COUPLED FIXED POINT THEOREMS UNDER CONTRACTIVE TYPE CONDITIONS IN S-METRIC SPACES WITH APPLICATION

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ABSTRACT. In this paper, we prove some coupled fixed point theorems via contractive type conditions in the framework of S-metric spaces. Furthermore, we give an example to validate the result. As an application of our main results we give some fixed point results for integral type contractions. Our results generalize, extend and enrich several results from the existing literature.

Mathematics Subject Classification (2010): 47H10, 54H25.
Key words: Coupled fixed point, contractive type condition, S-metric space.

Article history:
Received: May 07, 2023
Received in revised form: December 28, 2023
Accepted: December 29, 2023

1. Introduction

In 1922, *Stefan Banach* [3] formulated the concept of contraction and proved the famous theorem called Banach contraction mapping principle, which states that every self mapping $\mathcal{T}$ defined on a complete metric space (X, d) satisfying

$$d(\mathcal{T}(x), \mathcal{T}(y)) \leq t\, d(x, y), \tag{1.1}$$

for all $x, y \in X$, where $t \in (0, 1)$, has a unique fixed point and for every $x_0 \in X$ a sequence $\{\mathcal{T}^n x_0\}_{n \geq 1}$ is convergent to the fixed point. Scientist and mathematicians around the world are publishing new results that are related either to establish a generalization of metric space or to get a improvement of contractive conditions.

In the literature, there are many generalizations of the metric space exists. One of such generalizations is the generalized metric space or S-metric space. *Sedghi et al.* [24] (2012) introduced the notion of S-metric space as a generalization of G-metric (*Mustafa and Sims* [16]) and D^*-metric (*Sedghi et al.* [23]). They studied its some properties and also stated that S-metric space is a generalization of G-metric space. But *Dung et al.* [8] (2014) showed by an example that an S-metric space is not a generalization of G-metric space and conversely. Consequently, the class of S-metric spaces and the class of G-metric spaces are different. Many results which were proved earlier in metric spaces are valid in the framework of S-metric spaces.

Bhashkar and Lakshmikantham in [4] introduced the concepts of coupled fixed points and mixed monotone property and illustrated these results by proving the existence and uniqueness of the solution for a periodic boundary value problem. Later on, these results were further extended and generalized by *Ćirić and Lakshmikantham* [7] to coupled coincidence and coupled common fixed point results for nonlinear contractions in partially ordered metric spaces (see, also [5], [6], [15], [17], [18]).

Sabetghadam et al. [20] (2009) proved some coupled fixed point theorems in cone metric spaces for contractive type conditions. *Aydi* [2] (2011) proved some coupled fixed point theorems for various contractive type conditions in the setting of partial metric spaces and give some corollaries of the established results. Recently, *Saluja* [21] proved some coupled fixed point results for contractive type conditions in the framework of complex partial metric spaces (see, also [22]).

Motivated by the works of *Bhashkar and Lakshmikantham* [4] and *Sedghi et al.* [24], the purpose of this paper is to prove some coupled fixed point theorems for contractive type conditions in the setting of S-metric spaces. As an an application of our main results we give some fixed point results for integral type contractions. Our results in this paper extend, generalize and enrich several previously published results from the existing literature.

2. Preliminaries

In this section, we need the following definitions, lemmas and auxiliary results to prove our main results (see, [24]).

Definition 2.1. [24] Let X be a nonempty set and let $S\colon X^3 \to [0,\infty)$ be a function satisfying the following conditions for all x, y, z, $a \in X$:
$(S1)$ $0 < S(x,y,z)$ for all $x,y,z \in X$ with $x \neq y \neq z$;
$(S2)$ $S(x,y,z) = 0$ if and only if $x = y = z$;
$(S3)$ $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.
Then the function S is called an S*-metric* on X and the pair (X,S) is called an S*-metric space*.

Example 2.2. [24]
(1) Let $X = \mathbb{R}^n$ and $\|\cdot\|$ be a norm on X, then $S(x,y,z) = \|y+z-2x\| + \|y-z\|$ is an S-metric on X.
(2) Let $X = \mathbb{R}^n$ and $\|\cdot\|$ be a norm on X, then $S(x,y,z) = \|x-z\| + \|y-z\|$ is an S-metric on X.

Example 2.3. [25] Let $X = \mathbb{R}$ be the real line. Then $S(x,y,z) = |x-z| + |y-z|$ for all $x,y,z \in \mathbb{R}$ is an S-metric on X. This S-metric on X is called the usual S-metric on X.

Example 2.4. [13] Let X be a non-empty set and d be an ordinary metric on X. Then $S(x,y,z) = d(x,z) + d(y,z)$ for all $x,y,z \in \mathbb{R}$ is an S-metric on X.

Example 2.5. [26] Let X be a non-empty set and d_1, d_2 be two ordinary metrics on X. Then $S(x,y,z) = d_1(x,z) + d_2(y,z)$ for all $x,y,z \in X$ is an S-metric on X.

Example 2.6. [24] Let $X = \mathbb{R}^2$ and d an ordinary metric on X. Put $S(x,y,z) = d(x,y)+d(x,z)+d(y,z)$ for all $x,y,z \in \mathbb{R}^2$, that is, S is the perimeter of the triangle given x,y,z. Then S is an S-metric on X.

Definition 2.7. Let (X,S) be an S-metric space. For $r > 0$ and $x \in X$ we define the *open ball* $\mathcal{B}_S(x,r)$ and *closed ball* $\mathcal{B}_S[x,r]$ with center x and radius r as follows, respectively:

$$\mathcal{B}_S(x,r) = \{y \in X : S(y,y,x) < r\},$$

$$\mathcal{B}_S[x,r] = \{y \in X : S(y,y,x) \leq r\}.$$

Example 2.8. [25] Let $X = \mathbb{R}$. Denote by $S(x,y,z) = |y+z-2x| + |y-z|$ for all $x,y,z \in \mathbb{R}$. Then

$$\begin{aligned} \mathcal{B}_S(1,2) &= \{y \in \mathbb{R} : S(y,y,1) < 2\} = \{y \in \mathbb{R} : |y-1| < 1\} \\ &= \{y \in \mathbb{R} : 0 < y < 2\} = (0,2), \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}_S[2,4] &= \{y \in \mathbb{R} : S(y,y,2) \leq 4\} = \{y \in \mathbb{R} : |y-2| \leq 2\} \\ &= \{y \in \mathbb{R} : 0 \leq y \leq 4\} = [0,4]. \end{aligned}$$

Definition 2.9. *(*[24], [25]*)* Let (X, S) be an S-metric space and $A \subset X$.

(Θ_1) The subset A is said to be an *open subset* of X, if for every $x \in A$ there exists $r > 0$ such that $\mathcal{B}_S(x, r) \subset A$.

(Θ_2) A sequence $\{y_n\}$ in X it converges to $y \in X$ if $S(y_n, y_n, y) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $S(y_n, y_n, y) < \varepsilon$. We denote this by $\lim_{n\to\infty} y_n = y$ or $y_n \to y$ as $n \to \infty$.

(Θ_3) A sequence $\{y_n\}$ in X is called a *Cauchy sequence* if $S(y_n, y_n, y_m) \to 0$ as $n, m \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $S(y_n, y_n, y_m) < \varepsilon$.

(Θ_4) The S-metric space (X, S) is called *complete* if every Cauchy sequence in X is convergent in X.

(Θ_5) Let τ be the set of all $A \subset X$ with the property that for each $x \in A$ and there exists $r > 0$ such that $\mathcal{B}_S(x, r) \subset A$. Then τ *is a topology* on X (induced by the S-metric space).

(Θ_6) A nonempty subset A of X is S*-closed* if closure of A coincides with A.

Definition 2.10. [24] Let (X, S) be an S-metric space. A mapping $\mathcal{Q}\colon X \to X$ is said to be a *contraction* if there exists a constant $0 \leq \alpha < 1$ such that

$$S(\mathcal{Q}x, \mathcal{Q}y, \mathcal{Q}z) \leq \alpha\, S(x, y, z), \tag{2.1}$$

for all $x, y, z \in X$.

Remark 2.11. If the S-metric space (X, S) is complete then the mapping defined as above has a unique fixed point (see [24], Theorem 3.1).

Definition 2.12. [24] Let (X, S) and (Y, S') be two S-metric spaces. A function $P\colon X \to Y$ is said to be *continuous* at a point $x_0 \in X$ if for every sequence $\{x_n\}$ in X with $S(x_n, x_n, x_0) \to 0$, $S'(P(x_n), P(x_n), P(x_0)) \to 0$ as $n \to \infty$. We say that P is *continuous* on X if P is *continuous* at every point $x_0 \in X$.

Definition 2.13. Let X be a non-empty set and $A, B\colon X \to X$ be two self mappings of X. Then a point $z \in X$ is called

(1) a *fixed point* of operator A if $A(z) = z$.

(2) a *common fixed point* of A and B if $A(z) = B(z) = z$.

Definition 2.14. [1] Let A and B be single valued self-mappings on a set X. If $z = Av = Bv$ for some $v \in X$, then v is called a *coincidence point* of A and B, and z is called a *point of coincidence* of A and B. We denote the *coincidence point* of A and B by $C(A, B)$, that is, $C(A, B) = \{v \in X : Av = Bv\}$.

Definition 2.15. [11] Let A and B be single valued self-mappings on a set X. Mappings A and B are said to be *commuting* if $ABz = BAz$ for all $z \in X$.

Example 2.16. Let $X = [0, \frac{3}{4}]$ and define $A, B\colon X \to X$ defined by $A(x) = \frac{x^3}{4}$ and $B(x) = x^4$ for all $x, y \in X$. Then the mappings A and B have two coincidence points 0 and $\frac{1}{4}$. Clearly, they commute at 0 but not at $\frac{1}{4}$.

Definition 2.17. [12] Let A and B be single valued self-mappings on a set X. Mappings A and B are said to be *weakly compatible* if they commute at their coincidence points, i.e., if $Az = Bz$ for some $z \in X$ implies $ABz = BAz$.

Definition 2.18. An element $(x, y) \in X \times X$ is called:

(a) a *coupled fixed point* [2] of the mapping $F\colon X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$;

(b) a *coupled coincidence point* [7] of the mappings $F\colon X \times X \to X$ and $A\colon X \to X$ if $F(x, y) = A(x)$ and $F(y, x) = A(y)$;

(c) a *common coupled fixed point* [14] of the mappings $F\colon X \times X \to X$ and $A\colon X \to X$ if $x = F(x, y) = A(x)$ and $y = F(y, x) = A(y)$.

Example 2.19. Let $X = [0, +\infty)$ and $F\colon X \times X \to X$ defined by $F(x, y) = \frac{x+y}{3}$ for all $x, y \in X$. One can easily see that F has a unique coupled fixed point $(0, 0)$.

Example 2.20. Let $X = [0, +\infty)$ and $F\colon X \times X \to X$ be defined by $F(x, y) = \frac{x+y}{2}$ for all $x, y \in X$. Then we see that F has two coupled fixed point $(0, 0)$ and $(1, 1)$, that is, the coupled fixed point is not unique.

Lemma 2.21. *([24], Lemma 2.5) Let (X, S) be an S-metric space. Then, we have $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.*

Lemma 2.22. *([24], Lemma 2.12) Let (X, S) be an S-metric space. If $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $S(x_n, x_n, y_n) \to S(x, x, y)$ as $n \to \infty$.*

Lemma 2.23. *([9], Lemma 8) Let (X, S) be an S-metric space and A be a nonempty subset of X. Then A is said to be S-closed if and only if for any sequence $\{x_n\}$ in A such that $x_n \to x$ as $n \to \infty$, then $x \in A$.*

Lemma 2.24. *([24]) Let (X, S) be an S-metric space. If $r > 0$ and $x \in X$, then the ball $\mathcal{B}_S(x, r)$ is an open subset of X.*

Lemma 2.25. *([25]) The limit of a sequence $\{x_n\}$ in an S-metric space (X, S) is unique.*

Lemma 2.26. *([24]) Let (X, S) be an S-metric space. Then any convergent sequence $\{x_n\}$ in X is Cauchy.*

In the following lemma we see the relationship between a metric and S-metric.

Lemma 2.27. [10] *Let (X, d) be a metric space. Then the following properties are satisfied:*
(i) $S_d(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in X$ is an S-metric on X.
(ii) $x_n \to x$ in (X, d) if and only if $x_n \to x$ in (X, S_d).
(iii) $\{x_n\}$ is Cauchy in (X, d) if and only if $\{x_n\}$ is Cauchy in (X, S_d).
(iv) (X, d) is complete if and only if (X, S_d) is complete.

We call the function S_d defined in Lemma 2.27 (i) as the S-metric generated by the metric d. It can be found an example of an S-metric which is not generated by any metric in [10, 19].

Example 2.28. [10] Let $X = \mathbb{R}$ and the function $S\colon X^3 \to [0, \infty)$ be defined as

$$S(x, y, z) = |x - z| + |x + z - 2y|,$$

for all $x, y, z \in \mathbb{R}$. Then the function S is an S-metric on X and (X, S) is an S-metric space. Now, we prove that there does not exists any metric d such that $S = S_d$. On the contrary, suppose that there exists a metric d such that

$$S(x, y, z) = d(x, z) + d(y, z),$$

for all $x, y, z \in \mathbb{R}$. Hence, we obtain

$$S(x, x, z) = 2d(x, z) = 2|x - z|,$$

and

$$d(x, z) = |x - z|.$$

Similarly, we get

$$S(y, y, z) = 2d(y, z) = 2|y - z|,$$

and

$$d(y, z) = |y - z|,$$

for all $x, y, z \in \mathbb{R}$. Hence, we have

$$|x - z| + |x + z - 2y| = |x - z| + |y - z|,$$

which is a contradiction. Therefore, $S \neq S_d$ and $(\mathbb{R}, S)$ is a complete S-metric space.

3. Main Results

In this section, we prove some unique coupled fixed point theorems for contractive type conditions in the framework of S-metric spaces.

Theorem 3.1. *Let (X,S) be a complete S-metric space. Let $F\colon X\times X\to X$ be a mapping satisfying the following contractive condition: for all $x,y,u,v,z,w\in X$:*

$$\begin{aligned}
[S(F(x,y),F(u,v),F(z,w))]^2 \quad \le \quad & p_1\,[S(x,u,z)]^2+p_2\,[S(y,v,w)]^2\\
&+p_3\,[S(F(x,y),F(x,y),x)]^2\\
&+p_4\,[S(F(u,v),F(u,v),u)]^2\\
&+p_5\,[S(F(z,w),F(z,w),z)]^2\\
&+p_6\,[S(F(u,v),F(u,v),z)]^2 \qquad (3.1)
\end{aligned}$$

where p_1,p_2,p_3,p_4,p_5,p_6 are nonnegative constants such that $p_1+p_2+p_3+p_4+p_5+p_6<1$. If F is continuous, then F has a unique coupled point in X.

Proof. Choose $x_0,y_0\in X$. Set $x_1=F(x_0,y_0)$ and $y_1=F(y_0,x_0)$. Repeating this process, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $x_{n+1}=F(x_n,y_n)$ and $y_{n+1}=F(y_n,x_n)$. Assume that $u_n=[S(x_n,x_n,x_{n+1})]^2$, $v_n=[S(y_n,y_n,y_{n+1})]^2$ and $t_n=u_n+v_n$. Then, from equations (3.1), using $(S2)$ and Lemma 2.21, we have

$$u_n=[S(x_n,x_n,x_{n+1})]^2=[S(F(x_{n-1},y_{n-1}),F(x_{n-1},y_{n-1}),F(x_n,y_n))]^2$$

$$\begin{aligned}
&\le\; p_1\,[S(x_{n-1},x_{n-1},x_n)]^2+p_2\,[S(y_{n-1},y_{n-1},y_n)]^2\\
&\quad +p_3\,[S(F(x_{n-1},y_{n-1}),F(x_{n-1},y_{n-1}),x_{n-1})]^2\\
&\quad +p_4\,[S(F(x_{n-1},y_{n-1}),F(x_{n-1},y_{n-1}),x_{n-1})]^2\\
&\quad +p_5\,[S(F(x_n,y_n),F(x_n,y_n),x_n)]^2\\
&\quad +p_6\,[S(F(x_{n-1},y_{n-1}),F(x_{n-1},y_{n-1}),x_n)]^2\\
&=\; p_1\,[S(x_{n-1},x_{n-1},x_n)]^2+p_2\,[S(y_{n-1},y_{n-1},y_n)]^2\\
&\quad +p_3\,[S(x_n,x_n,x_{n-1})]^2+p_4\,[S(x_n,x_n,x_{n-1})]^2\\
&\quad +p_5\,[S(x_{n+1},x_{n+1},x_n)]^2+p_6\,[S(x_n,x_n,x_n)]^2\\
&=\; (p_1+p_3+p_4)\,[S(x_{n-1},x_{n-1},x_n)]^2\\
&\quad +p_2\,[S(y_{n-1},y_{n-1},y_n)]^2+p_5\,[S(x_n,x_n,x_{n+1})]^2\\
&=\; (p_1+p_3+p_4)\,u_{n-1}+p_2\,v_{n-1}+p_5\,u_n. \qquad (3.2)
\end{aligned}$$

Similarly, one can show that

$$v_n=[S(y_n,y_n,y_{n+1})]^2=[S(F(y_{n-1},x_{n-1}),F(y_{n-1},x_{n-1}),F(y_n,x_n))]^2$$

$$\le (p_1+p_3+p_4)\,v_{n-1}+p_2\,u_{n-1}+p_5\,v_n. \qquad (3.3)$$

From equations (3.2) and (3.3), we obtain

$$\begin{aligned}
t_n \;&=\; u_n+v_n\le (p_1+p_3+p_4)\,u_{n-1}+p_2\,v_{n-1}+p_5\,u_n\\
&\quad +(p_1+p_3+p_4)\,v_{n-1}+p_2\,u_{n-1}+p_5\,v_n\\
&=\; (p_1+p_3+p_4)\,(u_{n-1}+v_{n-1})+p_2\,(u_{n-1}+v_{n-1})\\
&\quad +p_5\,(u_n+v_n))\\
&=\; (p_1+p_2+p_3+p_4)\,t_{n-1}+p_5\,t_n. \qquad (3.4)
\end{aligned}$$

This implies that

$$t_n \leq \Big(\frac{p_1+p_2+p_3+p_4}{1-p_5}\Big)\, t_{n-1} = \nu\, t_{n-1}, \tag{3.5}$$

where $\nu = \Big(\frac{p_1+p_2+p_3+p_4}{1-p_5}\Big) < 1$, since $p_1+p_2+p_3+p_4+p_5 < 1$.

Consequently, for each $n \in \mathbb{N}$, we obtain

$$t_n \leq \nu\, t_{n-1} \leq \nu^2\, t_{n-2} \leq \cdots \leq \nu^n\, t_0. \tag{3.6}$$

If $t_0 = 0$, then $S(x_0, x_0, x_1) + S(y_0, y_0, y_1) = 0$. Hence, by condition $(S2)$, we get $x_0 = x_1 = F(x_0, y_0)$ and $y_0 = y_1 = F(y_0, x_0)$. Thus, (x_0, y_0) is a coupled fixed point of F. Now, we assume that $t_0 > 0$. For each $m > n$, where $n, m \in \mathbb{N}$, and using $(S3)$, we have

$$\begin{aligned}
&S(x_n, x_n, x_m) + S(y_n, y_n, y_m) \\
\leq\ & 2S(x_n, x_n, x_{n+1}) + S(x_m, x_m, x_{n+1}) \\
& +2S(y_n, y_n, y_{n+1}) + S(y_m, y_m, y_{n+1}) \\
=\ & 2(S(x_n, x_n, x_{n+1}) + S(y_n, y_n, y_{n+1})) \\
& +S(x_m, x_m, x_{n+1}) + S(y_m, y_m, y_{n+1}) \\
\leq\ & \ldots \\
\leq\ & 2\Big(t_n + t_{n+1} + \cdots + t_{m-1} + t_m\Big) \\
\leq\ & 2(\nu^n + \nu^{n+1} + \cdots + \nu^{m-1} + \nu^m)t_0 \\
\leq\ & 2\nu^n(1 + \nu + \nu^2 + \ldots)t_0 \\
\leq\ & \Big(\frac{2\nu^n}{1-\nu}\Big)t_0 \\
\to\ & 0 \text{ as } n \to \infty,
\end{aligned}$$

since $0 < \nu < 1$. Thus, $\{x_n\}$ and $\{y_n\}$ are S-Cauchy sequences in X. Since X is complete, we get $\{x_n\}$ and $\{y_n\}$ are S-convergent to some $x \in X$ and $y \in X$ respectively, that is, $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Since F is continuous, then we have

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = F\Big(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\Big) = F(x, y), \tag{3.7}$$

and

$$y = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(y_n, x_n) = F\Big(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n\Big) = F(y, x). \tag{3.8}$$

This shows that (x, y) is a coupled fixed point of F.

Now, we show the uniqueness of the coupled fixed point. Assume that (x_1, y_1) is another coupled fixed point of F such that $(x, y) \neq (x_1, y_1)$. Then from equation (3.1), using $(S2)$ and Lemma 2.21, we have

$$\begin{aligned}
[S(x,x,x_1)]^2 &= [S(F(x,y),F(x,y),F(x_1,y_1)]^2 \\
&\leq p_1\,[S(x,x,x_1)]^2 + p_2\,[S(y,y,y_1)]^2 \\
&\quad +p_3\,[S(F(x,y),F(x,y),x)]^2 \\
&\quad +p_4\,[S(F(x,y),F(x,y),x)]^2 \\
&\quad +p_5\,[S(F(x_1,y_1),F(x_1,y_1),x_1)]^2 \\
&\quad +p_6\,[S(F(x,y),F(x,y),x_1)]^2 \\
&= p_1\,[S(x,x,x_1)]^2 + p_2\,[S(y,y,y_1)]^2 \\
&\quad +p_3\,[S(x,x,x)]^2 + p_4\,[S(x,x,x)]^2 \\
&\quad +p_5\,[S(x_1,x_1,x_1)]^2 + p_6\,[S(x,x,x_1)]^2 \\
&= (p_1+p_6)\,[S(x,x,x_1)]^2 + p_2\,[S(y,y,y_1)]^2. \qquad (3.9)
\end{aligned}$$

Similarly, one can prove that

$$\begin{aligned}
[S(y,y,y_1)]^2 &= [S(F(y,x),F(y,x),F(y_1,x_1)]^2 \\
&\leq (p_1+p_6)\,[S(y,y,y_1)]^2 + p_2\,[S(x,x,x_1)]^2. \qquad (3.10)
\end{aligned}$$

Set

$$\mathcal{M} = [S(x,x,x_1)]^2,\ \mathcal{N} = [S(y,y,y_1)]^2,\ \mathcal{P} = \mathcal{M} + \mathcal{N}. \qquad (3.11)$$

From equations (3.9)-(3.11), we obtain

$$\begin{aligned}
\mathcal{P} &= [S(x,x,x_1)]^2 + [S(y,y,y_1)]^2 \\
&= \mathcal{M} + \mathcal{N} \\
&\leq (p_1+p_6)\,(\mathcal{M}+\mathcal{N}) + p_2\,(\mathcal{M}+\mathcal{N}) \\
&= (p_1+p_2+p_6)\,(\mathcal{M}+\mathcal{N}) = (p_1+p_2+p_6)\,\mathcal{P},
\end{aligned}$$

which is a contradiction, since $p_1 + p_2 + p_6 < 1$. Hence, we conclude that $\mathcal{P} = 0$, that is, $[S(x,x,x_1)]^2 + [S(y,y,y_1)]^2 = 0$ or $S(x,x,x_1) = 0$ and $S(y,y,y_1) = 0$ and so $x = x_1$ and $y = y_1$. This shows that the coupled fixed point of F is unique. This completes the proof. □

Theorem 3.2. *Let (X, S) be a complete S-metric space. Let $F\colon X \times X \to X$ be a mapping satisfying the following contractive condition: for all $x, y, u, v, z, w \in X$:*

$$\begin{aligned}
&[S(F(x,y),F(u,v),F(z,w))]^2 \\
&\leq \lambda \max\Big\{[S(x,u,z)]^2, [S(y,v,w)]^2, [S(F(x,y),F(x,y),x)]^2, \\
&\qquad [S(F(u,v),F(u,v),u)]^2, [S(F(z,w),F(z,w),z)]^2, \\
&\qquad [S(F(u,v),F(u,v),z)]^2\Big\}, \qquad (3.12)
\end{aligned}$$

where $\lambda \in [0, 1)$ is a constant. If F is continuous, then F has a unique coupled point in X.

Proof. Follows from Theorem 3.1 by noting that

$$p_1\,[S(x,u,z)]^2 + p_2\,[S(y,v,w)]^2 + p_3\,[S(F(x,y),F(x,y),x)]^2 + p_4\,[S(F(u,v),F(u,v),u)]^2$$
$$+p_5\,[S(F(z,w),F(z,w),z)]^2 + p_6\,[S(F(u,v),F(u,v),z)]^2$$

$$\begin{aligned}
&\leq (p_1+p_2+p_3+p_4+p_5+p_6)\max\Big\{[S(x,u,z)]^2,[S(y,v,w)]^2,[S(F(x,y),F(x,y),x)]^2,\\
&\qquad [S(F(u,v),F(u,v),u)]^2,[S(F(z,w),F(z,w),z)]^2,[S(F(u,v),F(u,v),z)]^2\Big\}\\
&= \lambda \max\Big\{[S(x,u,z)]^2,[S(y,v,w)]^2,[S(F(x,y),F(x,y),x)]^2,[S(F(u,v),F(u,v),u)]^2,\\
&\qquad [S(F(z,w),F(z,w),z)]^2,[S(F(u,v),F(u,v),z)]^2\Big\},
\end{aligned}$$

where $\lambda = p_1+p_2+p_3+p_4+p_5+p_6 < 1$.

□

Theorem 3.3. *Let (X,S) be a complete S-metric space. Let $F\colon X\times X\to X$ be a mapping satisfying the following contractive condition: for all $x,y,u,v,z,w\in X$:*

$$\begin{aligned}
[S(F(x,y),F(u,v),F(z,w))]^2 \leq\ & q_1\max\Big\{[S(x,u,z)]^2,[S(F(x,y),F(x,y),x)]^2,\\
&\qquad [S(F(x,y),F(x,y),u)]^2\Big\}\\
&+q_2\max\Big\{[S(F(u,v),F(u,v),z)]^2,\\
&\qquad [S(F(z,w),F(z,w),z)]^2\Big\}\\
&+q_3\, S(F(x,y),F(x,y),z)S(F(z,w),F(z,w),u),
\end{aligned} \tag{3.13}$$

where q_1,q_2,q_3 are nonnegative constants with $q_1+q_2+q_3<1$. If F is continuous, then F has a unique coupled point in X.

Proof. Choose $x_0,y_0\in X$. Set $x_1=F(x_0,y_0)$ and $y_1=F(y_0,x_0)$. Repeating this process, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $x_{n+1}=F(x_n,y_n)$ and $y_{n+1}=F(y_n,x_n)$. Assume that $u_n=[S(x_n,x_n,x_{n+1})]^2$, $v_n=[S(y_n,y_n,y_{n+1})]^2$ and $t_n=u_n+v_n$. Then, from equations (3.13), using $(S2)$ and Lemma 2.21, we have

$$\begin{aligned}
u_n &= [S(x_n,x_n,x_{n+1})]^2 = [S(F(x_{n-1},y_{n-1}),F(x_{n-1},y_{n-1}),F(x_n,y_n))]^2\\
&\leq q_1\max\Big\{[S(x_{n-1},x_{n-1},x_n)]^2,[S(F(x_{n-1},y_{n-1}),F(x_{n-1},y_{n-1}),x_{n-1})]^2,\\
&\qquad [S(F(x_{n-1},y_{n-1}),F(x_{n-1},y_{n-1}),x_{n-1})]^2\Big\}\\
&\quad +q_2\max\Big\{[S(F(x_{n-1},y_{n-1}),F(x_{n-1},y_{n-1}),x_n)]^2,\\
&\qquad [S(F(x_n,y_n),F(x_n,y_n),x_n)]^2\Big\}\\
&\quad +q_3\, S(F(x_{n-1},y_{n-1}),F(x_{n-1},y_{n-1}),x_n)S(F(x_n,y_n),F(x_n,y_n),x_{n-1})\\
&= q_1\max\Big\{[S(x_{n-1},x_{n-1},x_n)]^2,[S(x_n,x_n,x_{n-1})]^2,[S(x_n,x_n,x_{n-1})]^2\Big\}\\
&\quad +q_2\max\Big\{[S(x_n,x_n,x_n)]^2,[S(x_{n+1},x_{n+1},x_n)]^2\Big\}\\
&\quad +q_3\, S(x_n,x_n,x_n)S(x_{n+1},x_{n+1},x_{n-1})\\
&= q_1\max\Big\{[S(x_{n-1},x_{n-1},x_n)]^2,S(x_{n-1},x_{n-1},x_n)]^2,S(x_{n-1},x_{n-1},x_n)]^2\Big\}\\
&\quad +q_2\max\Big\{0,[S(x_n,x_n,x_{n+1})]^2\Big\}+q_3\,.0\\
&= q_1\, S(x_{n-1},x_{n-1},x_n)]^2+q_2\, S(x_n,x_n,x_{n+1})]^2\\
&= q_1\, u_{n-1}+q_2\, u_n.
\end{aligned} \tag{3.14}$$

Similarly, one can show that

$$\begin{aligned} v_n &= [S(y_n, y_n, y_{n+1})]^2 \\ &= [S(F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}), F(y_n, x_n))]^2 \\ &\leq q_1\, v_{n-1} + q_2\, v_n. \end{aligned} \tag{3.15}$$

From equations (3.14) and (3.15), we obtain

$$\begin{aligned} t_n &= u_n + v_n \\ &\leq q_1\,(u_{n-1} + v_{n-1}) + q_2\,(u_n + v_n) \\ &= q_1\, t_{n-1} + q_2\, t_n. \end{aligned} \tag{3.16}$$

This implies that

$$t_n \leq \big(\frac{q_1}{1-q_2}\big)\, t_{n-1} = \mu\, t_{n-1}, \tag{3.17}$$

where $\mu = \big(\frac{q_1}{1-q_2}\big) < 1$, since $q_1 + q_2 < 1$.

Consequently, for each $n \in \mathbb{N}$, we obtain

$$t_n \leq \mu\, t_{n-1} \leq \mu^2\, t_{n-2} \leq \cdots \leq \mu^n\, t_0. \tag{3.18}$$

If $t_0 = 0$, then $S(x_0, x_0, x_1) + S(y_0, y_0, y_1) = 0$. Hence, by condition $(S2)$, we get $x_0 = x_1 = F(x_0, y_0)$ and $y_0 = y_1 = F(y_0, x_0)$. Thus, (x_0, y_0) is a coupled fixed point of F. Now, we assume that $t_0 > 0$. For each $m > n$, where $n, m \in \mathbb{N}$, and using $(S3)$, we have

$$\begin{aligned} & S(x_n, x_n, x_m) + S(y_n, y_n, y_m) \\ \leq\ & 2S(x_n, x_n, x_{n+1}) + S(x_m, x_m, x_{n+1}) \\ & +2S(y_n, y_n, y_{n+1}) + S(y_m, y_m, y_{n+1}) \\ =\ & 2(S(x_n, x_n, x_{n+1}) + S(y_n, y_n, y_{n+1})) \\ & +S(x_m, x_m, x_{n+1}) + S(y_m, y_m, y_{n+1}) \\ \leq\ & \ldots \\ \leq\ & 2\Big(t_n + t_{n+1} + \cdots + t_{m-1} + t_m\Big) \\ \leq\ & 2(\mu^n + \mu^{n+1} + \cdots + \mu^{m-1} + \mu^m)t_0 \\ \leq\ & 2\mu^n(1 + \mu + \mu^2 + \ldots)t_0 \\ \leq\ & \Big(\frac{2\mu^n}{1-\mu}\Big)t_0 \\ \to\ & 0 \text{ as } n \to \infty, \end{aligned}$$

since $0 < \mu < 1$. Thus, $\{x_n\}$ and $\{y_n\}$ are S-Cauchy sequences in X. Since X is complete, we get $\{x_n\}$ and $\{y_n\}$ are S-convergent to some $j \in X$ and $k \in X$ respectively, that is, $\lim_{n\to\infty} x_n = j$ and $\lim_{n\to\infty} y_n = k$. Since F is continuous, then we have

$$\begin{aligned} j &= \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n) \\ &= F\Big(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\Big) = F(j, k), \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} k &= \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(y_n, x_n) \\ &= F\Big(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n\Big) = F(k, j). \end{aligned} \tag{3.20}$$

This shows that (j, k) is a coupled fixed point of F.

Now, we show the uniqueness of the coupled fixed point. Assume that (j_1, k_1) is another coupled fixed point of F such that $(j, k) \neq (j_1, k_1)$. Then from equation (3.13), using $(S2)$ and Lemma 2.21, we have

$$\begin{aligned}
[S(j,j,j_1)]^2 &= [S(F(j,k),F(j,k),F(j_1,k_1))]^2 \\
&\leq q_1 \max\Big\{[S(j,j,j_1)]^2, [S(F(j,k),F(j,k),j)]^2, \\
&\qquad\qquad [S(F(j,k),F(j,k),j)]^2\Big\} \\
&\quad + q_2 \max\Big\{[S(F(j,k),F(j,k),j_1)]^2, \\
&\qquad\qquad [S(F(j_1,k_1),F(j_1,k_1),j_1)]^2\Big\} \\
&\quad + q_3\, S(F(j,k),F(j,k),j_1) S(F(j_1,k_1),F(j_1,k_1),j) \\
&= q_1 \max\Big\{[S(j,j,j_1)]^2, [S(j,j,j)]^2, [S(j,j,j)]^2\Big\} \\
&\quad + q_2 \max\Big\{[S(j,j,j_1)]^2, [S(j_1,j_1,j_1)]^2\Big\} \\
&\quad + q_3\, S(j,j,j_1) S(j_1,j_1,j) \\
&= q_1 \max\Big\{[S(j,j,j_1)]^2, 0, 0\Big\} + q_2 \max\Big\{[S(j,j,j_1)]^2, 0\Big\} \\
&\quad + q_3\, [S(j,j,j_1)]^2 \\
&= (q_1 + q_2 + q_3)\, [S(j,j,j_1)]^2 \\
&< [S(j,j,j_1)]^2,
\end{aligned}$$

which is a contradiction, since $q_1 + q_2 + q_3 < 1$. Hence, we conclude that $[S(j, j, j_1)]^2 = 0$, that is, $S(j, j, j_1) = 0$ and so $j = j_1$. Similarly, we can show that $k = k_1$. Thus, we have shown that $(j, k) = (j_1, k_1)$. Consequently, the coupled fixed point of F is unique. This completes the proof. □

Remark 3.4. Our results extend and generalize the corresponding results of Aydi [2] from partial metric spaces to the setting of S-metric spaces.

4. Application

In this section, we state some applications of the main results of a self mapping which is involved in an integral type contraction.

Let us denote a set Ω of all of functions $\psi\colon [0, +\infty) \to [0, +\infty)$ satisfying the following properties:

(i) Each ψ is a Lebesgue-integrable mapping on every compact subset of $[0, +\infty)$,

(ii) For any $\varepsilon > 0$ we have $\int_0^{\varepsilon} \psi(t)dt > 0$.

Theorem 4.1. *Let (X, S) be a complete S-metric space. Suppose that the mapping $F\colon X \times X \to X$ satisfying the following contractive condition for all $x, y, u, v, z, w \in X$:*

$$\begin{aligned}
\int_0^{[S(F(x,y),F(u,v),F(z,w))]^2} \phi(t)dt &\leq p_1 \int_0^{[S(x,u,z)]^2} \phi(t)dt + p_2 \int_0^{[S(y,v,w)]^2} \phi(t)dt \\
&\quad + p_3 \int_0^{[S(F(x,y),F(x,y),x)]^2} \phi(t)dt + p_4 \int_0^{[S(F(u,v),F(u,v),u)]^2} \phi(t)dt \\
&\quad + p_5 \int_0^{[S(F(z,w),F(z,w),z)]^2} \phi(t)dt + p_6 \int_0^{[S(F(u,v),F(u,v),z)]^2} \phi(t)dt, \qquad (4.1)
\end{aligned}$$

where $p_1, p_2, \ldots, p_6$ are nonnegative constants with $p_1 + p_2 + p_3 + p_4 + p_5 + p_6 < 1$ and $\phi \in \Omega$. If F is continuous, then F has a unique coupled fixed point.

Theorem 4.2. *Let (X, S) be a complete S-metric space. Suppose that the mapping $F\colon X \times X \to X$ satisfying the following contractive condition for all $x, y, u, v, z, w \in X$:*

$$\int_0^{[S(F(x,y),F(u,v),F(z,w))]^2} \phi(t)dt \leq \lambda \int_0^{\Lambda(x,y,u,v,z,w)} \phi(t)dt, \tag{4.2}$$

where

$$\begin{aligned}\Lambda(x,y,u,v,z,w) \quad = \quad \max\Big\{&[S(x,u,z)]^2, [S(y,v,w)]^2, [S(F(x,y),F(x,y),x)]^2,\\ &[S(F(u,v),F(u,v),u)]^2, [S(F(z,w),F(z,w),z)]^2,\\ &[S(F(u,v),F(u,v),z)]^2\Big\},\end{aligned}$$

$\lambda \in [0, 1)$ is a constant and $\phi \in \Omega$. If F is continuous, then F has a unique coupled fixed point.

Theorem 4.3. *Let (X, S) be a complete S-metric space. Suppose that the mapping $F\colon X \times X \to X$ satisfying the following contractive condition for all $x, y, u, v, z, w \in X$:*

$$\begin{aligned}\int_0^{[S(F(x,y),F(u,v),F(z,w))]^2} \phi(t)dt \quad \leq \quad & q_1 \int_0^{\Lambda_1(x,y,u,v,z,w)} \phi(t)dt + q_2 \int_0^{\Lambda_2(x,y,u,v,z,w)} \phi(t)dt\\ & + q_3 \int_0^{\Lambda_3(x,y,u,v,z,w)} \phi(t)dt,\end{aligned} \tag{4.3}$$

where

$$\Lambda_1(x,y,u,v,z,w) = \max\Big\{[S(x,u,z)]^2, [S(F(x,y),F(x,y),x)]^2, [S(F(x,y),F(x,y),u)]^2\Big\},$$

$$\Lambda_2(x,y,u,v,z,w) = \max\Big\{[S(F(u,v),F(u,v),z)]^2, [S(F(z,w),F(z,w),z)]^2\Big\},$$

$$\Lambda_3(x,y,u,v,z,w) = S(F(x,y),F(x,y),z)S(F(z,w),F(z,w),u),$$

q_1, q_2, q_3 are nonnegative constants such that $q_1 + q_2 + q_3 < 1$ and $\phi \in \Omega$. If F is continuous, then F has a unique coupled fixed point.

Now, we give an example to validate the result.

Example 4.4. Let $X = [0, 1]$ and the function $S\colon X^3 \to [0, \infty)$ be defined as $S(x, y, z) = |y - z| + |y + z - 2x|$ for all $x, y, z \in X$. Then the function S is an S-metric on X and (X, S) is an S-metric space. Define the mapping $F\colon X \times X \to X$ by $F(x, y) = \frac{x+y}{5}$. Then, we have

$$[S(F(x,y),F(u,v),F(z,w))]^2$$

$$
\begin{aligned}
&= [|F(u,v)+F(z,w)-2F(x,y)| \\
&\quad +|F(u,v)-F(z,w)|]^2 \\
&= \Big[\Big|\frac{u+v}{5}+\frac{z+w}{5}-\frac{2(x+y)}{5}\Big| \\
&\quad +\Big|\frac{u+v}{5}-\frac{z+w}{5}\Big|\Big]^2 \\
&= \Big[\frac{1}{5}|u+z-2x|+\frac{1}{5}|v+w-2y| \\
&\quad +\frac{1}{5}|u-z|+\frac{1}{5}|v-w|\Big]^2 \\
&= \frac{1}{25}\Big[\big(|u+z-2x|+|u-z|\big) \\
&\quad +\big(|v+w-2y|+|v-w|\big)\Big]^2 \\
&= \frac{1}{25}\big[S(x,u,z)+S(y,v,w)\big]^2 \\
&\leq \frac{1}{12}\Big([(S(x,u,z)]^2+[S(y,v,w)]^2\Big) \\
&\leq \frac{1}{12}\Big([S(x,u,z)]^2+[S(y,v,w)]^2 \\
&\qquad +[S(F(x,y),F(x,y),x)]^2 \\
&\qquad +[S(F(u,v),F(u,v),u)]^2 \\
&\qquad +[S(F(z,w),F(z,w),z)]^2 \\
&\qquad +[S(F(u,v),F(u,v),z)]^2\Big),
\end{aligned}
$$

holds for all $x,y,z,u,v,w \in X$. It is easy to see that F satisfies all the conditions of Theorem 3.1 for $p_1=p_2=p_3=p_4=p_5=p_6=\frac{1}{12}$ with $p_1+p_2+p_3+p_4+p_5+p_6=\frac{1}{2}<1$. Thus F has a unique coupled fixed point, namely $F(0,0)=0$.

5. Conclusion

In this paper, we prove some unique coupled fixed point theorems via contractive type conditions in the setting of S-metric spaces and as an application of our main results we give some fixed point results for integral type contractions. Our results extend and generalize several previously published results from the existing literature.

6. Acknowledgement

The author would like to thank the Executive Editor A. Mihai for her careful reading and helpful comments to improve the quality of this manuscript.

References

[1] M. Abbas and B. E. Rhoades, *Common fixed point results for non-commuting mappings without continuity generalized metric spaces*, Appl. Math. Computation **215** (2009), 262-269.

[2] H. Aydi, *Some coupled fixed point results on partial metric spaces*, International J. Math. Math. Sci. 2011, art. no. 647091.

[3] S. Banach, *Sur les operation dans les ensembles abstraits et leur application aux equation integrals*, Fund. Math. **3** (1922), 133-181.

[4] T. Gnana Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Analysis: TMA **65(7)** (2006), 1379-1393.

[5] S. Chandok, T. D. Narang and M. A. Taudi, *Some coupled fixed point theorems for mappings satisfying a generalized contractive condition of rational type*, Palestine J. Math. **4(2)** (2015), 360-366.

[6] B. S. Choudhary and P. Maity, *Coupled fixed point results in generalized metric spaces*, Math. Comput. Model. **54(1-2)** (2011), 73-79.

[7] L. Ciric and V. Lakshmikantham, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Analysis: TMA **70(12)** (2009), 4341-4349.

[8] N. V. Dung, N. T. Hieu and S. Radojević, *Fixed point theorems for g-monotone maps on partially ordered S-metric spaces*, Filomat **28(9)** (2014), 1885-1898.

[9] A. Gupta, *Cyclic contraction on S-metric space*, Int. J. Anal. Appl. **3(2)** (2013), 119-130.

[10] N. T. Hieu, N. T. Ly and N. V. Dung, *A generalization of Ciric quasi-contractions for maps on S-metric spaces*, Thai J. Math. **13(2)** (2015), 369-380.

[11] G. Jungck, *Compatible mappings and common fixed points*, Int. J. Math. Math. Sci. **9** (1986), 771-779.

[12] G. Jungck, *Common fixed points for noncontinuous, nonself maps on nonnumetric spaces*, Far East J. Math. Sci. **4(2)** (1996), 195-215.

[13] J. K. Kim, S. Sedghi, A. Gholidahneh and M. M. Rezaee, *Fixed point theorems in S-metric spaces*, East Asian Math. J. **32(5)** (2016), 677-684.

[14] J. K. Kim, G. A. Okeke and W. H. Lim, *Common coupled fixed point theorems for w-compatible mappings in partial metric spaces*, Global J. Pure Appl. Math. **13(2)** (2017), 519-536.

[15] N. V. Luong and N. X. Thuan, *Coupled fixed points theorems for mixed monotone mappings and an application to integral equations*, Comput. Math. Appl. **62** (2011), 4238-4248.

[16] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal. **7** (2006), 289-297.

[17] H. K. Nashine, J. K. Kim, A. K. Sharma and G. S. Saluja, *Some coupled fixed point without mixed monotone mappings*, Nonlinear Funct. Anal. Appl. **21(2)** (2016), 235-247.

[18] J. O. Olaleru, G. A. Okeke and H. Akewe, *Coupled fixed point theorems for generalized φ-mappings satisfying contractive condition of integral type on cone metric spaces*, Int. J. Math. Model. Comput. **2(2)** (2012), 87-98.

[19] N. Y. Özgür and N. Tas, *Some new contractive mappings on S-metric spaces and their relationships with the mapping* ($S25$), Math. Sci. **11(7)** (2017), 7-16.

[20] F. Sabetghadam, H. P. Mashiha and A. H. Sanatpour, *Some coupled fixed point theorems in cone metric spaces*, Fixed Point Theory Appl. (2009), art. no. 125426.

[21] G. S. Saluja, *Coupled fixed point results on complex partial metric spaces for contractive type conditions*, Annals Math. Comput. Sci. **12** (2023), 34-45.

[22] G. S. Saluja, *Coupled fixed point results based on control function in complex partial metric spaces*, Funct. Anal. Apporx. Comput. **15(2)** (2023), 1-15.

[23] S. Sedghi, N. Shobe and H. Zhou, *A common fixed point theorem in metric spaces*, Fixed Point Theory Appl. **2007** (2007), art. no. 027906.

[24] S. Sedghi, N. Shobe and A. Aliouche, *A generalization of fixed point theorems in S-metric spaces*, Mat. Vesnik **64(3)** (2012), 258-266.

[25] S. Sedghi and N. V. Dung, *Fixed point theorems on S-metric spaces*, Mat. Vesnik **66(1)** (2014), 113-124.

[26] S. Sedghi, N. Shobkolaei, M. Shahraki and T. Dosenovic, *Common fixed point of four maps in S-metric space*, Math. Sci. **12** (2018), 137-143.

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