left(\sum_{t=-\infty}^{-1} 2^{(\frac{tn}{q'(0)} - \alpha(0)t)(p(1+\epsilon))'} (w-t)^{m(p(1+\epsilon))'} \right)^{\frac{p(1+\epsilon)}{(p(1+\epsilon))'}} \Bigg]^{\frac{1}{p(1+\epsilon)}}$$

$$\leq C\|b\|_{BMO}^m \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^\theta \left(\sum_{t=-\infty}^{k_0} 2^{\alpha(0)tp(1+\epsilon)} \|f\chi_t\|_{q(\cdot)}^{p(1+\epsilon)} \right) \right)^{\frac{1}{p(1+\epsilon)}}$$

$$\leq C\|b\|_{BMO}^m \|f\|_{M\dot{K}_{\lambda,q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)}.$$

Now we will find the estimate for E_2. Let $k \in \mathbb{Z}$ and $l \geq w+1$ and a.e. $x \in R_k$. to split E_2 we will use the Minkowski's inequality and using similar method of E_1 we have

$$E_2 \leq \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^\theta \sum_{w=-\infty}^{k_0} \left(\sum_{t=w+1}^{\infty} \|2^{w\alpha(\cdot)} \chi_w T_b^m(f\chi_t)\|_{q(\cdot)} \right)^{p(1+\epsilon)} \right)^{\frac{1}{p(1+\epsilon)}}$$

$$\leq \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^\theta \sum_{w=-\infty}^{-1} \left(\sum_{t=w+1}^{\infty} \|2^{w\alpha(\cdot)} \chi_w T_b^m(f\chi_t)\|_{q(\cdot)} \right)^{p(1+\epsilon)} \right)^{\frac{1}{p(1+\epsilon)}}$$

$$+ \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^\theta \sum_{k=0}^{k_0} \left(\sum_{t=w+1}^{\infty} \|2^{w\alpha(\cdot)} \chi_w T_b^m(f\chi_t)\|_{q(\cdot)} \right)^{p(1+\epsilon)} \right)^{\frac{1}{p(1+\epsilon)}}$$

$$:= E_{21} + E_{22}.$$

For E_{22} lemma (2.2) yields

$$2^{-tn} \|\chi_{B_w}\|_{q(\cdot)} \|\chi_{B_t}\|_{q'(\cdot)} \leq C 2^{-tn} 2^{\frac{wn}{q_\infty}} 2^{\frac{tn}{q'_\infty}} \leq C 2^{\frac{(w-t)n}{q_\infty}}. \tag{4.4}$$

We get

$$E_{22} \leq C \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^{\theta} \sum_{k=0}^{k_0} \left(\sum_{t=w+1}^{\infty} \|2^{w\alpha(\cdot)}\chi_w T_b^m(f\chi_t)\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}$$
$$\leq C\|b\|_{BMO}^m \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^{\theta} \sum_{k=0}^{k_0} \left(\sum_{t=w+1}^{\infty} (w-t)^m 2^{a_\infty t}\|f\chi_t\|_{q(\cdot)} 2^{d(w-t)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}},$$

where $\frac{n}{q_\infty} + \alpha_\infty = d > 0$. Again applying the Hölder's theorem and $2^{-p(1+\epsilon)} < 2^{-p}$ to obtain

$$E_{22} \leq C\|b\|_{BMO}^m \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left[\epsilon^{\theta} \sum_{k=0}^{k_0} \left(\sum_{t=w+2}^{\infty} 2^{t\alpha_\infty p(1+\epsilon)}\|f\chi_t\|_{q(\cdot)}^{p(1+\epsilon)} 2^{dp(1+\epsilon)(w-t)/2}\right)\right.$$
$$\left.\times \left(\sum_{t=w+2}^{\infty} 2^{d(p(1+\epsilon))'(w-t)/2}(w-t)^{m(p(1+\epsilon))'}\right)^{\frac{p(1+\epsilon)}{(p(1+\epsilon))'}}\right]^{\frac{1}{p(1+\epsilon)}}$$
$$\leq C\|b\|_{BMO}^m \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^{\theta} \sum_{k=0}^{k_0} \sum_{t=w+2}^{\infty} 2^{t\alpha_\infty p(1+\epsilon)}\|f\chi_t\|_{q(\cdot)}^{p(1+\epsilon)} 2^{dp(1+\epsilon)(w-t)/2}\right)^{\frac{1}{p(1+\epsilon)}}$$
$$\leq C\|b\|_{BMO}^m \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^{\theta} \sum_{k=0}^{k_0} \sum_{t=w+2}^{\infty} \sum_{j=-\infty}^{t} 2^{j\alpha_\infty p(1+\epsilon)}\|f\chi_j\|_{q(\cdot)}^{p(1+\epsilon)} 2^{dp(1+\epsilon)(w-t)/2}\right)^{\frac{1}{p(1+\epsilon)}}$$
$$\leq C\|b\|_{BMO}^m \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^{\theta} \sum_{k=0}^{k_0} \sum_{t=w+2}^{\infty} 2^{dp(1+\epsilon)(w-t)/2}\right)^{\frac{1}{p(1+\epsilon)}} \|f\|_{M\dot{K}_{\lambda,q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)}$$
$$\leq C\|b\|_{BMO}^m \|f\|_{M\dot{K}_{\lambda,q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)}.$$

For E_{21} by using Minkowski's inequality

$$E_{21} \leq \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^{\theta} \sum_{w=-\infty}^{-1} \left(\sum_{t=w+1}^{\infty} \|2^{w\alpha(\cdot)}\chi_w T_b^m(f\chi_t)\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}$$
$$E_{21} \leq \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^{\theta} \sum_{w=-\infty}^{-1} \left(\sum_{t=w+1}^{-1} \|2^{w\alpha(\cdot)}\chi_w T_b^m(f\chi_t)\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}$$
$$+ \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^{\theta} \sum_{w=-\infty}^{-1} \left(\sum_{t=0}^{\infty} \|2^{w\alpha(\cdot)}\chi_w T_b^m(f\chi_t)\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}$$
$$:= B_1 + B_2.$$

We can easily find the estimate for B_1 by similar way to E_{22}. We will simply replace q_∞ with $q(0)$ and the inequality $\frac{n}{q(0)} + \alpha(0) > 0$. For B_2 we have

$$2^{-tn}\|\chi_w\|_{q(\cdot)}\|\chi_t\|_{L^{q'(\cdot)}} \leq C 2^{-tn} 2^{\frac{wn}{q(0)}} 2^{\frac{tn}{q'\infty}} \leq C 2^{\frac{wn}{q(0)}} 2^{\frac{-tn}{q_\infty}}, \tag{4.5}$$

$$
\begin{aligned}
B_2 \leq &\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{w=-\infty}^{-1}2^{w\alpha(0)p(1+\epsilon)}\left(\sum_{t=0}^{\infty}\|\chi_w T_b^m(f\chi_t)\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
\leq &C\|b\|_{BMO}^m \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{w=-\infty}^{-1}2^{w\alpha(0)p(1+\epsilon)}\right.\\
&\qquad\times\left.\left(\sum_{t=0}^{\infty}2^{\frac{-tn}{q_\infty}}2^{\frac{wn}{q(0)}}\|f\chi_t\|_{q(\cdot)}(w-t)^m\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
\leq &C\|b\|_{BMO}^m \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{w=-\infty}^{-1}2^{w(\frac{n}{q(0)}+\alpha(0))p(1+\epsilon)}(w-t)^m\right)\\
&\qquad\times\left.\left(\sum_{t=0}^{\infty}2^{\frac{-tn}{q_\infty}}\|f\chi_t\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
\leq &C\|b\|_{BMO}^m \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^{\theta}\left(\sum_{t=0}^{\infty}2^{\frac{-tn}{q_\infty}}\|f\chi_t\|_{q(\cdot)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
\leq &C\|b\|_{BMO}^m \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^{\theta}\left(\sum_{t=0}^{\infty}2^{\alpha_\infty t}\|f\chi_t\|_{q(\cdot)}2^{-t(\frac{n}{q_\infty}+\alpha_\infty)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\\
\leq &C\|b\|_{BMO}^m \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^{\theta}\left(\sum_{t=0}^{\infty}\sum_{j=-\infty}^{t}2^{\alpha_\infty j}\|f\chi_j\|_{q(\cdot)}2^{-t(\frac{n}{q_\infty}+\alpha_\infty)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}.
\end{aligned}
$$

Now by applying Hölders inequality and using the fact that $\frac{n}{q(\infty)}+\alpha(\infty)>0$ we have

$$
\begin{aligned}
B_2 \leq &C\|b\|_{BMO}^m \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^{\theta}\left(\sum_{t=0}^{\infty}2^{-t(\frac{n}{q_\infty}+\alpha_\infty)}\right)^{p(1+\epsilon)}\right)^{\frac{1}{p(1+\epsilon)}}\|f\|_{M\dot{K}^{\alpha(\cdot),p),\theta}_{\lambda,q(\cdot)}(\mathbb{R}^n)}\\
\leq &C\|b\|_{BMO}^m\|f\|_{M\dot{K}^{\alpha(\cdot),p),\theta}_{\lambda,q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Combining the estimates for E_1 and E_2 yields

$$\|T_b^m f\|_{M\dot{K}^{\alpha(\cdot),p),\theta}_{\lambda,q(\cdot)}(\mathbb{R}^n)}\leq C\|b\|_{BMO}^m\|f\|_{M\dot{K}^{\alpha(\cdot),p),\theta}_{\lambda,q(\cdot)}(\mathbb{R}^n)},$$

which ends the proof. □

5. Conclusion

In this paper we used the idea of grand variable Herz-Morrey spaces to establish the boundedness of higher order commutators of sublinear operators in these spaces.

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