ON SOME CHARACTER HOMOLOGICAL PROPERTIES OF GENERALIZED MODULE EXTENSION BANACH ALGEBRAS

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Abstract. We consider several homological properties associated to characters of generalized module extension Banach algebras. These algebras were introduced to investigate questions concerning weak amenability, generalizing and collecting similar structures such as classical module extension algebras, θ-Lau and T-Lau products and direct sums of Banach algebras. We shall need to determine the structure of the corresponding character spaces. Then we shall focus our investigation on the character homological properties of left amenability, biprojectivity, biflatness, Johnson amenability and Johnson contractibility.

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1. Introduction

1.1. **Generalized module extension Banach algebras.** Throughout this article, let A and B be complex Banach algebras so that B is an algebraic Banach A-bimodule, i.e. it is a Banach A-bimodule and given $a \in A$ and $b_1, b_2 \in B$ the following identities hold

$$a(b_1b_2) = (ab_1)b_2, (b_1b_2)a = b_1(b_2a), (b_1a)b_2 = b_1(ab_2).$$

Let $A \bowtie B$ denote the cartesian product $A \times B$ endowed with the operations

$$z(a_1, b_1) + (a_2, b_2) = (za_1 + a_2, zb_1 + b_2),$$
$$(a_1, b_1)(a_2, b_2) = (a_1a_2, a_1b_2 + a_2b_1 + b_1b_2)$$

and the norm $\| (a, b) \|_L = \| a \| + \| b \|$ for any $z \in \mathbb{C}$, $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$. Then $A \bowtie B$ becomes a complex Banach algebra, which is called *the generalized module extension Banach algebra* (or GMEBA) *of A and B*.
In particular, $A \bowtie B$ admits the following A and B bimodule actions:

$$a'(a, b) \triangleq (a', 0_B)(a, b) \text{ and } (a, b)a' \triangleq (a, b)(a', 0_B),$$
$$b'(a, b) \triangleq (0_A, b')(a, b) \text{ and } (a, b)b' \triangleq (a, b)(0_A, b'),$$

respectively, with $a, a' \in A$ and $b, b' \in B$.
These algebras were introduced in 2017 [1]. Recent advances about biprojectivity and biflatness of GMEBA's were obtained in [2].

1.2. **Amenable, biprojective and biflat Banach algebras.** Amenable Banach algebras were introduced and studied by B. E. Johnson in his definitive monograph [3] in 1972. A Banach algebra A is amenable if given a Banach A-module X every continuous derivation D from A into X^* is inner, i.e. there exists $x' \in X^*$ so that $D(a) = ax' - x'a$ if $a \in A$. An equivalent condition of amenability of A is that A has a virtual diagonal, i.e. there exists $m \in (A\hat{\otimes}A)^{**}$ such that $am = ma$ and $\hat{\pi}_A^{**}(m)a = \kappa_A(a)$ if $a \in A$, where $\kappa_A : A \hookrightarrow A^{**}$ is the usual isometric immersion of A into its second dual space, $\pi_A : A \times A \to A$ is the product of A, $\hat{\otimes}$ denotes the projective tensor product and $\hat{\pi}_A \in \mathcal{B}(A\hat{\otimes}A, A)$ is the unique bounded linear operator so that $\hat{\pi}_A(a_1 \otimes a_2) = \pi_A(a_1, a_2)$ if $a_1, a_2 \in A$.
Initially, the classes of biprojective and biflat Banach algebras were considered by A. Ya. Helemskii [4] [5] [7]. For the structure theory of these algebras the reader can see [16] [17]. Recall that a Banach algebra A is biprojective if $\hat{\pi}_A$ has a bounded right inverse which is an A-bimodule homomorphism, while A is called biflat if $(\hat{\pi}_A)^*$ has a bounded left inverse which is an A-bimodule homomorphism. Plainly every biprojective Banach algebra is biflat. Besides it is worth mentioning that a Banach algebra A is amenable if and only if A is biflat and it has a bounded approximate identity [6], i.e. there is a bounded net $\{e_j\}_{j\in J}$ in A such that

$$\lim_{j\in J}[\| ae_j - a \| + \| e_j a - a \|] = 0$$

for all $a \in A$.

1.3. **Some notions of character amenability.** The following are the notions of character amenability of Banach algebras which we shall consider in connection with generalized module extension Banach algebras. Given a Banach algebra A then $\Delta(A)$ will denote the character space of A, eventually empty, consisting of non-zero complex multiplicative functionals of A. In what follows let ϕ be a fixed character on A.

Definition 1.1. [11] [12]. A Banach algebra A is called *left* (resp. *right*) ϕ-amenable if every continuous derivation from A into the dual space of a Banach A-module X is inner, where the left module action of A into X is defined by $ax = \phi(a)x$ (resp. $xa = \phi(a)x$) if $a \in A$ and $x \in X$.[1]

Definition 1.2. [13] A Banach algebra A is called *ϕ-biprojective* if there exists a bounded A-module morphism $\rho : A \to A\hat{\otimes}A$ such that $\phi \circ \hat{\pi}_A \circ \rho = \phi$.

Definition 1.3. [13] A Banach algebra A is called *ϕ-biflat* if there exists a bounded A-module morphism $\sigma : A \to (A\hat{\otimes}A)^{**}$ such that $\kappa_{A^*}(\phi) \circ \hat{\pi}_A^{**} \circ \sigma = \phi$.

Definition 1.4. [13] A Banach algebra A is called *ϕ-Johnson amenable* if there exists $J \in (A\hat{\otimes}A)^{**}$ such that $\langle \phi, \hat{\pi}_A^{**}(J)\rangle = 1$ and $aJ = Ja$ for every $a \in A$. Such an element J will be called a *ϕ-Johnson amenable element* of A.[2]

Definition 1.5. [9] A Banach algebra A is called *ϕ-Johnson contractible* if there exists $j \in A\hat{\otimes}A$ such that $\langle \hat{\pi}_A(j), \phi\rangle = 1$ and $aj = ja$ for every $a \in A$. Such an element j will be called a *ϕ-Johnson contractible element* of A.

Definition 1.6. [10] A Banach algebra A is called *ϕ-inner amenable* if there exists $a'' \in A^{**}$ such that $\langle \phi, a''\rangle = 1$ and $\langle aa' - a'a, a''\rangle = 0$ for all $a \in A$ and $a' \in A^*$.[3]

Remark 1.7. Let L-ϕ-A, R-ϕ-A, ϕ-JA, ϕ-JC, ϕ-IA, ϕ-BF and ϕ-BP be the classes of Banach algebras that are left ϕ-amenable, right ϕ-amenable, ϕ-Johnson amenable, ϕ-Johnson contractible, ϕ-inner amenable, ϕ-biflat and ϕ-biprojective respectively. Clearly ϕ-BP$\subseteq$ ϕ-BF and

[1]Or equivalently, there is a bounded net $(a_i)_{i\in I}$ in A such that $\phi(a_i) \to 1$ and $aa_i - \phi(a)a_i \to 0$ (resp. $a_ia - \phi(a)a_i \to 0$) for every $a \in A$.

[2]Here $\hat{\pi}_A^{**}(J) \in A^{**}$. As in forthcoming similar situations, $\langle \phi, \hat{\pi}_A^{**}(J)\rangle \in \mathbb{C}$ denotes the evaluation of the functional $\hat{\pi}_A^{**}(J)$ at the element $\phi \in A^*$.

[3]Or equivalently, there exists a bounded net $\{a_i\}_{i\in I}$ in A such that $aa_i - a_ia \xrightarrow{w} 0$ for every $a \in A$ and $\phi(a_i) \to 1$.

$$(\text{L-}\phi\text{-A})\cap(\text{R-}\phi\text{-A})=(\phi\text{-JA})=(\phi\text{-IA})\cap(\phi\text{-BF})$$

(cf. [13], Prop. 2.2, Lemma 2.3, Lemma 3.1 and Prop. 3.3). Besides ϕ-JC$\subseteq$ ϕ-BP, and the equality holds for either unital or commutative algebras (cf. [13], Lemma 3.2).

1.4. **Main results.** In this article we shall consider homological properties associated to characters of GMEBA's related to the notions introduced in Subsection 1.3. Our aim is to study to what extent such eventual amenability properties of a GMEBA affect the underlying pair of its constituent Banach algebras. To this end Section 2 is devoted to describe the general form of characters of GMEBA's. From Section 3 to Section 8 we shall consider the corresponding issues of left ϕ-amenability, ϕ-biprojectivity, ϕ-biflatness, ϕ-Johnson amenability, ϕ-Johnson contractibility and ϕ-inner amenability of GMEBA's respectively. Finally we close Section 9 with some final remarks and examples.

1.5. **Some additional notation.** We shall use the following notation: p_A, p_B (or p_1, p_2 if $A = B$) will be the natural projections of $A \bowtie B$ onto A and B respectively; ι_A, ι_B will be the natural injections of A and B into $A \bowtie B$ respectively.
Let us assume that B is a unital algebraic Banach A-bimodule with an A-central unit e_B, i.e. $ae_B = e_B a$ for all $a \in A$. Then, if $a \in A$, $b \in B$ we define

$$r_B : A \bowtie B \to B,\ r_B(a, b) \triangleq ae_B + b,$$
$$s_A : A \to A \bowtie B,\ s_A(a) \triangleq (a, -ae_B),$$
$$t : A \to B,\ t(a) \triangleq ae_B.$$

Note that p_A, ι_A, t are bounded A-bimodule maps and ι_B is a bounded B-bimodule map, while r_B and s_A are bounded B-bimodule and A-bimodule maps respectively (cf. [2], Lemma 2.1).

2. The character space of GMEBA's

Lemma 2.1. *Any character ϕ of $A \bowtie B$ has the form*

$$\phi = \alpha \circ p_A + \beta \circ p_B, \tag{2.1}$$

where α and β are unique multiplicative complex functionals, at least one of them not null, and

$$\beta(ab) = \beta(ba) = \alpha(a)\beta(b) \tag{2.2}$$

if $a \in A$ and $b \in B$. In particular, if $A = B$ then $\alpha = \beta$.

Proof. Let us assume that α, β are linear functionals on A and B so that (2.1) holds. Clearly $\alpha = \phi \circ \iota_A$, $\beta = \phi \circ \iota_B$, and they are uniquely determinated multiplicative complex linear functionals on A and B respectively. Since A and B are complete $\alpha \in A^*$ and $\beta \in B^*$.
Given $a_1, a_2 \in A$, $b_1, b_2 \in B$ we see that

$$\begin{aligned}\alpha(a_1a_2) + \beta(a_1b_2 + b_1a_2 + b_1b_2) &= \phi((a_1, b_1)(a_2, b_2))\\ &= \phi((a_1, b_1))\phi((a_2, b_2))\\ &= (\alpha(a_1) + \beta(b_1))(\alpha(a_2) + \beta(b_2)).\end{aligned}$$

Therefore $\beta(a_1b_2 + b_1a_2) = \alpha(a_1)\beta(b_2) + \beta(b_1)\alpha(a_2)$ and (2.2) follows immediately. Let us assume that the A-bimodule B is stable, i.e. the A-bimodule generated by B is dense in B. Then $\alpha \in \Delta(A)$. Precisely, if $\alpha = 0_{A^*}$ by (2.2) we see that $\beta = 0_{B^*}$. So, by (2.1) we would have $\phi_{(A\bowtie B)^*} = 0$, which contradicts that ϕ is non-zero.
If the A-bimodule B is not stable and $\alpha = 0_{A^*}$ we see that $\beta \in \Delta(B)$ and β vanishes on the A-bimodule generated by B in B. ☐

Remark 2.2. By Lemma 2.1 we shall denote any character $\phi \in \Delta(A \bowtie B)$ as $\phi = (\phi_A, \phi_B)$. So $\phi = \phi_A \circ p_A + \phi_B \circ p_B$, $\phi_A = \phi \circ \iota_A$ and $\phi_B = \phi \circ \iota_B$. *Here $\phi_A = \alpha$, $\phi_B = \beta$ are the multiplicative functionals defined in Lemma 2.1.*

Definition 2.3. Let A and B be Banach algebras so that B is an algebraic Banach A-bimodule and let α, β be multiplicative complex functionals on A and B respectively. We shall say that the pair (α, β) is *admissible* if it satisfies the conditions of Lemma 2.1.

Example 2.4. Let G be a locally compact group. Let $\mathrm{L}^1(G)$ be the Banach algebra of absolutely Haar integrable complex-valued functions on G and let $\mathrm{M}(G)$ be the Banach algebra of complex-valued bounded regular Borel measures on G. Then any character

$$\Theta \in \Delta(\mathrm{M}(G) \bowtie \mathrm{L}^1(G)) \cap [\Delta(\mathrm{M}(G)) \times \Delta(\mathrm{L}^1(G))]$$

is determined by a unique $\theta \in \hat{G}$ such that

$$\Theta(m, x) = \int_G \theta(g) dm(g) + \int_G \theta(g) x(g) dg \text{ for all } (m, x). \tag{2.3}$$

In fact, such a Θ defines an admissible pair $(\Theta_1, \Theta_2) \in \Delta(\mathrm{M}(G)) \times \hat{G}$. Hence there exists a unique $\theta \in \hat{G}$ such that $\Theta_2(x) = \int_G x(g)\theta(g)dg$ for all $x \in \mathrm{L}^1(G)$.
Let $(m, x) \in \mathrm{M}(G) \times \mathrm{L}^1(G)$. Then

$$\Theta_2(m * x) = \Theta_2(x * m) = \int_G \theta(g) dm(g) \Theta_2(x). \tag{2.4}$$

For, as θ is a group homomorphism, by Fubini's theorem and the left-invariance of the Haar measure on G we have

$$\begin{aligned}
\Theta_2(m * x) &= \int_G \theta(g) \int_G x(f^{-1}g) dm(f) dg \\
&= \int_G \int_G x(f^{-1}g)\theta(g) dg dm(f) \\
&= \int_G \int_G x(h)\theta(fh) dh dm(f) \\
&= \int_G \theta(f) dm(f) \Theta_2(x). \qquad (2.5)
\end{aligned}$$

Besides, if Δ_G denotes the modular function of G we also have

$$\begin{aligned}
\Theta_2(x * m) &= \int_G \theta(f) \int_G \Delta_G(g^{-1}) x(fg^{-1}) dm(g) df \\
&= \int_G \Delta_G(g^{-1}) \int_G \theta(f) x(fg^{-1}) df dm(g) \\
&= \int_G \Delta_G(g^{-1}) \int_G \Delta_G(f^{-1})\theta(f^{-1}) x((gf)^{-1}) df dm(g) \\
&= \int_G \Delta_G(g^{-1}) \int_G \Delta_G(h^{-1}g)\theta(h^{-1}g) x(h^{-1}) dh dm(g) \\
&= \int_G \theta(g) \int_G \Delta_G(h^{-1})\theta(h^{-1}) x(h^{-1}) dh dm(g) \\
&= \int_G \theta(g) \int_G \theta(h) x(h) dh dm(g) \\
&= \int_G \theta(g) dm(g) \Theta_2(x). \qquad (2.6)
\end{aligned}$$

So (2.4) follows by (2.5) and (2.6). Now, let $x_0 \in \mathrm{L}^1(G)$ such that $\Theta_2(x_0) = 1$. Consequently

$$\Theta_1(m) = \Theta_1(m)\Theta_2(x_0) = \Theta_2(m * x_0) = \int_G \theta(g) dm(g)$$

and (2.3) follows.

3. Left ϕ-Amenability of GMEBA's

Theorem 3.1. *(1) Let $L = A \bowtie B$ be the GMEBA of Banach algebras A and B. Let $\phi \in \Delta(L)$ so that L is left ϕ-amenable. Then: (i) B becomes left ϕ_B-amenable if $\phi_B \in \Delta(B)$ and (ii) A is becomes ϕ_A-amenable if $\phi_B = 0_{B^*}$.*

(2) Let A be a Banach algebra without non-zero characters and let B be a non-stable algebraic unital Banach A-bimodule with unit e_B. If B is left β-amenable for some $\beta \in \Delta(B)$ that vanishes on the A-bimodule generated by B then $A \bowtie B$ is left $(0_{A^*}, \beta)$-amenable.

(3) Let B be an algebraic Banach A-bimodule without non-zero characters and let A be a left α-amenable Banach algebra. Then $A \bowtie B$ is left $(\alpha, 0_{B^*})$-amenable.

Proof. (1) Since L is left ϕ-amenable let $\{(a_i, b_i)\}_{i\in I}$ be a bounded net in L such that

$$\phi_A(a_i) + \phi_B(b_i) \to 1 \tag{3.1}$$

and for every $(a, b) \in L$ we have

$$(a, b)(a_i, b_i) - (\phi_A(a) + \phi_B(b))(a_i, b_i) \to (0_A, 0_B).$$

Thus

$$aa_i - (\phi_A(a) + \phi_B(b))a_i \to 0_A, \tag{3.2}$$

$$ab_i + ba_i + bb_i - (\phi_A(a) + \phi_B(b))b_i \to 0_B. \tag{3.3}$$

If $\phi_B \in \Delta(B)$ there exists $b_1 \in B$ such that $\phi_B(b_1) = 1$. If we make $a = 0_A$ and $b = b_1$ in (3.2) then $a_i \to 0$. So, by (3.1), $\phi_B(b_i) \to 1$ and by (3.3) for any a and b we have

$$ab_i + bb_i - (\phi_A(a) + \phi_B(b))b_i \to 0_B. \tag{3.4}$$

If $a = 0_A$ in (3.4), $bb_i - \phi_B(b)b_i \to 0$. Since $\{b_i\}_{i\in I}$ becomes bounded then (i) holds.
If $\phi_B = 0_{B^*}$, by Lemma 2.1, $\phi_A \in \Delta(A)$. Besides $\{a_i\}_{i\in I}$ becomes bounded in A. By (3.1) and (3.2), $\phi_A(a_i) \to 1$ and given $a \in A$ we have $aa_i - \phi_A(a)a_i \to 0_A$, i.e. (ii) follows.

(2) Let $(b_j)_{j\in J}$ be a bounded net in B so that $\beta(b_j) \to 1$ and $bb_j - \beta(b)b_j \to 0_B$ for all $b \in B$. Now, $(0, \beta) \in \Delta(A \bowtie B)$ by Lemma 2.1. Evidently $(0_A, b_j)_{j\in J}$ is a bounded net in $A \bowtie B$. Given $a \in A$ and $b \in$ we have

$$ab_j = (ae_B)b_j - \beta(ae_B)b_j \to 0_B$$

and so

$$\begin{aligned}(a, b)(0_A, b_j) - \beta(b)(0_A, b_j) &= (0_A, ab_j + bb_j - \beta(b)b_j)\\ &\to (0_A, 0_B).\end{aligned}$$

Since $\beta(b_j) \to 1$ the assertion follows.

(3) Let $(a_i)_{i\in I}$ be a bounded net in A so that $\alpha(a_i) \to 1$ and $aa_i - \alpha(a)a_i \to 0$ for all $a \in A$. If $a \in A$ and $b \in B$, as $(ba_i)_{i\in I}$ is bounded, we have that

$$(a, b)(a_i, 0_B) - \alpha(a)(a_i, 0_B) = (aa_i - \alpha(a)a_i, ba_i) \to 0.$$

Since $\alpha(a_i) \to 1$ the assertion follows.

□

4. ϕ-biprojectivity of GMEBA's

Theorem 4.1. *Let $L = A \bowtie B$ be a ϕ-biprojective GMEBA.*

(1) Let us assume that B has an A-central unit e_B. Then $(0_A, e_B)$ is an idempotent element in L and $\phi(0, e_B) \in \{0, 1\}$. Further, (i) A is ϕ_A-biprojective if $\phi(0_A, e_B) = 0$ and (ii) B is ϕ_B-biprojective if $\phi(0_A, e_B) = 1$.

(2) If $A = B$ then A is ϕ-biprojective.

Proof. (1) (i) If $\phi(0_A, e_B) = 0$ given $a \in A$ and $b \in B$ we have

$$\phi(0_A, ae_B + b) = \phi((0_A, e_B)(a, b)) = \phi(0_A, e_B)\phi(a, b) = 0.$$

Hence $\phi_B = 0_{B^*}$, by Lemma 2.1 $\phi_A \neq 0_{A^*}$ and $\phi = \phi_A \circ p_A$.
Now, let $\rho_L \in_L \mathcal{B}_L(L, L\hat{\otimes}L)$ such that $\phi = \phi \circ \hat{\pi}_L \circ \rho_L$.[4] We define

$$\rho_A \triangleq (p_A \otimes p_A) \circ \rho_L \circ \iota_A.$$

Evidently $\rho_A \in_A \mathcal{B}_A(A\hat{\otimes}A)$ and

$$\begin{aligned}\phi_A &= \phi \circ \iota_A \\ &= \phi \circ \hat{\pi}_L \circ \rho_L \circ \iota_A \\ &= \phi_A \circ (p_A \circ \hat{\pi}_L) \circ \rho_L \circ \iota_A \\ &= \phi_A \circ (\hat{\pi}_A \circ (p_A \otimes p_A)) \circ \rho_L \circ \iota_A \\ &= \phi_A \circ \hat{\pi}_A \circ \rho_A\end{aligned}$$

and A becomes ϕ_A-biprojective.
(ii) If $\phi(0_A, e_B) = 1$ then $\phi_B \neq 0_{B^*}$. Let us write

$$\rho_B \triangleq (r_B \otimes r_B) \circ \rho_L \circ \iota_B.$$

Plainly $\rho_B \in_B \mathcal{B}_B(B, B\hat{\otimes}B)$. It is readily seen that $\hat{\pi}_B \circ (r_B \otimes r_B) = r_B \circ \hat{\pi}_L$ and $\phi_B \circ r_B = \phi$. Consequently

$$\begin{aligned}\phi_B &= \phi_B \circ \mathrm{Id}_B \\ &= \phi_B \circ (r_B \circ \iota_B) \\ &= \phi \circ \iota_B \\ &= [\phi \circ \hat{\pi}_L \circ \rho_L] \circ \iota_B \\ &= \phi_B \circ r_B \circ \hat{\pi}_L \circ \rho_L \circ \iota_B \\ &= \phi_B \circ \hat{\pi}_B \circ [(r_B \otimes r_B) \circ \rho_L \circ \iota_B] \\ &= \phi_B \circ \hat{\pi}_B \circ \rho_B\end{aligned}$$

and the assertion follows.

(2) The assertion follows by Lemma 3 and a little modification of the argument in (1)(i).

□

The next theorem generalizes Th. 3.2 of [2] to our context of character biprojectivity of Banach algebras.

Theorem 4.2. *Let $L = A \bowtie B$ be the GMEBA of Banach algebras A, B, and assume that B has an A-central unit e_B. Let (α, β) be a pair of admissible multiplicative complex functionals on A and B respectively. Then L is (α, β)-biprojective if there exist $\rho_A \in_A \mathcal{B}_A(A, A\hat{\otimes}A)$, $\rho_B \in_B \mathcal{B}_B(B, B\hat{\otimes}B)$ so that $\alpha = \alpha \circ \hat{\pi}_A \circ \rho_A$, $\beta = \beta \circ \hat{\pi}_B \circ \rho_B$ and $t \circ \hat{\pi}_A \circ \rho_A = \hat{\pi}_B \circ \rho_B \circ t$.*

Proof. Let $\rho_L : L \to L\hat{\otimes}L$ so that

$$\rho_L(a, b) \triangleq ((s_A \otimes s_A) \circ \rho_A \circ p_A)(a, b) + (a, b)(\iota_B \otimes \iota_B)(\rho_B(e_B)).$$

Then $\rho_L \in_L \mathcal{B}_L(L, L\hat{\otimes}L)$ [2] and, given $(a, b) \in L$, we see that

$$\begin{aligned}\hat{\pi}_L(\rho_L(a, b)) &= (\hat{\pi}_L \circ (s_A \otimes s_A))(\rho_A(a)) + (a, b)(\hat{\pi}_L \circ (\iota_B \otimes \iota_B))(\rho_B(e_B)) \\ &= (s_A \circ \hat{\pi}_A)(\rho_A(a)) + (a, b)(\iota_B \circ \hat{\pi}_B)(\rho_B(e_B)).\end{aligned}$$

As $p_A \circ s_A = \mathrm{Id}_A$ and $p_A \circ s_A = -t$ we have

$$\begin{aligned}((\alpha, \beta)\hat{\pi}_L\rho_L)(a, b) &= (\alpha\hat{\pi}_A\rho_A)(a) + \beta[-(t\hat{\pi}_A\rho_A)(a) + a\hat{\pi}_B(\rho_B(e_B)) + \hat{\pi}_B(\rho_B(b))] \\ &= \alpha(a) + \beta(b),\end{aligned}$$

[4]Hence ρ_L is a two-sided bounded homomorphism between the Banach L-bimodules L and $L\hat{\otimes}L$.

i.e. $(\alpha,\beta)\circ\hat{\pi}_L\circ\rho_L=(\alpha,\beta)$. □

5. ϕ-biflatness of GMEBA's

Theorem 5.1. *Let $L=A\bowtie B$ be a ϕ-biflat GMEBA of Banach algebras A and B.*

(1) Let us assume that B has an A-central unit e_B. Then $(0_A,e_B)$ is an idempotent element in L and $\phi(0,e_B)\in\{0,1\}$. Further, (i) If $\phi(0_A,e_B)=0$ then A is ϕ_A-biflat. (ii) B is ϕ_B-biflat if $\phi(0_A,e_B)=1$ and (iii) L is ϕ-biflat if B is ϕ_B-biflat.

(2) If $A=B$ then A is ϕ-biflat.

Proof. (1) (i) As in Th. 4.1 we know that $\phi_B=0_{B^*}$ and $\phi=\phi_A\circ p_A$. Given $\sigma_L\in_L\mathcal{B}_L(L,(L\hat{\otimes}L)^{**})$ such that $\kappa_{L^*}(\phi)\circ\hat{\pi}_L^{**}\circ\sigma_L=\phi$ let us write

$$\sigma_A:A\to(A\hat{\otimes}A)^{**},\ \sigma_A\triangleq(p_A\otimes p_A)^{**}\circ\sigma_L\circ\iota_A.$$

Besides $\sigma_A\in_A\mathcal{B}_A(A,(A\hat{\otimes}A)^{**})$ because σ_L becomes a bounded morphism of A-bimodules. Further,

$$\begin{aligned}\kappa_{A^*}(\phi_A)\circ\hat{\pi}_A^{**}\circ\sigma_A&=\kappa_{A^*}(\phi_A)\circ(\hat{\pi}_A\circ(p_A\otimes p_A))^{**}\circ\sigma_L\circ\iota_A\\&=\kappa_{A^*}(\phi_A)\circ(p_A\circ\hat{\pi}_L)^{**}\circ\sigma_L\circ\iota_A\\&=\kappa_{A^*}(\phi_A)\circ p_A^{**}\circ\hat{\pi}_L^{**}\circ\sigma_L\circ\iota_A\\&=\kappa_{L^*}(\phi)\circ\hat{\pi}_L^{**}\circ\sigma_L\circ\iota_A\\&=\phi\circ\iota_A\\&=\phi_A\end{aligned}$$

and A becomes ϕ_A-biflat.
(ii) Now we write

$$\sigma_B:B\to(B\hat{\otimes}B)^{**},\ \sigma_B\triangleq(r_B\otimes r_B)^{**}\circ\sigma_L\circ\iota_B.$$

Then $\sigma_B\in_B\mathcal{B}_B(B,(B\hat{\otimes}B)^{**})$. For,
Besides we know that $\phi_B\circ r_B=\phi$ and so

$$\begin{aligned}\kappa_{B^*}(\phi_B)\circ\hat{\pi}_B^{**}\circ\sigma_B&=\kappa_{B^*}(\phi_B)\circ(\hat{\pi}_B\circ(r_B\otimes r_B))^{**}\circ\sigma_L\circ\iota_B\\&=\kappa_{B^*}(\phi_B)\circ(r_B\circ\hat{\pi}_L)^{**}\circ\sigma_L\circ\iota_B\\&=\kappa_{B^*}(\phi_B)\circ r_B^{**}\circ\hat{\pi}_L^{**}\circ\sigma_L\circ\iota_B\\&=\kappa_{L^*}(\phi)\circ\hat{\pi}_L^{**}\circ\sigma_L\circ\iota_B\\&=\phi\circ\iota_B\\&=\phi_B\end{aligned}$$

i.e. B becomes ϕ_B-biflat.
(iii) Let $\sigma_B\in_B\mathcal{B}_B(B,(B\hat{\otimes}B)^{**})$ such that $\beta=\kappa_{B^*}(\beta)\circ\hat{\pi}_B^{**}\circ\sigma_B$. We shall write

$$\begin{aligned}&\sigma_L:L\to(L\hat{\otimes}L)^{**},\\&\sigma_L(a,b)\triangleq(a,b)(\iota_B\otimes\iota_B)^{**}(\sigma_B(e_B)).\end{aligned}\tag{5.1}$$

It is straightforward to see that

$$\begin{aligned}(a,b)(\iota_B\otimes\iota_B)^{**}(\sigma_B(e_B))&=(\iota_B\otimes\iota_B)^{**}(\sigma_B(ae_B+b))\\&=(\iota_B\otimes\iota_B)^{**}(\sigma_B(e_B))(a,b).\end{aligned}\tag{5.2}$$

Consequently $\sigma_L \in_L \mathcal{B}_L(L, (L\hat{\otimes}L)^{**})$. By (5.1) and (5.2) we see that

$$\begin{aligned}\kappa_{L^*}(\alpha,\beta)[\hat{\pi}_L^{**}(\sigma_L(a,b))] &= \kappa_{L^*}(\alpha,\beta)[(\hat{\pi}_L^{**}\circ(\iota_B\otimes\iota_B)^{**})(\sigma_B(ae_B+b))] \\ &= \kappa_{L^*}(\alpha,\beta)[\iota_B^{**}(\hat{\pi}_B^{**}(\sigma_B(ae_B+b)))] \\ &= \kappa_{B^*}(\beta)(\hat{\pi}_B^{**}(\sigma_B(ae_B+b)) \\ &= \beta(ae_B+b) \\ &= \alpha(a)+\beta(b) \\ &= (\alpha,\beta)(a,b).\end{aligned}$$

(2) The assertion follows by Lemma 2.1 and a little modification of the argument in (1)(i).

□

6. ϕ-Johnson amenability of GMEBA's

Theorem 6.1. *Let $L = A \bowtie B$ be the GMEBA of Banach algebras A and B.*

(1) If B has an A-central unit e_B, L is ϕ-Johnson amenable if and only if B is ϕ_B-Johnson amenable.

(2) If $A = B$ and L is ϕ-Johnson amenable then A is ϕ-Johnson amenable.

Proof. (1) ($\Rightarrow$) Let $J_L \in (L\hat{\otimes}L)^{**}$ be a ϕ-Johnson amenable element of L and let us write

$$J_B \triangleq (r_B\otimes r_B)^{**}(J_L)$$

in $(B\hat{\otimes}B)^{**}$. Given $l \in L$ we have

$$\begin{aligned}r_B(l)(r_B\otimes r_B)^{**}(J_L) &= (r_B\otimes r_B)^{**}(lJ_L) \\ &= (r_B\otimes r_B)^{**}(J_Ll) \\ &= (r_B\otimes r_B)^{**}(J_L)r_B(l).\end{aligned}$$

Since r_B is surjective we infer that $bJ_B = J_Bb$ for every $b \in B$. Besides

$$\begin{aligned}1 &= \langle\phi, \hat{\pi}_L^{**}(J_L)\rangle \\ &= \langle\phi_A\circ p_A + \phi_B\circ p_B, \hat{\pi}_L^{**}(J_L)\rangle \\ &= \langle\phi_A, (p_A^{**}\circ\hat{\pi}_L^{**})(J_L)\rangle + \langle\phi_B, (p_B^{**}\circ\hat{\pi}_L^{**})(J_L)\rangle \\ &= \langle\phi_A, (p_A^{**}\circ\hat{\pi}_L^{**})(J_L)\rangle + \langle\phi_B, \hat{\pi}_B^{**}(J_B) - t^{**}((p_A^{**}\circ\hat{\pi}_L^{**})(J_L))\rangle \\ &= \langle\phi_A - t^*(\phi_B), (p_A^{**}\circ\hat{\pi}_L^{**})(J_L)\rangle + \langle\phi_B, \hat{\pi}_B^{**}(J_B)\rangle \\ &= \langle\phi_B, \hat{\pi}_B^{**}(J_B)\rangle\end{aligned}$$

because $\phi_A = t^*(\phi_B)$.

($\Leftarrow$) Let $J_B \in (B\hat{\otimes}B)^{**}$ be a ϕ-Johnson amenable element of B. We define

$$J_L \triangleq (\iota_B\otimes\iota_B)^{**}(J_B) \text{ in } (L\hat{\otimes}L)^{**}.$$

Hence

$$\begin{aligned}\langle\phi, \hat{\pi}_L^{**}(J_L)\rangle &= \langle\phi, (\hat{\pi}_L\circ(\iota_B\otimes\iota_B))^{**}(J_B)\rangle \\ &= \langle\phi, \iota_B^{**}(\hat{\pi}_B^{**}(J_B))\rangle \\ &= \langle\phi_B, \hat{\pi}_B^{**}(J_B)\rangle \\ &= 1.\end{aligned}$$

Further, given $l \in L$ we see that

$$lJ_L = r_B(l)J_B = J_Br_B(l) = J_Ll.$$

(2) Given a ϕ-Johnson amenable element J_L of L let

$$J_A^1 \triangleq (p_1\otimes p_1)^{**}(J_L),\ J_A^2 \triangleq (p_1\otimes p_2 + p_2\otimes p_1 + p_2\otimes p_2)^{**}(J_L)$$

in $(A\hat{\otimes}A)^{**}$. If $a \in A$ we have

$$\begin{aligned} aJ_A^1 &= p_1(a, 0_A)J_A^1 \\ &= (p_1 \otimes p_1)^{**}((a, 0_A)J_L) \\ &= (p_1 \otimes p_1)^{**}(J_L(a, 0_A)) \\ &= J_A^1 a. \end{aligned}$$

Analogously one can see that $aJ_A^2 = J_A^2 a$ for every $a \in A$. Since L is ϕ-Johnson amenable we have

$$\begin{aligned} 1 &= \langle \phi \circ (p_1 + p_2), \hat{\pi}_L^{**} \rangle \\ &= \langle \phi, (p_1 \circ \hat{\pi}_L)^{**}(J_L) + (p_2 \circ \hat{\pi}_L)^{**}(J_L) \rangle \\ &= \langle \phi, (\hat{\pi}_A \circ (p_1 \otimes p_1))^{**}(J_L) + (\hat{\pi}_A \circ (p_1 \otimes p_2 + p_2 \otimes p_1 + p_2 \otimes p_2))^{**}(J_L) \rangle \\ &= \langle \phi, \hat{\pi}_A^{**}(J_A^1 + J_A^2) \rangle. \end{aligned}$$

Consequently $\langle \phi, \hat{\pi}_A^{**}(J_A^1) \rangle \neq 0$ or $\langle \phi, \hat{\pi}_A^{**}(J_A^2) \rangle \neq 0$, i.e. a suitable multiple of J_A^1 or J_A^2 provides a ϕ-Jhonson amenable element of A.

□

7. ϕ-Johnson contractibility of GMEBA's

Theorem 7.1. *Let $L = A \bowtie B$ be the GMEBA of Banach algebras A and B.*

(1) If B has an A-central unit e_B then L is ϕ-Johnson contractible if and only if B is ϕ_B-Johnson contractible.

(2) If $A = B$, L is (ϕ, ϕ)-Johnson contractible then A is ϕ-Johnson contractible.

Proof. (1) (⇒) With the above notation let $j_B \triangleq (r_B \otimes r_B)(j_L)$ in $B\hat{\otimes}B$. We write

$$j_L = \sum_{n=1}^{\infty} (a_n^1, b_n^1) \otimes (a_n^2, b_n^2),$$

with $\{(a_n^1, b_n^1), (a_n^2, b_n^2)\}_{n\in\mathbb{N}} \subseteq L$ and $\sum_{n=1}^{\infty} \| (a_n^1, b_n^1) \|_L \| (a_n^2, b_n^2) \|_L < \infty$. Thus, by Lemma 2.1,

$$\phi[(s_A \circ p_A)(\hat{\pi}_L(j_L))] = \sum_{n=1}^{\infty} \phi(a_n^1 a_n^2, -a_n^1 a_n^2) = 0.$$

As $\mathrm{Id}_L = s_A \circ p_A + \iota_B \circ r_B$ we have

$$\begin{aligned} \langle \hat{\pi}_B(j_B), \phi_B \rangle &= (\hat{\pi}_B \circ (r_B \otimes r_B))(j_L), \phi_B \rangle \\ &= \langle (r_B \circ \hat{\pi}_L)(j_L), \iota_B^*(\phi) \rangle \\ &= \langle (\iota_B \circ r_B)(\hat{\pi}_L(j_L)), \phi \rangle \\ &= \langle \hat{\pi}_L(j_L), \phi \rangle \\ &= 1. \end{aligned}$$

Given $\Theta \in (B\hat{\otimes}B)^*$ and $b \in B$ we see that

$$(r_B \otimes r_B)^*(\Theta b) = (r_B \otimes r_B)^*(\Theta)b \text{ and } (r_B \otimes r_B)^*(b\Theta) = b(r_B \otimes r_B)^*(\Theta)$$

and so $\langle bj_B - j_B b, \Theta \rangle = 0$, i.e. $bj_B = j_B b$.

(⇐) Let $j_B \in B\hat{\otimes}B$ be a ϕ_B-Johnson contractible element of B. We define $j_L \triangleq (\iota_B \otimes \iota_B)(j_B)$ in $L\hat{\otimes}L$. Now

$$\begin{aligned} \langle \hat{\pi}_L(j_L), \phi \rangle &= \langle \hat{\pi}_L \circ (\iota_B \otimes \iota_B)(j_B), \phi \rangle \\ &= \langle \iota_B(\hat{\pi}_B(j_B)), \phi \rangle \\ &= \langle \hat{\pi}_B(j_B), \phi_B \rangle \\ &= 1. \end{aligned}$$

Given $\Lambda \in (L\hat{\otimes}L)^*$ and $l \in L$ we have

$$(\iota_B \otimes \iota_B)^*(\Lambda l) = (\iota_B \otimes \iota_B)^*(\Lambda) r_B(l),$$
$$(\iota_B \otimes \iota_B)^*(l\Lambda) = r_B(l)(\iota_B \otimes \iota_B)^*(\Lambda).$$

Thus $\langle l j_L - j_L l, \Lambda\rangle = 0$, i.e. $l j_L = j_L l$.

(2) Given a ϕ-Johnson contractible element j_L of L let

$$j_A^1 \triangleq (p_1 \otimes p_1)(j_L),\ j_A^2 \triangleq (p_1 \otimes p_2 + p_2 \otimes p_1 + p_2 \otimes p_2)(j_L)$$

in $A\hat{\otimes}A$. The claim now follows similarly to Th. 6.1(2).

□

8. ϕ-inner amenability of GMEBA's

Theorem 8.1. *Let $L = A \bowtie B$ be a ϕ-inner amenable GMEBA of Banach algebras A, B, and let us assume that B has an A-central unit e_B. Then $(0_A, e_B)$ is idempotent in L and $\phi(0, e_B) \in \{0, 1\}$. Then,*

(1) A is ϕ_A-inner amenable if $\phi(0_A, e_B) = 0$.
(2) B is ϕ_B-inner amenable if $\phi(0_A, e_B) = 1$.

Proof. (1) Let $\{l_i = (a_i, b_i) : i \in I\}$ be a bounded net in L^{**} such that $l l_i - l_i l \xrightarrow{w} 0$ for all $l \in L$ and $\langle l_i, \phi\rangle \to 0$. Hence $\{a_i\}_{i\in I}$ becomes bounded in A. As $L^* \approx A^* \oplus_\infty B^*$ we have

$$a a_i - a_i a \xrightarrow{w} 0, \tag{8.1}$$
$$(a b_i - b_i a) + (b a_i - a_i b) + (b b_i - b_i b) \xrightarrow{w} 0 \tag{8.2}$$

for all $a \in A$ and all $b \in B$. We know that $\phi = \phi_A \circ p_A$ and

$$\langle a_i, \phi_A\rangle = \langle l_i, \phi\rangle \to 1.$$

So by (8.1) A becomes ϕ_A-inner amenable.

(2) Clearly $\{a_i e_B + b_i\}_{i\in I}$ becomes bounded in L. If $\phi(0_A, e_B) = 1$ then

$$\langle l_i, \phi\rangle = \langle (a_i, b_i)(0_A, e_B), \phi\rangle = \phi_B(a_i e_B + b_i) \to 1.$$

Choosing $a = 0_A$ and $b \in B$ by (8.2) we see that

$$b(a_i e_B + b_i) - (a_i e_B + b_i)b = (b a_i - a_i b) + (b b_i - b_i b) \xrightarrow{w} 0,$$

i.e. B is ϕ_B-inner amenable.

(3) Let $\{b_i\}_{i\in I}$ be a bounded net in B such that $\langle b_i, \phi_B\rangle \to 0$ and $b b_i - b_i b \xrightarrow{w} 0$ for every $b \in B$. Then $\{(0_A, b_i)\}_{i\in I}$ is a bounded net in L,

$$\langle (0_A, b_i), \phi\rangle = \langle b_i, \phi_B\rangle \to 0$$

and given $(a, b) \in L$ we have

$$(a, b)(0_A, b_i) - (0_A, b_i)(a, b) = (0_A, (a e_B + b) b_i - b_i(a e_B + b)) \xrightarrow{w} 0,$$

i.e. L becomes ϕ-inner amenable.

□

9. Some final remarks and examples

(1) A generalized module extension Banach algebra $A \bowtie B$ is unital if and only if A is unital and there exists $b_0 \in B$ such that $A b_0 = b_0 A = (0_B)$ and $b e_A + b b_0 = e_A b + b_0 b = b$ for every $b \in B$. Besides $A \bowtie B$ is commutative if and only A and B are commutative.

(2) Let G be a locally compact abelian group with Haar measure λ_G, and let w be a weight on G. Let ϕ_w be the corresponding augmentation character on $\mathrm{L}^1_w(G)$ (i.e. $\phi_w(x) \triangleq \int_G x(g) w(g) d\lambda_G(g)$ if $x \in \mathrm{L}^1_w(G)$). Then $L \triangleq \mathrm{L}^1_w(G) \bowtie \mathrm{L}^1_w(G)$ is (ϕ_w, ϕ_w)-Johnson contractible if and only if G is compact.

($\Rightarrow$) By Th. 7.1(2) $A \in \phi_w$-JC if $L \in (\phi_w, \phi_w)$-JC. Hence A becomes ϕ_w-biprojective and so G

must be compact (cf. [15], Th. 3.1).

($\Leftarrow$) Let G be a normalized compact abelian group. If $x, y \in \mathrm{L}^1_w(G)$ then $x *_w y = \dfrac{(xw) * (yw)}{w}$. Thus $x *_w w^{-1} = \phi_w(x)w^{-1}$ for all x. Let $j_L \triangleq (0_{\mathrm{L}^1_w(G)}, w^{-1}) \otimes (0_{\mathrm{L}^1_w(G)}, w^{-1})$ in $L\hat{\otimes}L$. Then

$$\begin{aligned}(x,y)j_L &= (0_{\mathrm{L}^1_w(G)}, \phi_w(x)w^{-1} + \phi_w(y)w^{-1})\hat{\otimes}(0_{\mathrm{L}^1_w(G)}, w^{-1})\\ &= (\phi_w, \phi_w)(x,y)j_L\\ &= (0_{\mathrm{L}^1_w(G)}, w^{-1})\hat{\otimes}(0_{\mathrm{L}^1_w(G)}, \phi_w(x)w^{-1} + \phi_w(y)w^{-1})\\ &= j_L(x,y).\end{aligned}$$

Further, as $\phi_w(w^{-1}) = \lambda_G(G) = 1$ then

$$(\phi_w, \phi_w)(\hat{\pi}_L(j_L)) = (\phi_w, \phi_w)(0_{\mathrm{L}^1_w(G)}, w^{-1}) = 1.$$

(3) It is known that $\mathrm{L}^1_w(G)$ is biprojective (or biflat) if and only if G is compact (or amenable) [8]. By Th. 4.1 (or Th. 5.1) we infer that a locally compact group G is compact (or amenable) if $\mathrm{L}^1_w(G) \bowtie \mathrm{L}^1_w(G)$ is ϕ-biprojective (or biflat).

(4) Let G be a locally compact group. Then $L \triangleq \mathrm{C}_0(G) \bowtie \mathrm{C}_0(G) \in \phi$-BP if and only if G is discrete. For, it is well known that $\Delta(\mathrm{C}_0(G))$ is homeomorphic to G. So, by Lemma 2.1,

$$\Delta(L) = \{\phi_g = (\delta_g, \delta_g) : g \in G\}$$

with $\delta_g(x) = x(g)$ if $x \in \mathrm{C}_0(G)$ and $g \in G$. Besides, G is discrete if and only if $\mathrm{C}_0(G) \in \delta_g$-BP for all $g \in G$ (cf. [14], Corollary 3.5). Now, by Th. 4.1(2), $\mathrm{C}_0(G) \in \delta_g$-BP if $L \in (\delta_g, \delta_g)$-BP and the necessity follows.

On the other hand, if G is discrete given $g \in G$ then $\chi_{\{g\}} \in \mathrm{C}_0(G)$. We define

$$\rho_g : L \to L\hat{\otimes}L,$$
$$\rho_g(x,y) \triangleq \phi_g(x,y)u_g,$$

with $x, y \in \mathrm{C}_0(G)$ and $u_g = (0_{\mathrm{C}_0(G)}, \chi_{\{g\}}) \otimes (0_{\mathrm{C}_0(G)}, \chi_{\{g\}})$. As $\phi_g \in \Delta(L)$ and

$$(x,y)u_g = u_g(x,y) = \phi_g(x,y)u_g$$

for every $(x,y) \in L$ it follows that ρ_g becomes a morphism of Banach L-bimodules. Indeed,

$$\begin{aligned}(\phi_g \circ \hat{\pi}_L \circ \rho_g)(x,y) &= (\phi_g \circ \hat{\pi}_L)(\phi_g(x,y)u_g)\\ &= \phi_g(x,y)\phi_g(\hat{\pi}_L(u_g))\\ &= \phi_g(x,y)\phi_g(0_{\mathrm{C}_0(G)}, \chi_{\{g\}})\\ &= \phi_g(x,y)\end{aligned}$$

and the assertion follows.

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