SOME ELEMENTARY REMARKS ON THE STRUCTURE OF COMPLETE TOTALLY ORDERED ABELIAN GROUPS

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Dedicated to the memory of our Teacher, Dr. Doc. Nicolae Popescu

Abstract. In this note we make some elementary remarks on complete totally ordered Abelian groups, and we prove in detail that such a group is isomorphic to the additive ordered group ($\mathbb{R}$,+, <). Some other additional results are given.

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1. Introduction, Notations, Definitions and Some Basic Results

In this paper we make some elementary remarks on the structure of complete (in the Dedekind meaning) ordered Abelian groups. Some basic interesting facts on Abelian totally ordered groups one can also find in [3].

An Abelian group $(G, \circ)$ is said to be a *totally ordered group* if there exists a total order relation "$\prec$" on it such that for any $x, y, z \in G$ with $x \prec y$, we have $x \circ z \prec y \circ z$. An abelian totally ordered group is *complete* if any upper bounded subset S of G has a least upper bound $\sup S$ in G. We say that G is *dense in itself* if for any $x, y \in G$ with $x \prec y$, there exists $w \in G$, such that $x \prec w \prec y$.

In the following we always denote by $G = (G, \circ, \prec)$ a complete totally ordered abelian group.

In this note we give all the auxiliary results (Section 1) to supply an elementary and self contained proof of the fact that any complete totally ordered abelian group $(G, \circ, \prec)$ is isomorphic and homeomorphic with the additive ordered group of real numbers $(\mathbb{R}, +, <)$ (Theorem 2.2). As a consequence we prove that any two complete totally ordered abelian groups are isomorphic and homeomorphic (Corollary 2.3).

Let us denote by e the unity element of G and, for any $x \in G$, we denote by x' the symmetric of x in G. An open interval (x, y), where $x, y \in G$, $x \prec y$, is the subset $\{z \in G : x \prec z \prec y\}$ of G. We similarly define $[x, y)$, $(x, y]$ and $[x, y]$. Since the mapping $x \to x'$ is strictly decreasing, a totally ordered abelian group $(G, \circ, \prec)$, dense in itself, is complete if and only if any lower bounded subset L of G has a greatest lower bound $\inf L$ in G.

A subset D of G is said to be *open* in G if for any $z \in D$, there exist $x, y \in G$, $x \prec y$, such that $(x, y) \subset D$ and $z \in (x, y)$. It is not difficult to see that all open subsets of G generate a topological group structure on G. We call this topology, the *interval topology.* It is easy to see that the set $\mathcal{V}_e = \{(\varepsilon', \varepsilon) : \varepsilon \succ e, \varepsilon \in G\}$ of open symmetric intervals which contain e is a fundamental system of neighborhoods of e. Then,

$$\mathcal{V}_a = \{(\varepsilon' \circ a, \varepsilon \circ a) : \varepsilon \succ e, \varepsilon \in G\}$$

is a fundamental system of neighborhoods of an arbitrary element a in G. In fact, $\mathcal{V}_a$ is the translation $\mathcal{V}_e \circ a$ of $\mathcal{V}_e$ in a. A mapping $f : G \to G$ is *continuous* at $a \in G$ if for any $\varepsilon \in G$, $\varepsilon \succ e$, there exists $\eta \in G$,

$\eta \succ e$, such that for any $x \in (\eta' \circ a, \eta \circ a)$, we have $f(x) \in (\varepsilon' \circ f(a), \varepsilon \circ f(a))$. If f is continuous at any point a of G, we say that f is *continuous on* G. It is not difficult to see that a function $f : G \to G$ is continuous if and only if for any open subset D of G, there exists an open subset E of G, such that $f(E) \subset D$. A mapping $g : G \times G \to G$ is continuous at $(a, b) \in G \times G$ if for any $\varepsilon \in G$, $\varepsilon > e$, there exist $\eta, \delta \in G$, $\eta, \delta \succ e$, such that for any $(x, y) \in ((\eta' \circ a, \eta \circ a), (\delta' \circ b, \delta \circ b)$, we have $g(x, y) \in (\varepsilon' \circ g(a, b), \varepsilon \circ g(a, b))$. Here we used the so called *product topology* on G. Namely, this last one is generated by the product subsets $D \times E$ of $G \times G$, where D and E are open subsets in G. Thus, a function $g : G \times G \to G$ is continuous if and only if for any open subset D of G, there exist two open subsets E and F of G, such that $g(E \times F) \subset D$.

In the following, for simplicity, if $w \in G$, we denote $\underbrace{w \circ w \circ \cdots \circ w}_{n-times}$ by $w^{\circ n}$.

Lemma 1.1. *With the above notation and hypotheses, the mapping $h : G \to G$, $h(x) = x'$ is a strictly decreasing continuous automorphism of G.*

Proof. Let x, y be in G such that $x \prec y$. Then, $e = x \circ x' \prec y \circ x'$, so $y' \prec x'$. Thus, h, h^{-1} are strictly decreasing group morphisms. Since h is a group morphism, it is sufficient to prove its continuity at e. Let ε be in G, with $e \prec \varepsilon$. Then $h((\varepsilon', \varepsilon)) = (\varepsilon', \varepsilon)$, so h is also a continuous mapping. □

Lemma 1.2. *Let $(G, \circ, \prec)$ be a totally ordered group which is dense in itself. Then, a) For any $x \succ e$, there exists $y \in G$, with $x \succ y \succ e$ and $x \succ y \circ y \succ e$. Moreover, for any $n \in \{2, 3, \cdots\}$, there exists $w \in G$, with $x \succ w \succ e$ and $x \succ w^{\circ 2^n} \succ e$. b) The mapping $g : G \times G \to G$, $g(x, y) = x \circ y$ is continuous relative to the above interval topology. Thus, $(G, \circ, \prec)$ is a topological group with respect to its interval topology.*

Proof. a) Let z be in (e, x), that is $e \prec z \prec x$. If $z \circ z \in (e, x)$, we write $y = z$, and we are done. If $z \circ z \succ x$, we see that $e \prec (x \circ z') \circ (x \circ z') \prec x$, because $x \circ z' \succ e$. Since $x \circ z' \prec x$, we can take $y = x \circ z'$. If $z \circ z = x$, we substitute z with $y \in (z, x)$. Then, $y \circ w \succ z \circ z = x$, etc. To prove the last assertion, we take $w_1 \in (e, x)$ such that $w_1 \circ w_1 \in (e, x)$ (see a)). Now, we take w_1 instead of x and find $w_2 \in (e, w_1)$ such that $w_2 \circ w_2 \in (e, w_1)$. Thus, $w_2^{\circ 4} \in (e, w_1 \circ w_1) \subset (e, x)$. We continue in this way and find $w_1, w_2, \cdots, w_k \cdots$ such that $w_k^{\circ 2^k} \in (e, x)$ for any $k = 1, 2, \cdots$.

b) It is sufficient to prove the continuity of g at (e, e). Let η be in G, $\eta \succ e$, and let us try to find $\varepsilon \succ e$, such that $g((\varepsilon', \varepsilon) \times (\varepsilon', \varepsilon)) \subset (\eta', \eta)$. It is sufficient to take $\varepsilon \in G$ with $e \prec \varepsilon \circ \varepsilon \prec \eta$ (see a)). Indeed, if $(x, y) \in ((\varepsilon', \varepsilon) \times (\varepsilon', \varepsilon))$, then

$$\eta' \prec \varepsilon' \circ \varepsilon' \prec x \circ y \prec \varepsilon \circ \varepsilon \prec \eta,$$

so g is continuous at (e, e). Hence, this lemma and the previous one say that $(G, \circ, \prec)$ is a topological group relative to its interval topology. □

Remark. If $(G, \circ, \prec)$ is a totally ordered Abelian group which is not dense in itself, then $(G, \circ, \prec)$ is isomorphic to $(\mathbb{Z}, +, <)$ (see Corollary 2.4), so it is also a topological group with the discrete topology.

Definition. We say that a sequence $\{x_n\}$, $x_n \in G$ for any $n = 0, 1, \cdots$ is *convergent to* $x \in G$ if for any interval (ε, η) which contains x, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, we have $x_n \in (\varepsilon, \eta)$.

For instance, any increasing (decreasing) upper (lower) bounded sequence $\{x_n\}$ in G, is convergent to the least upper bound (greater lower bound) of the set $\{x_n : n = 0, 1, \cdots\}$.

Lemma 1.3. [1] *Let $(G, \circ, \prec)$ be a complete totally ordered non-trivial Abelian group and let a be an element in G, with $e \prec a$. Then, the sequence $\{a^{\circ n}\}_n$, where $a^{\circ n} = a \circ a \circ \cdots \circ a$, n-times, is a strictly increasing not upper bounded sequence. Similar, if $b \prec e$, then the sequence $\{b^{\circ n}\}_n$ is a strictly decreasing not lower bounded sequence.*

Proof. Since $e \prec a$, we see that $a \prec a \circ a$, $a \circ a \prec a \circ a \circ a$, etc. So the sequence $\{a^{\circ n}\}_n$ is strictly increasing. Let us assume that $\{a^{\circ n}\}_n$ is upper bounded by $\gamma \in G$. So, $A = \sup\{a^{\circ n}\}_n$ is in G and, since the mapping $x \to a \circ x$ is continuous, we see that $a \circ A = A$, thus $a = e$, a contradiction. For the last statement, we take b', the symmetric of b, and we consider the sequence $\{b'^{\circ n}\}_n$, which is strictly increasing (see Lemma 1.1 and the first statement), so it cannot be upper bounded. Thus $\{b^{\circ n}\}_n$ cannot be lower bounded, otherwise, if δ is a lower bound of $\{b^{\circ n}\}_n$, then δ' is an upper bound for the strictly increasing sequence $\{b'^{\circ n}\}_n$, a contradiction from the first statement. ☐

Proposition 1.4. *Let $(G_1, \circ, \prec)$ and $(G_2, *, <)$ be two complete totally ordered Abelian groups which are dense in themselves, and let $f : G_1 \to G_2$ be a strictly increasing continuous (relative to their corresponding orders) mapping. Then, for any $a, b \in G_1$, $a \prec b$, we have $f((a, b)) = (f(a), f(b))$, that is f carries the open interval (a, b) of G_1 into the open interval $(f(a), f(b))$ of G_2.*

Proof. Let us take λ in G_2 with $f(a) < \lambda < f(b)$, and let us consider the following non-empty subsets in G_1 :

$$A^- = \{x \in [a, b] : f(x) \leq \lambda\}, \text{ and } A^+ = \{x \in [a, .b] : f(x) \geq \lambda\}.$$

We see that $A^- \cup A^+ = [a, b]$, and A^- and A^+ are bounded in G_1. In addition, $c_- = \sup A^- \preceq c_+ = \inf A^+$. Let us assume that $c_- \prec c_+$ and let us take $\xi \in (c_-, c_+)$ in G_1. Thus, $f(\xi) > \lambda$ and $f(\xi) < \lambda$, a contradiction. So, $c_- = c_+ = c$ in G_1. We prove that $f(c) = \lambda$. Otherwise, we suppose that $f(c) < \lambda$. We notice that $f(a) < f(c)$, otherwise $a = c$ and f was not continuous at $x = a$. Since $f(a) < f(c) < \lambda$, we can take $\delta, \mu \in G_2$ such that

$$f(a) < \delta < f(c) < \mu < \lambda.$$

Since f is continuous at $x = c$, there exist ε, η in G_1 such that $a < \varepsilon < c < \eta < b$ and $f((\varepsilon, \eta)) \subset (\delta, \mu)$. In particular, $f(\eta) \leq \mu < \lambda$, so $\eta \in A^-$, a contradiction, because $c = \sup A^-$ and $c < \eta$. We can similarly prove that the assumption $f(c) > \lambda$ is impossible. Thus $f(c) = \lambda$ and the proof is complete. ☐

Proposition 1.5. *Let $(G, \circ, \prec)$ be a non-trivial complete totally ordered Abelian group, and let $(H, \circ, \prec)$ be a non-trivial subgroup of it such that for any $x, y \in H$, $x \prec y$, the entire interval (x, y) (in G) is contained in H. Then $H = G$.*

Proof. Let us assume that $H \neq G$, that is there exists ξ in $G \setminus H$. We can suppose that $\xi \succ e$, otherwise we substitute ξ with ξ', the symmetric of ξ. We take now $a \in H$, $a \succ e$ and, from Lemma 1.3, we see that there exists $m \in \mathbb{N}^*$ such that $a^{\circ m} \succ \xi$, that is $\xi \in (e, a^{\circ m}) \subset H$, a contradiction. Thus, $H = G$. ☐

Proposition 1.6. *Let $(G, \circ, \prec)$ be a non-trivial complete totally ordered Abelian group, let u be an element in G and let n be a non-zero natural number. Then, the equation $x^{\circ n} = u$ has a unique solution in G. Here, as above, $x^{\circ n} = \underbrace{x \circ x \circ \cdots \circ x}_{n-times}$.*

Proof. We can assume that $u \succ e$, otherwise, we take u' and we consider the equation $y^{\circ n} = u'$, with $y = x'$, etc. It is sufficient to prove that the mapping $g_n : G \to G$, $g_n(x) = x^{\circ n}$ is a strictly increasing continuous automorphism of G. For $n = 1$, $g_1(x) = x$ is obviously strictly increasing. We assume that g_m is strictly increasing and we take $x \prec y$ in G. We see that $g_m(x) \prec g_m(y)$ implies

$$g_{m+1}(x) = g_m(x) \circ x \prec g_m(y) \circ x \prec g_m(y) \circ y = g_{m+1}(y).$$

So, g_{m+1} is also strictly increasing, that is g_n is strictly increasing for any $n = 1, 2, ...$ Since the mapping $(x, y) \to x \circ y$ is continuous (see Lemma 1.2, b) and the Remark from this section) we easily prove by mathematical induction on n that g_n is continuous for any $n = 1, 2, ...$ From Proposition 1.4 we see that $g_n((x, y)) = (g_n(x), g_n(y))$ for any $x \prec y$. Now, we consider the subgroup $H = g_n(G)$ and apply Proposition 1.5 to find that $H = G$, that is g_n is also onto. Since G is an abelian group, g_n is obviously an automorphism of ordered groups and so, the proof is complete. Since g_n is strictly increasing, we see that the solution x_n of the equation $x^{\circ n} = u$ is unique in G. ☐

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2. The main results

In the following we keep unchanged the notation, definitions and hypotheses given in Section 1.

Theorem 2.1. *Let $(G, \circ, \prec)$ be a non-trivial complete totally ordered Abelian group which is dense in itself, and let $(\mathbb{Q}, +, <)$ be the usual totally ordered additive group of rational numbers. Then, for any fixed element $u \succ e$ in G, there exists a unique continuous strictly increasing (one-to-one) group morphism $f_u^* : (\mathbb{Q}, +, <) \to (G, \circ, \prec)$, such that $f_u^*(1) = u$.*

Proof. (Here we use a basic idea of C.N. Beli [2] for the construction of f_u^*). First of all we define $f_u : (\mathbb{Z}, +, <) \to (G, \circ, \prec)$ by $f_u(n) = u^{\circ n}$, if $n > 0$, $f_u(n) = (u')^{\circ(-n)}$, if $n < 0$, and $f_u(0) = e$. It is not difficult to prove that f_u is a strictly increasing group morphism. Indeed, for instance, let us take $n > m > 0$ and let us prove that $f_u(n - m) = f_u(n) \circ [f_u(m)]'$:

$$f_u(n - m) = u^{\circ(n-m)} = u^{\circ n}(u')^{\circ m} = u^{\circ n}\,(u^{\circ m})' = f_u(n) \circ [f_u(m)]' .$$

Now, for $n > m > 0$, we see that $f_u(n - m) \succ e$, because

$$e \prec u \prec u \circ u \prec \cdots$$

So, $f_u(n) \circ [f_u(m)]' \succ e$, or, multiplying both sides in G by $[f_u(m)]$, we get $f_u(n) \succ f_u(m)$. The other situations of n, m in $\mathbb{Z}$ can be reduced to $n > m > 0$ by eventually considering $-n$, $-m$ instead of n and m.

Now we extend this mapping f_u to a strictly increasing continuous group morphism $f_u^* : (\mathbb{Q}, +, <) \to (G, \circ, \prec)$, such that $f_u^*(1) = u$. For any $n = 2, 3, ...$ we write $f_u^*\left(\frac{1}{n}\right) = x_n \in G$, where x_n is the unique solution of the equation $x^{\circ n} = u$ in G (see Proposition 1.6). Moreover, we also define: $f_u^*\left(\frac{m}{n}\right) = x_n^{\circ m}$ and $f_u^*\left(-\frac{m}{n}\right) = (x_n')^{\circ m}$ for $m = 2, 3, ...$ First of all we have to prove that the definition is correct. Namely, let $\frac{m}{n} = \frac{p}{q}$, where $m, n, p, q \in \mathbb{N}^*$, $n, q > 0$. So, $mq = np$, and we want to prove that $x_n^{\circ m} = x_q^{\circ p}$. We are in a group, so we can simplify with respect to the group law "$\circ$". Thus,

$$x_n^{\circ m} = x_q^{\circ p} \Leftrightarrow x_n^{\circ mq} = x_q^{\circ pq} \Leftrightarrow x_n^{\circ np} = x_q^{\circ pq} \Leftrightarrow (x_n^{\circ n})^p = \left(x_q^{\circ q}\right)^p \Leftrightarrow u^p = u^p,$$

and so, the definition of f_u^* is correct. Now we prove that the mapping f_u^* is a strictly increasing group morphism. Let $\frac{m}{n}$, $\frac{a}{b}$ be two rational numbers such that $\frac{m}{n} < \frac{a}{b}$. We can assume that $n = b > 0$ and that even m is a positive integer, otherwise we use x_n' instead of x_n. Since, in this last case, $m < a$, we can write: $f_u^*(\frac{m}{n}) = x_n^{\circ m} \prec x_n^{\circ a} = f_u^*(\frac{a}{b})$, because $x_n \succ e$. In addition, in general, if $m > 0$, $a > 0$, we have:

$$f_u^*\left(\frac{m}{n} + \frac{a}{n}\right) = f_u^*\left(\frac{m+a}{n}\right) = x_n^{\circ(m+a)} = x_n^{\circ m} \circ x_n^{\circ a} = f_u^*\left(\frac{m}{n}\right) \circ f_u^*\left(\frac{a}{n}\right).$$

If $m > 0$, $a < 0$, we can write:

$$f_u^*\left(\frac{m}{n} + \frac{a}{n}\right) = f_u^*\left(\frac{m}{n} - \frac{-a}{n}\right) = x_n^{\circ m} \circ \left(x_n^{\circ(-a)}\right)' = f_u^*\left(\frac{m}{n}\right) \circ f_u^*\left(\frac{a}{n}\right).$$

The case $m < 0$, $a > 0$ is symmetric with the previous one, because G is an abelian group.

If $m < 0$, $a < 0$, we can write:

$$f_u^*\left(\frac{m}{n} + \frac{a}{n}\right) = (x_n')^{\circ(-m-a)} = (x_n')^{\circ(-m)} \circ (x_n')^{\circ(-a)} = f_u^*\left(\frac{m}{n}\right) \circ f_u^*\left(\frac{a}{n}\right).$$

Hence, f_u^* is a group morphism.

We prove now that the mapping f_u^* is continuous. Since f_u^* is a group morphism, it is sufficient to prove the continuity of f_u^* at 0. For this, we take $\eta \succ e$ and a η-neighborhood (η', η) of $f_u^*(0) = e$. Since f_u^* is strictly increasing, we see that it is sufficient to prove that $f_u^*\left(\frac{1}{n}\right) = x_n \in (e, \eta)$ for any $n \geq n_0$. Since $\eta \succ e$, the sequence $\{\eta^{\circ n}\}_n$ is not upper bounded in G (see Lemma 1.3). Thus, there exists $n_0 \in \mathbb{N}^*$ such that if $n \geq n_0$, we have $u \prec \eta^{\circ n}$. So, $u = x_n^{\circ n} \prec \eta^{\circ n}$, that is $x_n \prec \eta$ for any $n \geq n_0$. If we change $\frac{1}{n}$ with $-\frac{1}{n}$, we have to change x_n with x_n' and η with η'. Hence, f_u^* is a strictly increasing continuous group morphism. □

Theorem 2.2. *With the notation and hypotheses of Theorem 2.1, the strictly increasing continuous group morphism $f_u^*:(\mathbb{Q},+<)\to(G,\circ,\prec)$ can be uniquely extended up to a strictly increasing continuous group isomorphism $\widetilde{f}_u:(\mathbb{R},+<)\to(G,\circ,\prec)$, where $(\mathbb{R},+,<)$ is the usual totally ordered additive group of real numbers. Moreover, $\widetilde{f}_u^{-1}:(G,\circ,\prec)\to(\mathbb{R},+<)$ is also a strictly increasing continuous group isomorphism, that is $\widetilde{f}_u$ is also a homeomorphism of abelian ordered groups relative to the interval topologies induced by their corresponding orders.*

Proof. Let r be a real number and let us define two subsets of rational numbers in connection with r:

$$A_r^-=\{q\in\mathbb{Q}:q\le r\}\text{ and }A_r^+=\{p\in\mathbb{Q}:p\ge r\}.$$

Thus, $r=\sup A_r^-=\inf A_r^+$. We see that any $p\in A_r^+$ is an upper bound for A_r^-, and any $q\in A_r^-$ is a lower bound for A_r^+. Since f_u^* is strictly increasing, we see that the set $f_u^*(A_r^-)$ is upper bounded by $f_u^*(p)$ for any $p\in A_r^+$, and $f_u^*(A_r^+)$ is lower bounded by $f_u^*(q)$ for any $q\in A_r^-$. Since G is complete, we see that there exist $\widetilde{f}_u^-(r)=\sup f_u^*(A_r^-)$, and $\widetilde{f}_u^+(r)=\inf f_u^*(A_r^+)$ in G and $\widetilde{f}_u^-(r)\preceq\widetilde{f}_u^+(r)$. We shall see later that $\widetilde{f}_u^-(r)=\widetilde{f}_u^+(r)$, but, for the moment we define: $\widetilde{f}_u(r)=\widetilde{f}_u^-(r)=\sup f_u^*(A_r^-)$. It is clear that $\widetilde{f}_u(r)=f_u^*(r)$ if r is a rational number. If $r_1<r_2$, we can find three rational number s_1,s and s_2, such that $r_1<s_1<s<s_2<r_2$. Thus,

$$\widetilde{f}_u(r_1)\preceq\widetilde{f}_u(s_1)\prec\widetilde{f}_u(s)\prec\widetilde{f}_u(s_2)\preceq\widetilde{f}_u(r_2),$$

and so, $\widetilde{f}_u$ is strictly increasing.

We prove now that $\widetilde{f}_u$ is a group morphism. For this, it is sufficient to prove that for any $r_1,r_2\in\mathbb{R}$,

$$\sup A_{r_1+r_2}^-=\sup A_{r_1}^-\circ\sup A_{r_2}^-.$$

We take $q_1\in A_{r_1}^-$ and $q_2\in A_{r_2}^-$, so $q_1+q_2\in A_{r_1+r_2}^-$ and

$$f_u^*(q_1+q_2)=f_u^*(q_1)\circ f_u^*(q_2)\preceq\widetilde{f}_u(r_1)\circ\widetilde{f}_u(r_2).$$

Hence,

$$\widetilde{f}_u(r_1+r_2)\preceq\widetilde{f}_u(r_1)\circ\widetilde{f}_u(r_2).\tag{2.1}$$

Conversely, we take an arbitrary $\varepsilon\in G$, $\varepsilon\succ e$, and we chose $q_1,q_2\in\mathbb{Q}$, such that $q_1\le r_1$, $q_2\le r_2$, $\widetilde{f}_u(r_1)\circ\varepsilon'\preceq f_u^*(q_1)\preceq\widetilde{f}_u(r_1)$ and $\widetilde{f}_u(r_2)\circ\varepsilon'\preceq f_u^*(q_2)\preceq\widetilde{f}_u(r_2)$. Thus,

$$\widetilde{f}_u(r_1)\circ\widetilde{f}_u(r_2)\circ\varepsilon'\circ\varepsilon'\preceq f_u^*(q_1+q_2)\preceq\widetilde{f}_u(r_1+r_2).\tag{2.2}$$

Since the mapping $x\to x\circ x$ is continuous at e (see Lemma 1.2), we see that if $\varepsilon_n'\nearrow e$ (see the Definition from the first section), then $\varepsilon_n'\circ\varepsilon_n'\nearrow e$. So, from (2.2), we find that

$$\widetilde{f}_u(r_1)\circ\widetilde{f}_u(r_2)\preceq\widetilde{f}_u(r_1+r_2).\tag{2.3}$$

Thus, (2.1) and (2.3) implies that $\widetilde{f}_u$ is a group morphism.

Let us prove now that $\widetilde{f}_u$ is a continuous mapping. Since $\widetilde{f}_u$ is a group morphism, it is sufficient to prove that $\widetilde{f}_u$ is continuous at $r=0$. But $\widetilde{f}_u(0)=f_u^*(0)=e$, so let us take $\eta\in G$, $\eta\succ e$ and let us try to find $\varepsilon\in\mathbb{Q}_+$ such that $\widetilde{f}_u((-\varepsilon,\varepsilon))\subset(\eta',\eta)$, where $(-\varepsilon,\varepsilon)$ is considered as an open interval in $\mathbb{R}$. It is sufficient to find such an ε with $\widetilde{f}_u(0,\varepsilon)\subset(e,\eta)$. Let us take $\eta_1\in G$ with $e\prec\eta_1\prec\eta$. Since $f_u^*:(\mathbb{Q},+,<)\to(G,\circ,\prec)$ is continuous, there exists an $\varepsilon\in\mathbb{Q}_+$ such that $f_u^*((0,\varepsilon)\cap\mathbb{Q})\subset(e,\eta_1)$. Now we take $r\in(0,\varepsilon)\subset\mathbb{R}$, and a sequence $q_n\nearrow r$, $q_n\in(0,\varepsilon)\cap\mathbb{Q}$. By the definition of $\widetilde{f}_u(r)$, we see that $f_u^*(q_n)=\widetilde{f}_u(q_n)\nearrow\widetilde{f}_u(r)$. But $e\prec\widetilde{f}_u(q_n)\preceq\eta_1\prec\eta$ for any $n=1,2,...$ So, $e\prec\widetilde{f}_u(r)\preceq\eta_1\prec\eta$, otherwise, there is $m\in\mathbb{N}^*$ with $f_u^*(q_m)\succ\eta_1$, a contradiction. If $r\in(-\varepsilon,0)\subset\mathbb{R}$, since $\widetilde{f}_u(r)=\left(\widetilde{f}_u(-r)\right)'$, we see that $\widetilde{f}_u((-\varepsilon,0))\subset(\eta',e)$. Hence, $\widetilde{f}_u$ is also a continuous one-to-one group morphism.

From Proposition 1.4 we see that $\widetilde{f}_u((a,b))=\left(\widetilde{f}_u(a),\widetilde{f}_u(b)\right)$ for any $a,b\in\mathbb{R}$, $a<b$. This means that the subgroup $H=\widetilde{f}_u(\mathbb{R})$ of G has the property given in Proposition 1.5. Therefore, $\widetilde{f}_u(\mathbb{R})=G$ (see

Proposition 1.5), and so $\widetilde{f}_u : (\mathbb{R}, + <) \to (G, \circ, \prec)$ is a strictly increasing continuous group isomorphism. For the second statement, since $\widetilde{f}_u^{-1}$ is also a group morphism, it is sufficient to prove its continuity at e. Moreover, since $\widetilde{f}_u$ and $\widetilde{f}_u^{-1}$ are strictly increasing mappings, it is sufficient to prove that for any $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there exists $\eta \in G$, $\eta \succ e$, such that $\widetilde{f}_u^{-1}((e, \eta)) \subset (0, \varepsilon)$. We assume contrary, namely, that there exists a sequence $\{\eta_n\}$, $\eta_n \in G$, $\eta_n \succ e$ for any $n = 1, 2, ...$, such that $\eta_n \searrow e$ and $\widetilde{f}_u^{-1}(\eta_n) \geq \varepsilon$, that is $\eta_n \succeq \widetilde{f}_u(\varepsilon) \succ e$ for any $n = 1, 2, ...$, a contradiction with $\eta_n \searrow e$. Hence, $\widetilde{f}_u^{-1}$ is also a continuous mapping, and the proof of the theorem is now complete. ☐

Corollary 2.3. *Let $(G_1, \circ, \prec_1)$ and $(G_2, *, \prec_2)$ be two complete totally ordered Abelian groups which are dense in themselves, and e_1, e_2 be their unity elements. Let $u_1 \in G_1$, $u_1 \succ_1 e_1$ and $u_2 \in G_2$, $u_2 \succ_2 e_2$ be two fixed elements in G_1 and G_2 respectively. Then there exists a unique strictly increasing bicontinuous group isomorphism $f_{u_1,u_2} : (G_1, \circ, \prec_1) \to (G_2, *, \prec_2)$, such that $f_{u_1,u_2}(u_1) = u_2$.*

Proof. From Theorem 2.2 we can construct two strictly increasing bicontinuous group isomorphisms,

$$\widetilde{f}_{u_1} : (\mathbb{R}, +, <) \to (G_1, \circ, \prec_1), \text{ and } \widetilde{f}_{u_2} : (\mathbb{R}, +, <) \to (G_2, \circ, \prec_2).$$

So, $f_{u_1,u_2} = \widetilde{f}_{u_2} \circ \widetilde{f}_{u_1}^{-1}$ is the required isomorphism. ☐

Remark. If in the above considerations, we replace $u \succ e$ with $v \prec e$, $v \in G$, we get strictly decreasing bicontinuous group isomorphisms $f_v^*, \widetilde{f}_v, f_{v_1,v_2}$, where $v_1 \prec e_1$ and $v_2 \prec e_2$.

Corollary 2.4. *Any complete totally ordered Abelian group $(G, \circ, \prec)$ is isomorphic to $(\mathbb{Z}, +, <)$ or to $(\mathbb{R}, +, <)$, where "$<$" is the usual total order relation on $\mathbb{Z}$ and on $\mathbb{R}$ respectively.*

Proof. *a*) First of all we assume that $(G, \circ, \prec)$ has a minimal positive element $c \succ e$, where e is the unity element in G. Let us take another positive element a in G. We want to prove that there exists $n \in \mathbb{N}$ such that $a = c^{\circ n} = c \circ c \circ \cdots \circ c$, n-times. If $a = c$, we are done; if $a \succ c$, let us take the largest natural number n such that $c^{\circ n} \preceq a$. If $c^{\circ n} = a$, we are done; if $c^{\circ n} \prec a$, then $a \circ c^{\circ(-n)} \succ e$ and $a \circ c^{\circ(-n)} \prec c$, otherwise, $a \circ c^{\circ(-n)} \succeq c$, or $a \succeq c^{\circ(n+1)}$, a contradiction with the maximality of n with $c^{\circ n} \preceq a$. Thus, $a = c^{\circ n}$. If $b \prec e$, we consider its symmetric $b' \succ e$ and take $m \in \mathbb{N}$ such that $b' = c^{\circ m}$. So, $b = c^{\circ(-m)}$, that is for any element x in G, there exists a unique $k \in \mathbb{Z}$ so that $x = c^{\circ k}$. Now, it is not difficult (see the proof of Theorem 2.1) to see that the mapping $x \to k$ is an isomorphism of ordered groups between $(G, \circ, \prec)$ and $(\mathbb{Z}, +, <)$.

b) Now we assume that $(G, \circ, \prec)$ has no positive minimal element, that is $(G, \circ, \prec)$ is dense in itself. In this last situation we directly apply Theorem 2.2 and find that $(G, \circ, \prec)$ is isomorphic to $(\mathbb{R}, +, <)$. ☐

Remark. Let $(G, \circ, \prec)$ be a totally ordered Abelian group which has no minimal positive element, that is it is dense in itself. If $(G, \circ, \prec)$ is not complete, we cannot conclude that it is isomorphic to $(\mathbb{R}, +, <)$. Indeed, we take for instance $G = \mathbb{Q} \times \mathbb{Q}$ with the lexicographic order " $\prec$ " induced by the usual order " $<$ " of $\mathbb{Q}$. It is clear that $(G, +, \prec)$ is a totally ordered Abelian group which has no minimal positive element and it is not complete, so it cannot be isomorphic to $(\mathbb{R}, +, <)$.

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