ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2012, VOLUME 2, ISSUE 2, p.61-69

BOX-COUNTING DIMENSION OF A KIND OF FRACTAL INTERPOLATION SURFACE ON RECTANGULAR GRIDS

Cholhui Yun*

Abstract

We estimate a Box-counting dimension of fractal surfaces, which are generated by iterated function systems with a vertical contraction factor function on an arbitrary data set over rectangular grids and can express well many natural surfaces with very complicated structures.

Mathematical Subject Classification: 37C45; 28A80
Keywords: Iterated function system (IFS); Fractal interpolation function (FIF); Fractal interpolation surface (FIS); Box-counting dimension

1. Introduction

A fractal interpolation surface (FIS) is a fractal set which is a graph of an interpolation function. Therefore, constructions of FISs are closely associated with one of fractal interpolation function, i.e. interpolation functions whose graph is fractal sets. By Barnsley[1], FIFs were introduced in 1986 and after that have widely been studied in approximation theory, image compression, computer graphics and so on. In many papers constructions of fractal interpolation surfaces on the basis of IFSs were studied ([2, 3, 4, 5, 6, 7, 8]). Massopust presented the construction of FISs on rectangular data sets, at which the interpolation points on the boundary are coplanar. Geronimo and Hardin generalized this construction to allow more general boundary data.

Constructions of fractal interpolation surfaces, which interpolate a data set over rectangular grid, were studied in [2, 3, 5, 7]. A lack of constructions is to use IFSs constructed with a restricted data set, whose data points on the boundary are collinear, constant contraction factor, quadratic polynomials. In [7], these constructions were generalized with an arbitrary data set, Lipschitz function, contraction factor function, lower and upper bounds for the Box-counting of the constructed surface.

In this paper, we consider the problem on improving estimation of the box counting dimension of fractal surfaces with vertical contraction factor function as fractal surfaces constructed in [7].

2. FISs over the rectangular grids

We consider the construction of fractal surfaces presented in [7]. Let the data

**Faculty of Mathematics, Kim Il Sung University, D.P.R.K*
mathcci@yahoo.com

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set over the rectangular grid be

$$P = \{(x_i, y_j, z_{ij}) \in \mathrm{R}^3; i = 0, 1, \cdots, n,\ j = 0, 1, \cdots, m\}$$

$$(x_0 < x_1 < \cdots < x_n,\ \ y_0 < y_1 < \cdots < y_m)$$

and denote

$$N_{nm} = \{1, \cdots, n\} \times \{1, \cdots, m\},\ \ I_x = [x_0, x_n],\ \ I_y = [y_0, y_m], E = I_x \times I_y,$$

$$I_{x_i} = [x_{i-1}, x_i],\ \ I_{y_j} = [y_{j-1}, y_j],\ \ E_{ij} = I_{x_i} \times I_{y_j}, (i, j) \in N_{nm},$$

$$P_{x_\alpha} = \{(x_\alpha, y_l, z_{\alpha l}) \in P; l = 0, 1, \cdots, m\},\ \ \ \alpha = 0, 1, \cdots, n,$$

$$P_{y_\beta} = \{(x_k, y_\beta, z_{k\beta}) \in P; k = 0, 1, \cdots, n\},\ \ \beta = 0, 1, \cdots, m.$$

We define the domain contraction transformations $L_{ij} : E \to E_{ij}$ for $(i, j) \in N_{nm}$ by $L_{ij}(x, y) = (L_{x_i}(x), L_{y_j}(y))$ where $L_{x_i} : I_x \to I_{x_i}$ and $L_{y_j} : I_y \to I_{y_j}$ are contractive homeomorphisms with contraction factors c_{x_i}, c_{y_j} satisfying the following conditions.

(1) $L_{x_i} : \{x_0, x_n\} \to \{x_{i-1}, x_i\},\ L_{y_j} : \{y_0, y_m\} \to \{y_{j-1}, y_j\}$

(2) For any $i \in \{1, \cdots, n-1\},\ j \in \{1, \cdots, m-1\}$, there are $x_k \in \{x_0, x_n\}$, $y_l \in \{y_0, y_m\}$ such that

$$L_{x_{i+1}}(x_k) = L_{x_i}(x_k) = x_i,\ \ \ L_{y_{j+1}}(y_l) = L_{y_j}(y_l) = y_j. \tag{1}$$

Then $c_{ij} = \mathrm{Max}\{c_{x_i}, c_{y_j}\}, (i, j) \in N_{nm}$ are contractivity factors of the transformations L_{ij}. Functions $F_{ij} : E \times \mathrm{R} \to \mathrm{R}, (i, j) \in N_{nm}$ are defined by $F_{ij}(x, y, z) = s(L_{ij}(x, y))(z - g(x, y)) + h(L_{ij}(x, y))$, where $s(x, y)$ is a vertical continuous contraction function such that $0 <| s(x, y) |< 1$ on E, $h(x, y)$, $g(x, y)$ are continuous Lipschitz functions on E with the Lipschitz constants L_h, L_g satisfying

$$g(x_\alpha, y_\beta) = z_{\alpha\beta},\ \ (\alpha, \beta) \in \{0, n\} \times \{0, m\},$$

$$h(x_i, y_j) = z_{ij},\ \ \ \ \ (i, j) \in \{0, 1, \cdots, n\} \times \{0, 1, \cdots, m\}.$$

Then transformations $W_{ij} = (L_{ij}, F_{ij})^T, (i, j) \in N_{nm}$ coincide on common borders and are contractions with respected to some metric that is equivalent to the Euclidean metric on R^3. And the attractor A of the IFS $\{\mathrm{R}^3; W_{ij}, i = 1, \cdots, n,\ \ j = 1, \cdots, m\}$ is a graph of a continuous function $f : E \to \mathrm{R}$, i.e., a surface in R^3. The type of f is as follows.

$$
\begin{aligned}
f(x, y) &= s(x, y) f(L_{ij}^{-1}(x, y)) + Q(x, y) \\
Q(x, y) &= -s(x, y) g(L_{ij}^{-1}(x, y)) + h(x, y)
\end{aligned} \tag{2}
$$

3. Box-counting dimension of interpolation surface

In this section, we calculate lower and upper bounds for Box-counting dimension of the surface constructed above with the data set

$$P = \left\{ \left(x_0 + \frac{x_n - x_0}{n} i,\ y_0 + \frac{y_n - y_0}{n} j,\ z_{ij} \right) \in \mathrm{R}^3;\ i,\ j = 0, 1, \cdots, n \right\}$$

Since there is a bi-Lipschitz mapping between some rectangular in R^2 and $[0, 1] \times [0, 1]$, and Box-counting dimension is invariant under bi- Lipschitz mapping, we can assume that $E = [0, 1] \times [0, 1]$. Then

$$P = \left\{ \left(\frac{i}{n}, \frac{j}{n}, z_{ij} \right) \in \mathrm{R}^3;\ i,\ j = 0, 1, \cdots, n \right\}.$$

For D a compact subset of R^2, let the maximum range of f on D be denoted by $R_f[D] := \sup\{| f(x_2, y_2) - f(x_1, y_1) |; (x_1, y_1), (x_2, y_2) \in D\}$.

Lemma 1. *Let $W : D \times \mathrm{R} \to D \times \mathrm{R}$ be the form*

$$W\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} L(x, y) \\ F(x, y, z) \end{pmatrix} = \begin{pmatrix} L_x(x) \\ L_y(y) \\ s(L(x, y))z + Q(x, y) \end{pmatrix}$$

where Q is Lipschitz function with the Lipschitz constants L_Q, L is the domain contraction transformation with contraction factor c_L defined as L_{ij} and $s(x, y)$ is also a contraction function with $0 <| s(x, y) |< 1$.

Then, for any continuous function $f : D \to \mathrm{R}$, $R_{F(L^{-1},\, f \circ L^{-1})}[L(D)] \le \bar{s} R_f[D] + \mathrm{diam}(D)(c_s \overline{f} + L_Q)$, where $\mathrm{diam}(D)$ is a diameter of the set D , $\bar{s} = \operatorname{Max}_D | s(x, y) |$, c_s is a contraction factor of $s(x, y)$, $\overline{f} = \operatorname{Max}_D | f(x, y) |$.

Proof. For $(x, y), (x', y')(\in L(D))$, let denote $L^{-1}(x, y) =: (\tilde{x}, \tilde{y})$, $L^{-1}(x', y') =: (\tilde{x}', \tilde{y}')(\in D)$. Then

$$
\begin{aligned}
&| F(L^{-1}, f \circ L^{-1})(x, y) - F(L^{-1}, f \circ L^{-1})(x', y') | = \\
&=| F(L^{-1}(x, y), f \circ L^{-1}(x, y)) - F(L^{-1}(x', y'), f \circ L^{-1}(x', y')) | \\
&=| s(x, y) f(\tilde{x}, \tilde{y}) + Q(\tilde{x}, \tilde{y}) - s(x', y') f(\tilde{x}', \tilde{y}') - Q(\tilde{x}', \tilde{y}') |
\end{aligned}
$$

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$$=|s(x,y)f(\tilde{x},\tilde{y})-s(x,y)f(\tilde{x}',\tilde{y}')+s(x,y)f(\tilde{x}',\tilde{y}')-s(x',y')f(\tilde{x}',\tilde{y}')$$
$$+Q(\tilde{x},\tilde{y})-Q(\tilde{x}',\tilde{y}')|$$
$$\le \bar{s}R_f[D]+c_s d((x,y),(x',y'))\overline{f}+L_Q d((\tilde{x},\tilde{y}),(\tilde{x}',\tilde{y}'))$$
$$\le \bar{s}R_f[D]+\mathrm{diam}(D)(c_s\overline{f}+L_Q). \qquad \square$$

For the $N\times N$ matrices $U=(u_{ij})_{N\times N}$, $V=(v_{ij})_{N\times N}$, a relation "<" is defined by $U<V\overset{d}{\Leftrightarrow}u_{ij}<v_{ij},\ i,j=1,2,\cdots,N$ And let denote $\bar{s}_{ij}=\underset{E_{ij}}{\mathrm{Max}}|s(x,y)|$, $\tilde{s}_{ij}=\underset{E_{ij}}{\mathrm{Min}}|s(x,y)|$.

Theorem 1. *If there is* $\alpha_0\in\{0,1,\cdots,n\}$ *(or* $\beta_0\in\{0,1,\cdots,n\}$*) such that the points of* $P_{x_{\alpha_0}}$ *(or* $P_{y_{\beta_0}}$*) are not collinear, then the Box-dimension* $\dim_B A$ *of the attractor* A *is as follows.*

(1) *If* $\sum_{i,j=1}^{n}\tilde{s}_{ij}>n$, *then* $1+\log_n^{\sum_{i,j=1}^{n}\tilde{s}_{ij}}\le\dim_B A\le 1+\log_n^{\sum_{i,j=1}^{n}\bar{s}_{ij}}$.

(2) *If* $\sum_{i,j=1}^{n}\bar{s}_{ij}\le n$, $\dim_B A=2$.

Proof. We first prove (1). By the hypothesis of the theorem, maximum vertical distance calculated only with respect to Z axis from the points of $P_{x_{\alpha_0}}$ (or $P_{y_{\beta_0}}$) to the line which passes through two end points of $P_{x_{\alpha_0}}$ (or $P_{y_{\beta_0}}$) is positive. This is called a height and denoted by H.

After each W_{ij} is applied to the interpolation points in E, we obtain $(n+1)^2$ new points in every E_{ij} and the vertical lines parallel to z axis are mapped to the vertical lines parallel to z axis by W_{ij}. Hence all of vertical lines with length H are mapped to vertical lines in E_{ij} whose length is greater than $\tilde{s}_{ij}H$. And using Lemma 1 for $R_f[E_{ij}]$, we have $R_f[E_{ij}]\le\bar{s}_{ij}R_f[E]+b$, where $b=\sqrt{2}(c_s\overline{f}+L_Q)$. Let denote $R_f[E_{ij}]$ by R_{ij}.

Let an injection $\tau:\{1,\cdots,n\}\times\{1,\cdots,n\}\to\{1,\cdots,n^2\}$ be defined by $\tau(i,j)=(i-1)n+j$. Let $n^2\times n^2$ the diagonal matrices $\overline{S}$, $\widetilde{S}$ and let the vectors $\mathbf{h}_1,\mathbf{r},\mathbf{u}_1,\mathbf{i}$ be as follows.

$$\overline{S}=\mathrm{diag}\left(\bar{s}_{\tau^{-1}(1)},\cdots,\bar{s}_{\tau^{-1}(n^2)}\right),\quad \widetilde{S}=\mathrm{diag}\left(\tilde{s}_{\tau^{-1}(1)},\cdots,\tilde{s}_{\tau^{-1}(n^2)}\right),$$

$$\mathbf{h}_1 = \begin{pmatrix} \tilde{s}_{\tau^{-1}(1)}H \\ \vdots \\ \tilde{s}_{\tau^{-1}(n^2)}H \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} \bar{s}_{\tau^{-1}(1)}R_{\tau^{-1}(1)} \\ \vdots \\ \bar{s}_{\tau^{-1}(n^2)}R_{\tau^{-1}(n^2)} \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad \mathbf{u}_1 = \mathbf{r} + b\mathbf{i}.$$

For $r(>0)$ let $\varepsilon_r := \dfrac{1}{n^r}$, then $\varepsilon_r \to 0 \Leftrightarrow r \to \infty$. And let $N(\varepsilon_r)$ be defined by the smallest number of ε_r-mesh cubs that covers A. Then, since A is the graph of continuous function on E, the smallest number of ε_r-mesh cubs that cover $(E_{ij} \times \mathrm{R}) \cap A$ is greater than one of ε_r-mesh cubs that cover vertical lines with the length $\tilde{s}_{ij}H$, and it is less than one of ε_r-mesh cubs that cover parallelepiped $E_{ij} \times [\tilde{f}_{ij}, \bar{f}_{ij}]$, where $\tilde{f}_{ij} = \underset{E_{ij}}{\mathrm{Min}} | f(x, y) |$, $\bar{f}_{ij} = \underset{E_{ij}}{\mathrm{Max}} | f(x, y) |$. Hence,

$$\sum_{i,j=1}^{n} [\tilde{s}_{ij} H \varepsilon_r^{-1}] \le N(\varepsilon_r) \le \sum_{i,j=1}^{n} ([(\bar{s}_{ij} R_{ij} + b)\varepsilon_r^{-1}] + 1)\left(\left[\frac{\varepsilon_r^{-1}}{n}\right] + 1\right)^2,$$

$$\sum_{k=1}^{n^2} (\tilde{s}_{\tau^{-1}(k)} H \varepsilon_r^{-1}) - n^2 \le N(\varepsilon_r) \le \sum_{k=1}^{n^2} ((\bar{s}_{\tau^{-1}(k)} R_{\tau^{-1}(k)} + b)\varepsilon_r^{-1} + 1)\left(\left[\frac{\varepsilon_r^{-1}}{n}\right] + 1\right)^2,$$

that is, $\Phi(\mathbf{h}_1 \varepsilon_r^{-1}) - n^2 \le N(\varepsilon_r) \le \Phi(\mathbf{u}_1 \varepsilon_r^{-1} + \mathbf{i})\left(\left[\dfrac{\varepsilon_r^{-1}}{n}\right] + 1\right)^2$, where for vector $\mathbf{a} = (a_1, \cdots, a_m)$, $\Phi(\mathbf{a}) = a_1 + \cdots + a_m$ and $\varepsilon_r^{-1} \ge n$.

After applying W_{ij} to E two times, we get n^2 squares of side $1/n^2$ in E_{ij}. Since each square is obtained from each E_{kl} lying inside E by transformation W_{kl}, the sum of maximum ranges of f on n^2 squares of side $1/n^2$ contained in E_{ij} is less than or equal to the coordinates of vector $\mathbf{u_2} = \bar{S}C\mathbf{u_1} + nb\mathbf{i}$ and greater than or equal to the coordinates of vector $\mathbf{h_2} = \tilde{S}C\mathbf{h_1}$. And we have

$$\Phi(\mathbf{h_2}\varepsilon_r^{-1}) - n^4 \le N(\varepsilon_r) \le \Phi(\mathbf{u_2}\varepsilon_r^{-1} + n^2\mathbf{i})\left(\left[\frac{\varepsilon_r^{-1}}{n^2}\right] + 1\right)^2,$$

where C is $n^2 \times n^2$ matrix whose entries are all 1 and $\varepsilon_r^{-1} \ge n^2$.

After applying W_{ij} to E three times, we get $\mathbf{u_3} = \bar{S}C\mathbf{u_2} + n^2 b\mathbf{i}$, $\mathbf{h_3} = \tilde{S}C\mathbf{h_2}$. Hence, after taking k such that

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$$\varepsilon_r < \frac{1}{n^k} \le n\varepsilon_r, \tag{3}$$

that is,

$$r > k \ge r-1 \tag{4}$$

and applying W_{ij} to E k times, we get $n^{2(k-1)}$ squares of side $1/n^k$ contained in E_{ij} and

$$\Phi(\mathbf{h}_k\varepsilon_r^{-1}) - n^{2k} \le N(\varepsilon_r) \le \Phi(\mathbf{u}_k\varepsilon_r^{-1} + n^{2(k-1)}\mathbf{i})\left(\left[\frac{\varepsilon_r^{-1}}{n^k}\right]+1\right)^2, \tag{5}$$

where $\mathbf{u}_k = \overline{S}C\mathbf{u}_{k-1} + n^{k-1}b\mathbf{i}$, $\mathbf{h}_k = \widetilde{S}C\mathbf{h}_{k-1}$. Then

$$\mathbf{u}_k = (\overline{S}C)^{k-1}\mathbf{r} + (\overline{S}C)^{k-1}b\mathbf{i} + (\overline{S}C)^{k-2}nb\mathbf{i} + \cdots + (\overline{S}C)n^{k-2}b\mathbf{i} + n^{k-1}b\mathbf{i} \tag{6}$$

$$\mathbf{h}_k = (\widetilde{S}C)^{k-1}\mathbf{h}_1 \tag{7}$$

Since $\widetilde{S}C$ and $\overline{S}C$ are non-negative irreducible matrices, from Frobenius' theorem there are strictly positive eigenvectors of $\widetilde{S}C$ and $\overline{S}C$ which correspond to eigenvalues $\widetilde{a} := \sum_{i,j=1}^{n}\widetilde{s}_{ij}$ and $\overline{a} := \sum_{i,j=1}^{n}\overline{s}_{ij}$. Therefore we can choose strictly positive eigenvectors $\widetilde{\mathbf{a}}$ and $\overline{\mathbf{a}}$ which correspond to eigenvalues $\widetilde{a}$ and $\overline{a}$ so that

$$0 < \widetilde{\mathbf{a}} < \mathbf{h_1}, \tag{8}$$

$$\mathbf{r} < \overline{\mathbf{a}}, \quad b\mathbf{i} < \overline{\mathbf{a}}n. \tag{9}$$

Then by (5),

$$\begin{aligned} N(\varepsilon_r) &\le \Phi(\mathbf{u}_k\varepsilon_r^{-1} + n^{2(k-1)}\mathbf{i})\left(\left[\frac{\varepsilon_r^{-1}}{n^k}\right]+1\right)^2 \le \Phi(\mathbf{u}_k\varepsilon_r^{-1} + n^{2(k-1)}\mathbf{i})(n+1)^2 \\ &\le \Phi((\overline{S}C)^{k-1}\mathbf{r}\varepsilon_r^{-1} + (\overline{S}C)^{k-1}b\mathbf{i}\varepsilon_r^{-1} + (\overline{S}C)^{k-2}nb\mathbf{i}\varepsilon_r^{-1} + \cdots + (\overline{S}C)n^{k-2}b\mathbf{i}\varepsilon_r^{-1} + \\ &+ n^{k-1}b\mathbf{i}\varepsilon_r^{-1} + n^{2(k-1)}\mathbf{i})(n+1)^2 \\ &\le \Phi((\overline{S}C)^{k-1}\overline{\mathbf{a}}\varepsilon_r^{-1} + (\overline{S}C)^{k-1}\overline{\mathbf{a}}\varepsilon_r^{-1}n + (\overline{S}C)^{k-2}\overline{\mathbf{a}}\varepsilon_r^{-1}n^2 + \cdots + (\overline{S}C)\overline{\mathbf{a}}\varepsilon_r^{-1}n^{k-1} + \\ &\le (\overline{a}^{r-1}\overline{\mu}\varepsilon_r^{-1} + \overline{a}^{r-1}\overline{\mu}\varepsilon_r^{-1}n + \overline{a}^{r-2}\overline{\mu}\varepsilon_r^{-1}n^2 + \cdots + \overline{a}\overline{\mu}\varepsilon_r^{-1}n^{r-1} + \\ &+ \overline{\mu}\varepsilon_r^{-1}n^r + \varepsilon_r^{-1}n^r)(n+1)^2, \end{aligned} \tag{10}$$

where $\overline{\mu} = \Phi(\overline{\mathbf{a}})$.

On the other hand, since $0 < \widetilde{s}_{ij} \le \overline{s}_{ij}$, $i, j = 1, \cdots, n$, we have $\widetilde{a} = \sum_{i,j=1}^{n}\widetilde{s}_{ij} \le \sum_{i,j=1}^{n}\overline{s}_{ij} = \overline{a}$. If $\widetilde{a} > n$, then $1 > \frac{n}{\widetilde{a}} \ge \frac{n}{\overline{a}}$. Therefore,

$$N(\varepsilon_r) \le \overline{a}^{r-1}\varepsilon_r^{-1}\overline{\mu}\left(1 + n + \frac{n^2}{\overline{a}} + \cdots + \frac{n^r}{\overline{a}^{r-1}} + \frac{n^r}{\overline{\mu}\overline{a}^{r-1}}\right)(n+1)^2$$

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$$= \bar{a}^{r-1}\varepsilon_r^{-1}\bar{\mu}\left(1+\frac{n\left(1-\left(\frac{n}{\bar{a}}\right)^r\right)}{1-\frac{n}{\bar{a}}}+\frac{n^r}{\bar{\mu}\bar{a}^{r-1}}\right)(n+1)^2.$$

But since

$$\gamma := 1+\frac{n\left(1-\left(\frac{n}{\bar{a}}\right)^r\right)}{1-\frac{n}{\bar{a}}}+\frac{n^r}{\bar{\mu}\bar{a}^{r-1}} > 0,$$

we have

$$\log N(\varepsilon_r) \le (r-1)\log\bar{a} + \log\varepsilon_r^{-1} + \log(\bar{\mu}\gamma(n+1)^2),$$

$$\frac{\log N(\varepsilon_r)}{-\log\varepsilon_r} \le (r-1)\frac{\log\bar{a}}{r\log n} + 1 + \frac{\log(\bar{\mu}\gamma(n+1)^2)}{-\log\varepsilon_r}$$
$$= \log_n^{\bar{a}} - \frac{\log\bar{a}}{r\log n} + 1 + \frac{\log(\bar{\mu}\gamma(n+1)^2)}{-\log\varepsilon_r},$$

where $\log x$ implies $\log_a^x$ with $a > 1$. Hence

$$\dim_B A = \lim_{\varepsilon_r\to 0}\frac{\log N(\varepsilon_r)}{-\log\varepsilon_r} \le 1+\log_n^{\bar{a}} = 1+\log_n^{\sum_{i,j=1}^{n}\bar{s}_{ij}}. \tag{11}$$

By (5),

$$N(\varepsilon_r) \ge \Phi(\mathbf{h}_k\varepsilon_r^{-1}) - n^{2k} = \Phi((\tilde{S}C)^{k-1}\mathbf{h}_1\varepsilon_r^{-1}) - n^{2k}$$
$$\ge \Phi((\tilde{S}C)^{k-1}\tilde{\mathbf{a}}\varepsilon_r^{-1}) - n^{2k} = \tilde{a}^{k-1}\Phi(\tilde{\mathbf{a}})\varepsilon_r^{-1} - n^{2k}$$
$$\ge \tilde{a}^{r-2}\varepsilon_r^{-1}\tilde{\mu} - n^{2r} = \varepsilon_r^{-1}\tilde{a}^{r-2}\left(\tilde{\mu} - \frac{n^r}{\tilde{a}^{r-2}}\right),$$

where $\tilde{\mu} = \Phi(\tilde{\mathbf{a}})$. But since $\tilde{a} > n$, there is r_0 such that for all $r(> r_0)$

$$\tilde{\mu} - \frac{n^r}{\tilde{a}^{r-2}} > 0.$$

Hence for $r(> r_0)$

$$\frac{\log N(\varepsilon_r)}{-\log \varepsilon_r} \ge 1 + (r-2)\frac{\log \tilde{a}}{r \log n} + \frac{\log\left(\tilde{\mu} - \frac{n^r}{\tilde{a}^{r-2}}\right)}{-\log \varepsilon_r}$$

$$\dim_B A = \lim_{\varepsilon_r \to 0} \frac{\log N(\varepsilon_r)}{-\log \varepsilon_r} \ge 1 + \log_n^{\tilde{a}} = 1 + \log_n^{\sum_{i,j=1}^{n} \tilde{s}_{ij}} . \tag{12}$$

By (11), (12) if $\sum_{i,j=1}^{n} \tilde{s}_{ij} > n$, we get $1 + \log_n^{\sum_{i,j=1}^{n} \tilde{s}_{ij}} \le \dim_B A \le 1 + \log_n^{\sum_{i,j=1}^{n} \bar{s}_{ij}}$.

We now prove (2). In the case of $\sum_{i,j=1}^{n} \bar{s}_{ij} = \bar{a} \le n$, by (10)

$$N(\varepsilon_r) \le (\bar{a}^{r-1}\bar{\mu}\varepsilon_r^{-1} + \bar{a}^{r-1}\bar{\mu}\varepsilon_r^{-1}n + \bar{a}^{r-2}\bar{\mu}\varepsilon_r^{-1}n^2 + \cdots + \bar{a}\bar{\mu}\varepsilon_r^{-1}n^{r-1} +$$
$$+ \bar{\mu}\varepsilon_r^{-1}n^r + \varepsilon_r^{-1}n^r)(n+1)^2 \le \varepsilon_r^{-1}n^r\bar{\mu}\left(n^{-1} + r + \frac{1}{\bar{\mu}}\right)(n+1)^2 ,$$

$$\frac{\log N(\varepsilon_r)}{-\log \varepsilon_r} \le 1 + \frac{\log n^r}{\log n^r} + \frac{\log\left(\bar{\mu}\left(n^{-1} + r + \frac{1}{\bar{\mu}}\right)(n+1)^2\right)}{-\log \varepsilon_r}$$

$$= 2 + \frac{1}{r}\log_n^{\left(\bar{\mu}\left(n^{-1}+r+\frac{1}{\bar{\mu}}\right)(n+1)^2\right)},$$

and therefore $\dim_B A = \lim_{\varepsilon_r \to 0} \frac{\log N(\varepsilon_r)}{-\log \varepsilon_r} \le 2$. On the other hand, since A is a surface in R^2, we have $\dim_B A \ge 2$. Hence $\dim_B A = 2$. □

Remark 1. The result of the Theorem 1 improves estimation of Box-counting dimension in [7]. In fact, let denote $\tilde{s} := \operatorname*{Min}_E |s(x, y)|$, $\bar{s} := \operatorname*{Max}_E |s(x, y)|$, and then since $\tilde{s} \le \tilde{s}_{ij} \le \bar{s}_{ij} \le \bar{s}$, $(i, j = 1, \cdots, n)$, if $\tilde{s} > \frac{1}{n}$ then $\sum_{i,j=1}^{n} \underline{s}_{ij} > n$,

$$1 + \log_n^{\sum_{i,j=1}^{n} \tilde{s}_{ij}} \ge 1 + \log_n^{\sum_{i,j=1}^{n} \tilde{s}} \ge 1 + \log_n^{n^2\tilde{s}} = 3 + \log_n^{\tilde{s}},$$
$$1 + \log_n^{\sum_{i,j=1}^{n} \bar{s}_{ij}} \le 1 + \log_n^{\sum_{i,j=1}^{n} \bar{s}} = 3 + \log_n^{\bar{s}} .$$

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That is, $3+\log_n^{\tilde{s}} \le \dim_B A \le 3+\log_n^{\bar{s}}$. And if $\tilde{s} \le \dfrac{1}{n}$, then since $\sum_{i,j=1}^{n} \bar{s}_{ij} \le n$, we have $\dim_B A = 2$. This is just estimation of Box-counting dimension in [7].

Remark 2. For the surface $f(x, y) = s_{ij}(x, y) f(L^{-1}(x, y)) + Q(x, y)$ we get the same results as one of the Theorem 1. In this case, $s_{ij}(x, y)$ is contraction function on E_{ij} with $0 < |s_{ij}(x, y)| < 1$, and $L(x, y)$, $Q(x, y)$ are defined as in (2). In fact, we can choose $\bar{s}_{ij}$, $\tilde{s}_{ij}$ as $\bar{s}_{ij} = \operatorname*{Max}_{E_{ij}} |s_{ij}(x, y)|$, $\underset{\sim}{s}_{ij} = \operatorname*{Min}_{E_{ij}} |s_{ij}(x, y)|$ in the proof.

References

1. Barnsley M.F., *Fractal functions and interpolation*. Constr. Approx. 1986;2:2303-329.

2. Bouboulis P, Dalla L and Drakopoulos V. *Construction of recurrent bivariate fractal interpolation surfaces and computation of their box-counting dimension*. J. Approx. Theory, 2006;141:99-117.

3. Bouboulis P, Dalla L. *A general construction of fractal interpolation functions on grids of* R^n. Euro. Jnl of Applied Mathematics, 2007;18:449-476.

4. Bouboulis P, Dalla L. *Fractal interpolation surfaces derived from fractal interpolation functions*. J. Math. Anal. Appl. 2007;336:919-936.

5. Malysz R. *The Minkowski dimension of the bivariate fractal interpolation surfaces. Chaos, Solitions and Fractals*, 2006;27:1147-1156.

6. Massopust P.R., *Fractal surfaces*. J. Math. Anal. Appl. 1990;151:275-290.

7. Metzler W, Yun C.H. *Construction of fractal interpolation surfaces on rectangular grids*. Internat. J. Bifur. Chaos, 2010;12: 4079-4086.

8. Zhao N. *Construction and application of fractal interpolation surfaces*. The Visual Computer, 1996; 12:132-146.