ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2015, VOLUME 5, ISSUE 1, p.1-12

POWER BOUNDED COMPOSITION OPERATORS IN SEVERAL VARIABLES

ELKE WOLF

ABSTRACT. Let ϕ be an analytic self-map of the open unit polydisk $\mathbb{D}^N$, $N \in \mathbb{N}$. Such a map induces a composition operator C_ϕ acting on weighted Banach spaces of holomorphic functions. We study when such operators are power bounded resp. uniformly mean ergodic.

Mathematics Subject Classification (2010): 47B33, 47B38
Key words: weighted composition operator, weighted Banach spaces of holomorphic functions, power bounded, uniformly mean ergodic

Article history:
Received 15 January 2015
Received in revised form 17 February 2015
Accepted 18 February 2015

1. Introduction

Let $\mathbb{D}^N$ denote the unit polydisk in $\mathbb{C}^N$, $N \geq 1$, and $H(\mathbb{D}^N)$ the set of all analytic functions on $\mathbb{D}^N$. In the case of the unit disk in the complex plane we simply write $\mathbb{D}$. An analytic self-map ϕ of $\mathbb{D}^N$ induces through composition a linear composition operator

$$C_\phi : H(\mathbb{D}^N) \to H(\mathbb{D}^N),\ f \mapsto f \circ \phi.$$

In the classical setting of the Hardy space H^2 these operators link operator theory with complex analysis. Moreover they provide a large class of operators which gives many examples on basic operator theoretical questions. Therefore there exists a wide literature on the topic of composition operators. On the top of the list of literature we would like to mention the excellent monographs of Cowen and MacCluer [8] and of Shapiro [15] and refer the reader to even these monographs for more and deeper information.

Now, let $v : \mathbb{D}^N \to (0, \infty)$ be a continuous and bounded function. Such a map is called a *weight*. We will study composition operators that act on the *weighted Banach spaces of holomorphic functions* given by

$$H_v^\infty(\mathbb{D}^N) := \left\{ f \in H(\mathbb{D}^N);\ \|f\|_v = \sup_{z \in \mathbb{D}^N} v(z)|f(z)| < \infty \right\}.$$

Endowed with the weighted sup-norm $\|.\|_v$ these are Banach spaces. Again in the setting of the open unit disk we use a special notation for such spaces and write H_v^∞. They arise naturally in a number of problems related to functional analysis, Fourier analysis, partial differential equations and convolution equations. Later, they became of interest themselves, see [3] and the references therein for further and deeper information.

Let X be a Banach space and $L(X)$ be the space of all continuous linear operators from X into itself endowed with the operator norm topology. The Cesàro means of an operator $T \in L(X)$ are given by

$$T_{[n]} := \frac{1}{n} \sum_{m=1}^{n} T^m,\ n \in \mathbb{N}.$$

Now, an operator $T \in L(X)$ is called

(a) *uniformly mean ergodic* if and only if $(T_{[n]})_n$ is a convergent sequence in $L(X)$.
(b) *power bounded* if and only if there is a constant $C > 0$ such that $\sup_{n\in\mathbb{N}} \|T^n\| \leq C$.
(c) *similar to a contraction* if there is an invertible operator S on X such that $T = S^{-1} \circ C \circ S$. A contraction C on X is *strict* if $\|C\| < 1$.

For more background information on ergodic theory we refer the reader to the monograph [12]. Interesting articles related to the topic are [2] and [5].

In [6] Bonet and Ricker studied when multiplication operators acting on spaces of type H_v^∞ are power bounded resp. uniformly mean ergodic. Reading this article motivated us to investigate the same questions for composition operators. In [18] we characterized power bounded composition operators acting on spaces of type H_v^∞ by studying the fixed point behaviour of the symbol ϕ. Moroever, we analyzed when such operators are uniformly mean ergodic. In this article it is our aim to extend our studies to the setting of several variables. Doing this there occur several new and quite different phenomena.

2. Preliminaries

2.1. **Geometry of the unit disk and the unit polydisk.** Let $p \in \mathbb{D}$ and α_p denote the Möbius transformation that interchanges p with 0, i.e.

$$\alpha_p(z) = \frac{z-p}{1-\overline{p}z} \text{ for every } z \in \mathbb{D}.$$

As a corollary to the Schwarz lemma, every automorphism α of $\mathbb{D}$ is of the form

$$\alpha(z) = \xi\alpha_p(z) \text{ for every } z \in \mathbb{D}, \text{ where } \xi \in \partial\mathbb{D} \text{ and } p \in \mathbb{D}.$$

The following facts are well-known:

(1) $\alpha_p^{-1} = \alpha_{-p}$.
(2) $1 - |\alpha_p(z)|^2 = \frac{(1-|z|^2)(1-|p|^2)}{|1-\overline{p}z|^2}$.

An important classical result which plays a main role in this article is the Denjoy-Wolff-Theorem. The n-th iterate of an analytic self-map ϕ of $\mathbb{D}$ is denoted by ϕ^n.

Theorem 2.1 (Denjoy-Wolff Theorem)**.** *Let ϕ be an analytic self-map of $\mathbb{D}$. If ϕ is not the identity and not an automorphism with exactly one fixed point, then there is a unique point $p \in \overline{\mathbb{D}}$ such that $(\phi^n)_n$ converges to p uniformly on the compact subsets of $\mathbb{D}$.*

Let us close this section with some facts about the unit polydisk. Let ϕ be an automorphism of the unit polydisk. Then we can find conformal maps $\phi_1, ..., \phi_N$ of $\mathbb{D}$ onto $\mathbb{D}$ as well as a permutation π of $\{1, ..., N\}$ such that

$$\phi(z) = \phi(z_1, ..., z_N) = (\phi_1(z_{\pi(1)}), ..., \phi_N(z_{\pi(N)})).$$

In other words, each automorphism of the unit polydisk is of the form

$$(\xi_1\alpha_{p_1}(z_{\pi(1)}), ..., \xi_k\alpha_{p_N}(z_{\pi(N)}))$$

for every $z \in \mathbb{D}^N$ with $p_i \in \mathbb{D}$ and $\xi_i \in \partial\mathbb{D}$ for every $1 \leq i \leq N$, where π is a permutation of $\{1, ..., N\}$. See e.g. [10] Theorem 6.1.25.

2.2. **Weighted spaces and composition operators.** The first difficulty is to generalize the concept of a typical weight to the unit polydisk. It turned out that there are two main possibilities. But let us start with general definitions and notations. A very important tool when dealing with weights and weighted spaces is the so called *associated weight*, given by

$$\tilde{v}(z) = \frac{1}{\sup\{|f(z)|;\ f \in H_v^\infty,\ \|f\|_v \leq 1\}},\ z \in \mathbb{D}^N.$$

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2015, VOLUME 5, ISSUE 1, p.1-12

The concept of associated weights was implicitly introduced by Anderson and Duncan in [1] and thoroughly studied by Bierstedt, Bonet and Taskinen in [3]. They showed that the associated weight $\tilde{v}$ has the following useful properties (see [3]):

(1) $\tilde{v} \geq v > 0$,
(2) $\tilde{v}$ is continuous,
(3) for every $z \in \mathbb{D}^N$ there is $f_z \in H_v^\infty(\mathbb{D}^N)$ with $\|f_z\|_v \leq 1$ such that

$$|f_z(z)| = \frac{1}{\tilde{v}(z)}.$$

Especially interesting are weights which satisfy $v(z) = v(|z|)$ for every $z \in \mathbb{D}$. We say that weights of this type are *radial*. Every radial weight which is non-increasing with respect to $|z|$ and such that $\lim_{|z|\to 1} v(z) = 0$ is called a *typical* weight.

In the setting of the open unit disk Bonet, Domański, Lindström and Taskinen, [7], showed that $C_\phi : H_v^\infty \to H_v^\infty$ is bounded if and only if $\sup_{z\in\mathbb{D}} \frac{v(z)}{\tilde{v}(\phi(z))} < \infty$.

Moreover, in case that v is a typical weight they proved that the boundedness of C_ϕ on H_v^∞ for some automorphism ϕ with $\phi(0) \neq 0$ implies the boundedness of all composition operators on H_v^∞. In [17] we showed that this remains true in the following two settings:

(1) Let v be of the form $v : \mathbb{D}^N \to (0, \infty)$, $v(z) = v(z_1, ..., z_N) = \prod_{i=1}^N v_i(z_i)$ such that each v_i is a typical weight on the unit disk $\mathbb{D}$, where $z = (z_1, ..., z_N) \in \mathbb{D}^N$.
(2) Consider a weight v of type $v : \mathbb{D}^N \to (0, \infty), v(z) = v(z_1, ...z_N) = v(\prod_{i=1}^N |z_i|)$ such that $\lim_{\min_{1\leq i\leq N} |z_i|\to 1} v(z) = 0$, where $z = (z_1, ..., z_N) \in \mathbb{D}^N$.

Example 2.2. (1) The weight v given by $v : \mathbb{D}^N \to (0, \infty)$, $v(z) = v(z_1, ..., z_N) = (1 - |z_1|) \cdots (1 - |z_N|) = \prod_{i=1}^N (1 - |z_i|)$ is an example of a weight that belongs to the first of the classes introduced above.
(2) The weight $v : \mathbb{D}^N \to (0, \infty)$, $v(z) = 1 - |z_1| \cdots |z_k|$ provides an example for the second class.

On the unit disk weights which satisfy condition $(L1)$ (due to Lusky [13]) given by

$$(L1) \quad \inf_{n\in\mathbb{N}} \frac{v(1 - 2^{-n-1})}{v(1 - 2^{-n})} > 0$$

are of special interest for more than one reason. E.g. if a weight v satisfies condition $(L1)$ then it is *essential*, i.e. there is $k > 0$ such that

$$v(z) \leq \tilde{v}(z) \leq kv(z) \text{ for every } z \in \mathbb{D}.$$

This is an important property since it is quite hard to really compute the associated weight $\tilde{v}$. Moreover, Bonet, Domański, Lindström and Taskinen showed that each composition operator acting on a space H_v^∞ induced by a typical weight which satisfies $(L1)$ must be bounded. Now, the question is to introduce a property like $(L1)$ in our two classes of weights. We obtain:

- In the setting of Class 1 we consider weights $v(z) = \sum_{i=1}^N v_i(z_i)$ such that each v_i enjoys $(L1)$.
- In the setting of Class 2 we say that a weight $v(z) = v(\prod_{i=1}^N z_i)$ satisfies condition $(L1_N)$ if and only if

$$\inf_{n\in\mathbb{N}} \frac{v((1 - 2^{-n-1}), ..., (1 - 2^{-n-1}))}{v((1 - 2^{-n}), ..., (1 - 2^{-n}))} > 0.$$

We close this section by showing that in the setting of Class 2 the condition $(L1_N)$ is equivalent with condition

$$(A) \quad \exists 0 < r < 1 \; \exists 1 < C < \infty : \quad \frac{v(z)}{v(p)} \leq C \quad \forall z, p \in \mathbb{D}^N \text{ with } \max_{1\leq i\leq N} \rho(z_i, p_i) \leq r.$$

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2015, VOLUME 5, ISSUE 1, p.1-12

To do this we first assume that v satisfies the condition $(L1_N)$. By [9] p.42 we can conclude that $\max_{1\le i\le N}\rho(p_i,z_i)\le r$ gives the inequality

$$\frac{1-\prod_{i=1}^N|z_i|}{1-\prod_{i=1}^N|p_i|}\le\frac{8}{1-r}.$$

Thus if $\max_{1\le i\le N}\rho(p_i,z_i)\le\frac{1}{3}$, then we get that $\frac{1}{12}\le\frac{1-\prod_{i=1}^N|z_i|}{1-\prod_{i=1}^N|p_i|}\le 12$. For $z,p\in\mathbb{D}^N$ with $\max_{1\le i\le N}\rho(p_i,z_i)\le\frac{1}{3}$ we select n such that $1-2^{-n}\le\prod_{i=1}^N|p_i|\le 1-2^{-n-1}$. Hence $1-\prod_{i=1}^N|z_i|\le 12(1-\prod_{i=1}^N|p_i|)\le 2^{-n+4}$, and we have that $\prod_{i=1}^N|z_i|\ge 1-2^{-n+4}$ and $\prod_{i=1}^N|p_i|\le 1-2^{-n-1}$. Thus, if $\frac{v(1-2^{-k-1},...,1-2^{-k-1})}{v(1-2^{-k},...,1-2^{-k})}>C$ for all k, then for $r=\frac{1}{3}$ it follows that

$$\frac{v(z)}{v(p)}\le\frac{v(1-2^{-n+4},...,1-2^{-n+4})}{v(1-2^{-n-1},...,1-2^{-n-1})}<C^{-5}.$$

Thus, (A) is satisfied.

Conversely, if $\frac{v(z)}{v(p)}\le C$ for $\max_{1\le i\le N}\rho(z_i,p_i)\le r$, then

$$\frac{v(1-t(1+r)^{-1},...,1-t(1+r)^{-1})}{v(1-t,...,1-t)}\ge\frac{1}{C}\text{ for all }t\in[0,1).$$

By letting k be the smallest integer such that $k\ge\frac{\ln 2}{\ln(1+r)}$ we obtain

$$\frac{v(1-2^{-n-1},...,1-2^{-n-1})}{v(1-2^{-n},...,1-2^{-n})}\ge\frac{v(1-2^{-n}(1+r)^{-k},...,1-2^{-n}(1+r)^{-k})}{v(1-2^{-n},...,1-2^{-n})}\ge\frac{1}{C^k}.$$

for all n. Hence, the claim follows.

Example 2.3. (1) Class 1: The weight $v:\mathbb{D}^N\to(0,\infty)$, $v(z_1,...,z_N)=(1-|z_1|)\cdots(1-|z_N|)=\prod_{i=1}^N(1-|z_i|)$ provides an example of a weight v of type $v(z)=\prod_{i=1}^N v_i(z_i)$ with the property that each v_i, $1\le i\le N$, enjoys the Lusky condition $(L1)$, more precisely, for every $n\in\mathbb{N}$ we have that

$$\frac{v((1-2^{-n-1}),...,(1-2^{-n-1}))}{v((1-2^{-n}),...,(1-2^{-n}))}=\prod_{i=1}^N\frac{v_i(1-2^{-n-1})}{v_i(1-2^{-n})}=\prod_{i=1}^N\frac{1-(1-2^{-n-1})}{1-(1-2^{-n})}=\left(\frac{1}{2}\right)^N>0.$$

(2) Class 2: The weight $v:\mathbb{D}^2\to(0,\infty)$, $v(z_1,z_2)=1-|z_1||z_2|$ has condition $(L1_2)$. For every $n\in\mathbb{N}$ we obtain that

$$\frac{v((1-2^{-n-1}),(1-2^{-n-1}))}{v((1-2^{-n}),(1-2^{-n}))}=\frac{1-(1-2^{-n-1})^2}{1-(1-2^{-n})^2}$$

$$=\frac{1-(1-2^{-n}+2^{-2n-2})}{1-(1-2^{-n+1}+2^{-2n})}=\frac{2^{-n}+2^{-2n-2}}{2^{-n+1}+2^{-2n}}=\frac{2^{-n}(1+2^{-n-2})}{2^{-n}(2+2^{-n})}.$$

Now,

$$\inf_{n\in\mathbb{N}}\frac{v((1-2^{-n-1}),(1-2^{-n-1}))}{v((1-2^{-n}),(1-2^{-n}))}=\frac{1}{2}>0.$$

Hence the claim follows. Applying the Binomial theorem we can easily show that weights $v:\mathbb{D}^N\to(0,\infty)$, $v(z)=1-|z_1|\cdots|z_N|$ enjoy the condition $(L1_N)$.

As mentioned above the following result by Bonet, Domański, Lindström and Taskinen [7] is quite important for the progress of this paper.

Theorem 2.4 (Bonet, Domański, Lindström, Taskinen [7]). *Let v be a typical weight on the open unit disk $\mathbb{D}$. Then all operators $C_\phi:H_v^\infty\to H_v^\infty$ are bounded if and only if the weight v satisfies the Lusky condition $(L1)$.*

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2015, VOLUME 5, ISSUE 1, p.1-12

We close this section by showing that if the weight belongs to Class 1 or Class 2 all operators

$$C_\phi : H_v^\infty(\mathbb{D}^N) \to H_v^\infty(\mathbb{D}^N)$$

are bounded if the analytic map $\phi : \mathbb{D}^N \to \mathbb{D}^N$ is of type $\phi(z) = (\phi_1(z_1), ..., \phi_N(z_N))$ for every $z = (z_1, ..., z_N) \in \mathbb{D}^N$.

(1) Class 1: If v is a weight of class 1 and $\phi : \mathbb{D}^N \to \mathbb{D}^N$ an analytic map with $\phi(z) = (\phi_1(z_1), ..., \phi_N(z_N)$ for every $z = (z_1, ..., z_N) \in \mathbb{D}^N$, then we obtain

$$\sup_{z\in\mathbb{D}^N} \frac{v(z)}{v(\phi(z))} = \sup_{z\in\mathbb{D}^N} \frac{\prod_{i=1}^N v_i(z_i)}{\prod_{i=1}^N v_i(\phi_i(z_i))} = \prod_{i=1}^N \sup_{z_i\in\mathbb{D}} \frac{v_i(z_i)}{v_i(\phi_i(z_i))}.$$

Hence $C_\phi : H_v^\infty(\mathbb{D}^N) \to H_v^\infty(\mathbb{D}^N)$ may - in a sense - be considered as a combination of operators $C_{\phi_i} : H_v^\infty \to H_v^\infty$. By Theorem 2.4 each of the operators C_{ϕ_i} must be bounded. Hence C_ϕ is bounded.

(2) Class 2: Let v be a weight of class 2 and $\phi : \mathbb{D}^N \to \mathbb{D}^N$ an analytic map with $\phi(z) = (\phi_1(z_1), ..., \phi_N(z_N)$ for every $z = (z_1, ..., z_N) \in \mathbb{D}^N$. We reduce the problem to the one-dimensional case in the following way. We put

$$u := \prod_{i=1}^N z_i \in \mathbb{D} \text{ and } \psi(u) := \prod_{i=1}^N \phi_i(z_i).$$

Hence $\sup_{u\in\mathbb{D}} \frac{v(u)}{v(\psi(u)} < \infty$ by Theorem 2.4 since v satisfies $(L1_N)$.

3. Polydisk

In this section we study composition operators acting on weighted spaces generated by two different classes of weights.

3.1. Weights of type $v(z) = \prod_{i=1}^N v_i(z_i)$.

Throughout this subsection let v be a weight of type $v(z) = \prod_{i=1}^N v_i(z_i)$ where each v_i is a typical weight that satisfies $(L1)$.

Lemma 3.1. *Let ϕ with $\phi(z) = (\phi_1(z_1), ..., \phi_N(z_N))$ be an analytic self-map of $\mathbb{D}^N$ such that each ϕ_i has a fixed point $a_i \in \mathbb{D}$. Then the composition operator $C_\phi : H_v^\infty(\mathbb{D}^N) \to H_v^\infty(\mathbb{D}^N)$ is similar to a contraction.*

Proof. Let $a_i \in \mathbb{D}$ be a fixed point of ϕ_i, $1 \le i \le N$. Now we consider maps $\psi : \mathbb{D}^N \to \mathbb{D}^N$, $\psi(z) = (\psi_1(z_1), ..., \psi_N(z_N))$ with

$$\psi_i := \alpha_{-a_i} \circ \phi_i \circ \alpha_{a_i} \text{ for every } 1 \le i \le N,$$

and

$$\alpha : \mathbb{D}^N \to \mathbb{D}^N, \alpha(z) = (\alpha_{a_1}(z_1), ..., \alpha_{a_N}(z_N)).$$

By definition we see that

$$\alpha^{-1}(z) = (\alpha_{a_1}^{-1}(z_1), ..., \alpha_{a_N}^{-1}(z_N)) = (\alpha_{-a_1}(z_1), ..., \alpha_{-a_N}(z_N))$$

for every $z \in \mathbb{D}^N$. Moreover, $\psi_i(0) = 0$ for every $1 \le i \le N$. Thus, by the Schwarz Lemma we have that $|\psi_i(w)| \le |w|$ for every $w \in \mathbb{D}$ and every $1 \le i \le N$. Hence $\frac{v_i(w)}{v_i(\psi_i(w))} \le 1$ for every $w \in \mathbb{D}$ and every $1 \le i \le N$. Finally, we can conclude

$$\sup_{z\in\mathbb{D}^N} \frac{v(z)}{v(\psi(z))} = \sup_{z\in\mathbb{D}^N} \frac{v_1(z_1)}{v_1(\psi_1(z_1))} \cdots \frac{v_N(z_N)}{v_N(\psi(z_N))} \le 1$$

Since $C_\phi = C_\alpha \circ C_\psi \circ C_{\alpha^{-1}}$, we obtain the claim. □

Theorem 3.2. *Let ϕ with $\phi(z) = (\phi_1(z_1), ..., \phi_N(z_N))$ be an analytic self-map but not a conformal automorphism of $\mathbb{D}^N$. The operator $C_\phi : H_v^\infty(\mathbb{D}^N) \to H_v^\infty(\mathbb{D}^N)$ is power bounded if and only if it is similar to a contraction.*

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2015, VOLUME 5, ISSUE 1, p.1-12

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2015, VOLUME 5, ISSUE 1, p.1-12

Proof. By Lemma 3.1 it is enough to show that each ϕ_i has a fixed point in $\mathbb{D}$. We prove this indirectly and assume that there is $1 \leq i_0 \leq N$ such that ϕ_{i_0} has no fixed point in $\mathbb{D}$. By the Denjoy-Wolff Theorem we know that in this case the sequence $(|\phi_{i_0}^n|)_n$ tends to 1 uniformly on the compact subsets of $\mathbb{D}$. Hence

$$\|C_\phi^n\| = \|C_{\phi^n}\| = \sup_{z\in\mathbb{D}^N} \frac{v(z)}{v(\phi^n(z))} \geq \frac{v_1(0)}{v(\phi_1^n(0))} \cdots \frac{v_{i_0}(0)}{v(\phi_{i_0}^n(0))} \cdots \frac{v_N(0)}{v(\phi_N^n(0))} \to \infty,$$

if $n \to \infty$. This is a contradiction.
Conversely, by hypothesis, there exists an invertible operator S on $H_v^\infty(\mathbb{D}^N)$ such that $C_\phi = S^{-1} \circ C \circ S$, where C is a contraction on $H_v^\infty(\mathbb{D}^N)$, i.e. $\|C\| \leq 1$. Hence $C_\phi^n = S^{-1} \circ C^n \circ S$ and thus $\|C_\phi^n\| \leq \|S^{-1}\|\|S\|$ for every $n \in \mathbb{N}$. This means that C_ϕ is power bounded. □

Example 3.3. We consider the composition operator $C_\phi : H_v^\infty(\mathbb{D}^2) \to H_v^\infty(\mathbb{D}^2)$ induced by $\phi(z_1, z_2)) = (\frac{z_1+1}{2}, z_2)$ and $v(z_1, z_2) = (1 - |z_1|)(1 - |z_2|)$ for every $z \in \mathbb{D}^2$. Thus, we obtain $\phi^n(z) = (\frac{z_1+1+\cdots+2^{n-1}}{2^n}, z_2)$ for every $n \in \mathbb{N}$ and every $z \in \mathbb{D}^2$. With that easy calculations (see e.g. [16]) show that

$$\|C_\phi^n\| = \sup_{z\in\mathbb{D}^2} \frac{v(z)}{v(\phi_n(z))} = \sup_{z\in\mathbb{D}} \frac{1 - |z_1|}{1 - |\phi_1^n(z)|} = 2^n.$$

Thus, C_ϕ obviously is not power bounded.

The following result is completely analogous to the one-dimensional case. For the sake of completeness we give it here.

Proposition 3.4. *Let v be a weight and ϕ with $\phi(z) = (\phi_1(z_1), ..., \phi_N(z_N))$ be an analytic self-map of $\mathbb{D}^N$. If $C_\phi : H_v^\infty(\mathbb{D}^N) \to H_v^\infty(\mathbb{D}^N)$ is similar to a strict contraction, then C_ϕ is uniformly mean ergodic.*

Proof. We will show that $\|(C_\phi)_{[n]}\| \to 0$ if $n \to \infty$. By hypothesis, we can find an invertible operator S on $H_v^\infty(\mathbb{D}^N)$ and an operator C on $H_v^\infty(\mathbb{D}^N)$ with $\|C\| < 1$ such that $C_\phi = S^{-1} \circ C \circ S$. Thus, we arrive at the following estimate

$$\|(C_\phi)_{[n]}\| \leq \frac{1}{n} \sum_{m=1}^{n} \|C_\phi^m\| \leq \frac{1}{n} \sum_{m=1}^{n} \|S^{-1}\|\|C\|^m\|S\| \leq \frac{1}{n}\|S^{-1}\|\|S\|M \to 0$$

if $n \to \infty$, where $M := \sum_{m=1}^\infty \|C\|^m = \frac{1}{1-\|C\|} < \infty$. □

The converse is not true, as the following trivial example shows.

Example 3.5. If we take $v(z) = \prod_{i=1}^N (1 - |z_i|)$ and $\phi(z) = \mathrm{id}(z) = z$ for every $z \in \mathbb{D}^N$ we obtain $\phi^n(z) = z$ for every $n \in \mathbb{N}$. Obviously we have that $\|C_\phi\| = 1$ and $(C_\phi)_{[n]} = \frac{1}{n}\sum_{m=1}^n C_\phi^m = C_\phi$ for every $n \in \mathbb{N}$. Hence C_ϕ is uniformly mean ergodic.

To deal with uniformly mean ergodicity we need the following result given in [16].

Theorem 3.6. *Let v be a weight of type $v(z) = \prod_{i=1}^N v_i(z_i)$, where each v_i is a typical weight on $\mathbb{D}$ satisfying the Lusky condition $(L1)$. Moreover, let ϕ and ψ be analytic self-maps of the unit polydisk $\mathbb{D}^N$. Then the norm of the operator $C_\phi - C_\psi : H_v^\infty \to H_v^\infty$ is equivalent with*

$$\sup_{z\in\mathbb{D}^N} \max\left\{\frac{v(z)}{v(\phi(z))}, \frac{v(z)}{v(\psi(z))}\right\} \max_{1\leq i\leq N} \rho(\phi_i(z), \psi_i(z)).$$

Theorem 3.7. *Let ϕ with $\phi(z) = (\phi_1(z_1), ..., \phi_N(z_N))$ be an analytic self-map but not a conformal automorphism of $\mathbb{D}^N$. Let us assume that each ϕ_i has an* attracting *fixed point a_i in $\mathbb{D}$, i.e. $\phi_i'(a_i) \neq 0$ for every $1 \leq i \leq N$, then $C_\phi : H_v^\infty(\mathbb{D}^N) \to H_v^\infty(\mathbb{D}^N)$ is uniformly mean ergodic.*

Proof. Let ϕ be a map as described above. W.l.o.g. we may assume that the maps ϕ_i, $1 \leq i \leq N$, are given by $\phi_i(w) = \lambda_i w$ for every $w \in \mathbb{D}$, where $\lambda_i \in \mathbb{C}$ with $|\lambda_i| < 1$. For details we refer the reader to the article [14]. However, the original proof of this fact goes back to Koenigs [11]. Now, $\phi_i^n(w) = \lambda_i^n w$ for every $w \in \mathbb{D}$ and every $n \in \mathbb{N}$ as well as $\|C_{\phi^n}\| = \|C_\phi\| = 1$ for every $n \in \mathbb{N}$. If C_0 is the composition operator defined by $C_0 : H_v^\infty(\mathbb{D}^N) \to H_v^\infty(\mathbb{D}^N)$, $(C_0 f)(z) = f(0)$ for every $z \in \mathbb{D}^N$ we obtain by using Theorem 3.6

$$\|(C_\phi)_{[n]} - C_0\| = \left\|\frac{1}{n}\sum_{m=1}^{n} C_{\phi^m} - C_0\right\| \leq \frac{1}{n}\sum_{m=1}^{n}\|C_{\phi^m} - C_0\|$$
$$\leq C\frac{1}{n}\sum_{m=1}^{n}\sup_{z\in\mathbb{D}^N}\max\left\{\frac{v(z)}{v(\phi^m(z))}, \frac{v(z)}{v(0)}\right\}\max_{1\leq i\leq N}\rho(\phi_i^m(z), 0)$$
$$\leq C\frac{1}{n}\sum_{m=1}^{n}\sup_{z\in\mathbb{D}^N}|\phi_{i_0}^m(z)| \leq C\frac{1}{n}\sum_{m=1}^{n}|\lambda_{i_0}|^m \to 0$$

since $|\lambda_{i_0}| < 1$. Hence, in this case, $((C_\phi)_{[n]})_{n\in\mathbb{N}}$ tends to C_0 in $L(H_v^\infty(\mathbb{D}^N))$. Thus, C_ϕ is uniformly mean ergodic, and the claim follows. □

Theorem 3.8. *Let $v(z) = \prod_{i=1}^N(1-|z_i|)$ for every $z \in \mathbb{D}^N$. Moreover, let ϕ with $\phi(z) = (\phi_1(z_1), ..., \phi_N(z_N))$ be an analytic self-map but not a conformal automorphism of $\mathbb{D}^N$ such that each ϕ_i has a* super-attracting *fixed point $a_i \in \mathbb{D}$, i.e. $\phi_i'(a) = 0$ for every $1 \leq i \leq N$. Then $C_\phi : H_v^\infty(\mathbb{D}^N) \to H_v^\infty(\mathbb{D}^N)$ is uniformly mean ergodic.*

Proof. Let ϕ be a map as described above. W.l.o.g. we may consider the model maps $\phi_i(z) = z_i^{n_i}$, $1 \leq i \leq N$, for every $z_i \in \mathbb{D}$, $n_i \geq 2$. Again for details we refer the reader to [14]. Originally, this was shown by Böttcher [4]). We will show that the sequence $((C_\phi)_{[k]})_k$ tends to C_0 with respect to the operator norm $\|.\|$, where C_0 is given by $(C_0 f)(z) = f(0)$ for every $z \in \mathbb{D}^N$.
The function $f_i : [0,1) \to \mathbb{R}$, $f_i(r_i) = \frac{1-r_i}{1-r_i^{n_i^k}}$ is monotone decreasing since

$$f_i'(r_i) = \frac{-1 + (1-n_i^k)r_i^{n_i^k} + n_i^k r_i^{n_i^k - 1}}{(1-r_i^{n_i^k})^2} \leq 0 \text{ for every } r_i \in [0,1).$$

Moreover, we have that $\lim_{r_i\to 1}\frac{1-r_i}{1-r_i^{n_i^k}} = \lim_{r_i\to 1}\frac{1}{n_i^k r_i^{n_i^k-1}} = \frac{1}{n_i^k}$ and $\sum_{k=1}^\infty \frac{1}{n_i^k} = \frac{1}{n-1}$. Hence there have to be $0 < r_{0,i} < 1$ such that $\sum_{k=1}^\infty\prod_{i=1}^N \frac{1-r_{0,i}}{1-r_{0,i}^{n_i^k}} = M < \infty$. Now, we choose such $0 < r_{0,i} < 1$ and obtain with an application of Theorem 3.6

$$\|(C_\phi)_{[k]} - C_0\| \leq \frac{1}{k}\sum_{m=1}^{k}\|C_{\phi^m} - C_0\|$$
$$\leq \frac{1}{k}\sum_{m=1}^{k}\sup_{|z_i|\leq r_{0,i},\, 1\leq i\leq N}\max\left\{\prod_{i=1}^N\frac{1-|z_i|}{1-|z_i|^{n_i^m}}, \prod_{i=1}^N(1-|z_i|)\right\}\max_{1\leq i\leq N}\rho(\phi_i^m(z), 0)$$
$$+\frac{1}{k}\sum_{m=1}^{k}\sup_{|z_i|> r_{0,i},\, 1\leq i\leq N}\max\left\{\prod_{i=1}^N\frac{1-|z_i|}{1-|z_i|^{n_i^m}}, \prod_{i=1}^N(1-|z_i|)\right\}\max_{1\leq i\leq N}\rho(\phi_i^m(z), 0)$$
$$\leq \frac{1}{k}\sum_{m=1}^{k}\prod_{i=1}^N|r_{0,i}|^{n_i^m} + \frac{1}{k}\sum_{m=1}^{k}\prod_{i=1}^N\frac{1-r_{0,i}}{1-r_{0,i}^{n_i^m}} \leq \frac{1}{k}\prod_{i=1}^N\frac{1}{1-r_{0,i}^{n_i}} + \frac{1}{k}M \to 0$$

if $k \to \infty$. Thus, the claim follows. □

Remark 3.9. If v is a weight of type $v(z) = \prod_{i=1}^N v_i(z_i)$ for every $z \in \mathbb{D}^N$, where each v_i is a typical weight with $(L1)$ such that $\frac{v_i(r_i)}{v(r_i^n)}$ is monotone decreasing with respect to r_i and such that there is $C_i < 1$ with $\lim_{r_i \to 1} \frac{v_i(r_i)}{v(r_i^n)} \le C_i^n$ for every $n \in \mathbb{N}$, then - with the same proof as above - we can show that an analytic self-map of $\mathbb{D}^N$ such that each ϕ_i has a super-attracting fixed point $a_i \in \mathbb{D}$ induces a uniformly mean ergodic composition operator $C_\phi : H_v^\infty(\mathbb{D}^N) \to H_v^\infty(\mathbb{D}^N)$. An example of this is e.g. the weight $v(z) = \prod_{i=1}^N \frac{1}{1-\ln(1-|z_i|)}$ for every $z \in \mathbb{D}^N$.

Next, let us turn our attention to automorphisms.

Proposition 3.10. *For $a = (a_1, ..., a_N)$ let ϕ_a be an automorphism of the form $\phi_a(z) = (\alpha_{a_1}(z_1), ..., \alpha_{a_N}(z_N))$.*

(a) *For every $a = (a_1, ..., a_N) \in \mathbb{D}^N$ the operator $C_{\phi_a} : H_v^\infty(\mathbb{D}^N) \to H_v^\infty(\mathbb{D}^N)$ is power bounded.*
(b) *For every $a = (a_1, ..., a_N) \in \mathbb{D}^N$ the operator $C_{\phi_a} : H_v^\infty(\mathbb{D}^N) \to H_v^\infty(\mathbb{D}^N)$ is uniformly mean ergodic.*

Proof. (a) If we compute $C_{\phi_a}^n$ we see that

$$C_{\phi_a}^n = \begin{cases} C_{\phi_a} \text{ if } n \in \mathbb{N} \text{ is even }, \\ C_{\text{id}} \text{ if } n \in \mathbb{N} \text{ is odd} \end{cases}$$

where C_{id} is defined by $C_{\text{id}} f(z) = f(z)$ for every $z \in \mathbb{D}^N$. Hence we obtain

$$\sup_{n \in \mathbb{N}} \|C_{\phi_a}^n\| \le \max\{\|C_{\text{id}}\|, \|C_{\phi_a}\|\} = \max\{1, \|C_{\phi_a}\|\}$$

(b) We want to prove that

$$\left\| (C_{\phi_a})_{[n]} - \frac{1}{2}\left(C_{\phi_a} - C_{\text{id}}\right) \right\| \to 0 \text{ if } n \to \infty.$$

Using the proof of (a) we obtain

$$(C_{\phi_a})_{[n]} - \frac{1}{2}\left(C_{\phi_a} - C_{\text{id}}\right) = \begin{cases} \frac{1}{2n}(C_{\phi_a} - C_{\text{id}}) \text{ if } n \in \mathbb{N} \text{ is odd} \\ 0 \text{ if } n \in \mathbb{N} \text{ is even} \end{cases}.$$

Hence

$$\left\| (C_{\phi_a})_{[n]} - \frac{1}{2}\left(C_{\phi_a} - C_{\text{id}}\right) \right\| \le \frac{1}{2n}\|C_{\phi_a} - C_{\text{id}}\| \le \frac{1}{n}\max\{\|C_{\phi_a}\|, \|C_{\text{id}}\|\} \to \infty$$

if $n \to \infty$. □

It remains an open question what happens in the case of automorphisms of the form

$$\phi(z) = (\alpha_{p_1}(z_{\pi(1)}), ..., \alpha_{p_N}(z_{\pi(N)}))$$

for every $z \in \mathbb{D}^N$ where π is a permutaion of $\{1, ..., N\}$. However, since -as stated above- each automorphism of the unit polydisk is of the form

$$(\xi_1 \alpha_{p_1}(z_{\pi(1)}), ..., \xi_N \alpha_{p_N}(z_{\pi(N)}))$$

for every $z \in \mathbb{D}^N$ with π a permutation of $\{1, ..., N\}$, $p_i \in \mathbb{D}$ and $\xi_i \in \partial D$ for every $1 \le i \le N$ we turn our attention to rotaions.

Obviously, each rotation is a contraction and therefore powerbounded. Thus, it remains to investigate when rotations are uniformly mean ergodic.

Proposition 3.11. *Let $k_1, ..., k_N \in \mathbb{N}_0$ and $\phi(z) = (\phi_1(z_1), ..., \phi_N(z_N))$ with $\phi_l(z_l) = e^{\frac{i}{k_l}\pi} z_l$ for every $z_l \in \mathbb{D}$ and every $1 \le l \le N$. Then the operator $C_\phi : H_v^\infty(\mathbb{D}^N) \to H_v^\infty(\mathbb{D}^N)$ is uniformly mean ergodic.*

Proof. Let m be the lowest common multiple of $k_1, ..., k_N$, that is,

$$m := \operatorname{lcm}(k_1, ..., k_N).$$

We show that

$$\left\| (C_\phi)_{[n]} - \frac{1}{2m}\sum_{r=1}^{2m} C_\phi^r \right\| \to 0 \text{ if } n \to \infty.$$

Computing $(C_\phi)_{[n]}$ we obtain

$$(C_\phi)_{[n]} - \frac{1}{2m}\sum_{r=1}^{2m} C_\phi^r = \begin{cases} \frac{2m-1}{2mn} C_\phi - \frac{1}{2mn}\sum_{r=2}^{2m} C_\phi^r, \text{ if } n = 2ml+1 \\ \frac{2m-2}{2mn}(C_\phi + C_\phi^2) - \frac{2}{2mn}\sum_{r=3}^{2m} C_\phi^r, \text{ if } n = 2ml+2 \\ \vdots \\ \frac{1}{2mn}\sum_{r=1}^{2m-1} C_\phi^r - \frac{2m-1}{2mn} C_\phi^{2m}, \text{ if } n = 2ml+m-1 \\ 0, \text{ if } n = 2ml. \end{cases}$$

Since all involved operators are bounded, this sequence tends to zero if $n \to \infty$. $\square$

Proposition 3.12. *Let $\Theta_l = \frac{p_l}{q_l} \in [0, 2)$ be rational for every $1 \le l \le N$ and $\phi(z) = (\phi_1(z), ..., \phi_N(z_N))$ for every $z \in \mathbb{D}^N$ with $\phi_l(z_l) = e^{i\Theta_l \pi} z_l$ for every $z_l \in \mathbb{D}$, $1 \le l \le N$. Then $C_\phi : H_v^\infty(\mathbb{D}^N) \to H_v^\infty(\mathbb{D}^N)$ is uniformly mean ergodic.*

Proof. Let m denote the lowest common multiple of $q_1, ..., q_N$. Obviously, we need $2m$ steps to get back to the operator C_ϕ. Thus, proceeding as in the previous proposition we obtain the claim. $\square$

It is still an open question whether each rotation is uniformly mean ergodic.

3.2. **Weights of type $v(z) = v(\prod_{i=1}^N |z_i|)$.** Throughout this section let v be a weight of type $v(z) = v(\prod_{i=1}^N |z_i|)$ with $\lim_{\min_{1\le i\le N} |z_i| \to 1} v(z) = 0$ that satisfies condition $(L1_N)$.

Lemma 3.13. *Let ϕ with $\phi(z) = (\phi_1(z_1), ..., \phi_N(z_N))$ be an analytic self-map of $\mathbb{D}^N$ such that each ϕ_i has a fixed point $a_i \in \mathbb{D}$. Then the composition operator $C_\phi : H_v^\infty(\mathbb{D}^N) \to H_v^\infty(\mathbb{D}^N)$ is similar to a contraction.*

Proof. Let ϕ be an analytic self-map of $\mathbb{D}^N$ such that each ϕ_i has a fixed point $a_i \in \mathbb{D}$. Now we consider maps $\psi : \mathbb{D}^N \to \mathbb{D}^N$, $\psi(z) = (\psi_1(z_1), ..., \psi_N(z_N))$ given by

$$\psi_i := \alpha_{-a_i} \circ \phi_i \circ \alpha_{a_i},\ 1 \le i \le N,$$

and $\alpha : \mathbb{D}^N \to \mathbb{D}^N$, $\alpha(z) = (\alpha_{a_1}(z_1), ..., \alpha_{a_N}(z_N))$. Since $\alpha_{a_i}^{-1} = \alpha_{-a_i}$ for every $1 \le i \le N$, we obviously have $\alpha^{-1}(z) = (\alpha_{-a_1}(z_1), ..., \alpha_{-a_N}(z_N))$. Moreover, $\psi_i(0) = 0$, $1 \le i \le N$, and by the Schwarz Lemma $|\psi_i(w)| \le |w|$ for every $w \in \mathbb{D}$. Then $\frac{v_i(w)}{v_i(\psi_i(w))} \le 1$ for every $w \in \mathbb{D}$ and hence

$$\sup_{z \in \mathbb{D}^N} \frac{v(z)}{v(\psi(z))} = \sup_{z \in \mathbb{D}^N} \frac{v(\prod_{i=1}^N |z_i|)}{v(\prod_{i=1}^N |\psi_i(z_i)|)} \le 1$$

Since $C_\phi = C_\alpha \circ C_\psi \circ C_{\alpha^{-1}}$, we obtain the claim. $\square$

Theorem 3.14. *Let ϕ with $\phi(z) = (\phi_1(z_1), ..., \phi_N(z_N))$ be an analytic self-map but not a conformal automorphism of $\mathbb{D}^N$.*

(a) *If $C_\phi : H_v^\infty(\mathbb{D}^N) \to H_v^\infty(\mathbb{D}^N)$ is power bounded, then there is $1 \le i \le N$ such that ϕ_i has a fixed point in $\mathbb{D}$.*
(b) *If $C_\phi : H_v^\infty(\mathbb{D}^N) \to H_v^\infty(\mathbb{D}^N)$ is similar to a contraction, then it is power bounded.*

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2015, VOLUME 5, ISSUE 1, p.1-12

Proof. The proof of (b) is the same as in Theorem 3.2 above, hence we omit it here. It remains to show (a). We prove (a) indirectly and assume to the contrary that no ϕ_i has a fixed point in $\mathbb{D}$. By the Denjoy-Wolff Theorem we know that in this case the sequence $(|\phi_{i_0}^n|)_n$ tends to 1 uniformly on the compact subsets of $\mathbb{D}$ for every $1 \leq i \leq N$. Hence

$$\|C_\phi^n\| = \sup_{z\in\mathbb{D}^N} \frac{v(z)}{v(\phi^n(z))} \geq \frac{v(0)}{v(\prod_{i=1}^N \phi_i^n(0))} \to \infty,$$

if $n \to \infty$. This is a contradiction. □

Proposition 3.15. *Let $v(z) = 1 - \prod_{k=1}^N |z_k|$ for every $z \in \mathbb{D}^N$ and ϕ with $\phi(z) = (\phi_1(z_1), ..., \phi_N(z_N))$ an analytic self-map of the unit polydisk $\mathbb{D}^N$. If there is $1 \leq i \leq N$ such that ϕ_i has a fixed point in $\mathbb{D}$, then C_ϕ is power bounded.*

Proof. Let ϕ be an analytic self-map of $\mathbb{D}^N$ as described above such that ϕ_i has a fixed point a_i inside the disk $\mathbb{D}$. Now, we have to distinguish the following two cases:

Case 1: a_i is an attracting fixed point of ϕ_i. In this case the model maps are given by $\phi_i(w) = \lambda_i w$ for every $w \in \mathbb{D}$, where $\lambda_i \in \mathbb{C}$ with $|\lambda_i| < 1$. Now, we get for every $n \in \mathbb{N}$

$$\begin{aligned}\|C_{\phi^n}\| &= \sup_{z\in\mathbb{D}^N} \frac{v(z)}{v(\phi^n(z))} = \sup_{z\in\mathbb{D}^N} \frac{1-\prod_{k=1}^N |z_k|}{1-\prod_{k=1}^N |\phi_k(z)^n|} \\ &\leq \sup_{z\in\mathbb{D}^N} \frac{1-\prod_{k=1}^N |z_k|}{1-|\lambda_i^n|} \leq \frac{1}{1-|\lambda_i|}.\end{aligned}$$

Hence the corresponding operator C_ϕ must be power bounded.

Case 2: a_i is a super-attracting fixed point of ϕ_i. We have the model maps $\phi_i(w) = w^l$ for every $w \in \mathbb{D}$ and obtain for every $n \in \mathbb{N}$.

$$\|C_{\phi^n}\| = \sup_{z\in\mathbb{D}^N} \frac{v(z)}{v(\phi^n(z)} \leq \sup_{z\in\mathbb{D}^N} \frac{1-\prod_{k=1}^N |z_k|}{1-|z_i|^{ln}}.$$

To deal with this we consider the following function

$$f : [0,1]^N \to mathbbR,\ f(r_1, ..., r_N) = \frac{1-\prod_{k=1}^N r_k}{1-r_i^{ln}}$$

and determine the extremal points. We get

$$\nabla f(r_1,...,r_N) = \begin{pmatrix} -\frac{\prod_{k=2}^N r_k}{1-r_i^{ln}} \\ \vdots \\ \frac{-\prod_{k=1,k\neq i} r_k(1-r_i^{ln})+(1-\prod_{k=1}^N r_k) ln r_i^{ln-1}}{(1-r_i^{ln})^2} \\ \vdots \\ -\frac{\prod_{k=1}^{N-1} r_k}{1-r_i^{ln}} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

which is equivalent with $(r_1, ..., r_N) = (0, ..., 0)$. Moreover,
$\lim_{r_i\to 1} \frac{1-\prod_{k=1}^N r_k}{1-r_i^n} = \lim_{r_i\to 1} \frac{-\prod_{k=1,k\neq i}^N}{-nr_i^{n-1}} \leq \frac{1}{n} \to 0$ if $n \to \infty$. Hence the composition operator must be power bounded. □

Example 3.16. (a) Let $G = \mathbb{D}^2$, $v(z) = (1-|z_1|)(1-|z_2|)$ and $\phi(z) = \left(\frac{z_1+1}{2}, z_2\right)$. Then we obtain

$$\sup_{z\in\mathbb{D}^N} \frac{v(z)}{v(\phi^n(z))} = \sup_{z_1\in\mathbb{D}} \frac{1-|z_1|}{1-|\phi_1^n(z_1)|} = 2^n \to \infty.$$

Obviously, the composition operator is not power bounded.

(b) Let $G = \mathbb{D}^2$, $v(z) = 1 - |z_1||z_2|$ and $\phi(z) = \left(\frac{z_1+1}{2}, z_2\right)$. With the same arguments as above we can show, that the corresponding composition operator ist power bounded. Thus, we have found an example of a power bounded composition operator that does not have a fixed point inside the polydisk. Hence in the polydisk setting there occur other phenomena than in the disk setting.

Completely analogous to the setting in Class 1 we obtain the following proposition. Hence we omit the proof.

Proposition 3.17. *Let v be a weight of type $v(z) = v(\prod_{i=1}^N |z_i|)$ and ϕ be an analytic self-map of $\mathbb{D}^N$. If $C_\phi : H_v^\infty(\mathbb{D}^N) \to H_v^\infty(\mathbb{D}^N)$ is similar to a strict contraction, then C_ϕ is uniformly mean ergodic.*

Again, the converse is not true. For a deeper study when such operators are uniformly mean ergodic we need the following theorem.

Theorem 3.18. *Let v be a weight of type $v(z) = v(\prod_{i=1}^N |z_i|)$ such that $\lim_{\min_{1\le i\le N}|z_i|\to 1} v(z) = 0$ and such that v satisfies the condition $(L1_N)$. Moreover, let ϕ and ψ be analytic self-maps of the unit polydisk $\mathbb{D}^N$. Then the norm of the operator $C_\phi - C_\psi : H_v^\infty(\mathbb{D}^N) \to H_v^\infty(\mathbb{D}^N)$ is equivalent with*

$$\sup_{z\in\mathbb{D}^N} \max\left\{\frac{v(z)}{v(\phi(z))}, \frac{v(z)}{v(\psi(z))}\right\} \max_{1\le i\le N} \rho(\phi_i(z), \psi_i(z)).$$

The proof given in [16] can be used to show the above theorem if we take into account that condition $(L1_N)$ is equivalent to condition (A) which has been shown at the beginning of this article. Again, completely analogous to Case 1 we obtain the following two theorems.

Theorem 3.19. *Let v be a weight of type $v(z) = v(\prod_{i=1}^N |z_i|)$ for every $z \in \mathbb{D}^N$ such that v satisfies $(L1_N)$ and $\phi(z) = (\phi_1(z_1), ..., \phi_N(z_N))$ be an analytic self-map but not a conformal automorphism of $\mathbb{D}^N$. Let us assume that each ϕ_i has an* attracting *fixed point a_i in $\mathbb{D}$, i.e. $\phi_i'(a_i) \neq 0$ for every $1 \le i \le N$, then $C_\phi : H_v^\infty(\mathbb{D}^N) \to H_v^\infty(\mathbb{D}^N)$ is uniformly mean ergodic.*

Theorem 3.20. *Let $v(z) = 1 - \prod_{i=1}^N |z_i|$ for every $z \in \mathbb{D}^N$ and let ϕ with $\phi(z) = (\phi_1(z_1), ..., \phi_N(z_N))$ be an analytic self-map but not a conformal automorphism of $\mathbb{D}^N$ such that each ϕ_i has a* super-attracting *fixed point $a_i \in \mathbb{D}$, i.e. $\phi_i'(a_i) = 0$ for every $1 \le i \le N$. Then $C_\phi : H_v^\infty(\mathbb{D}^N) \to H_v^\infty(\mathbb{D}^N)$ is uniformly mean ergodic.*

Again we turn our attention to automorphisms. Completely analogous to Class 1 we obtain the following results.

Proposition 3.21. *For $a = (a_1, ..., a_N)$ let α be an automorphism of the form $\alpha(z) = (\alpha_{a_1}(z_1), ..., \alpha_{a_N}(z_N))$.*

(a) *For every $a = (a_1, ..., a_N) \in \mathbb{D}^N$ the operator $C_\alpha : H_v^\infty(\mathbb{D}^N) \to H_v^\infty(\mathbb{D}^N)$ is power bounded.*

(b) *For every $a = (a_1, ..., a_N) \in \mathbb{D}^N$ the operator $C_\alpha : H_v^\infty(\mathbb{D}^N) \to H_v^\infty(\mathbb{D}^N)$ is uniformly mean ergodic.*

Proposition 3.22. *Let $k_1, ..., k_N \in \mathbb{N}_0$ and $\phi(z) = (\phi_1(z_1), ..., \phi_N(z_N))$ with $\phi_l(z) = e^{\frac{i}{k_l}\pi} z$ for every $z \in \mathbb{D}$ and every $1 \le l \le N$. Then the operator $C_\phi : H_v^\infty(\mathbb{D}^N) \to H_v^\infty(\mathbb{D}^N)$ is uniformly mean ergodic.*

Proposition 3.23. *Let $\Theta_l = \frac{p_l}{q_l} \in [0, 2)$ be rational for every $1 \le l \le N$ and $\phi(z) = (\phi_1(z), ..., \phi_N(z_N))$ for every $z \in \mathbb{D}^N$ with $\phi_l(z_l) = e^{i\Theta_l\pi} z_l$ for every $z_l \in \mathbb{D}$, $1 \le l \le N$. Then $C_\phi : H_v^\infty(\mathbb{D}^N) \to H_v^\infty(\mathbb{D}^N)$ is uniformly mean ergodic.*

References

[1] J. M. Anderson, J. Duncan, Duals of Banach spaces of entire functions, *Glasgow Math. J.* **32** (1991), no. 2, 215-220.

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2015, VOLUME 5, ISSUE 1, p.1-12

[2] A. A. Albanese, J. Bonet, W. Ricker, On mean ergodic operators, *Ann. Acad. Sci. Fenn. Math.* **34** (2009), no. 2, 401-436.

[3] K.D. Bierstedt, J. Bonet, J. Taskinen, Associated weights and spaces of holomorphic functions, *Studia Math.* **127** (1998), no. 2, 137-168.

[4] L.E. Böttcher, The principal laws of convergence of iterates and their applications to analysis (russian), *Izv. Kazan. Fiz.-Mat. Obshch.* **14** (1904), 155-234.

[5] J. Bonet, P. Domański, A note on mean ergodic composition operators on spaces of holomorphic functions, *Rev. R. Acad. Cienc. Exactas Fis. Nat, Ser. A Mat, RACSAM* **105** (2011), no. 2, 389-396.

[6] J. Bonet, W. Ricker, Mean ergodicity of multiplication operators in weighted spaces of holomorphic functions, *Archiv der Mathematik* **92** (2009), 428-437.

[7] J. Bonet, P. Domański, M. Lindström, J. Taskinen, Composition operators between weighted Banach spaces of analytic functions, *J. Austral. Math. Soc. Ser. A* **64** (1998), no. 1, 101-118.

[8] C. Cowen, B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, 1995.

[9] P. Duren, A. Schuster, *Bergman Spaces*, Providence RI, Amer. Math. Soc. 2004.

[10] I. Graham, G. Kohr, *Geometric Function Theory in One and Higher Dimensions*, Pure and Applied Mathematics, Marcel Dekker, 255, New York, 2003.

[11] G. Koenigs, Recherches sur les intégrals de certaines équations fonctionnelles, *Ann. Sci. École Norm. Sup. (3rd series)* **1** (1884), Supplément 3-41.

[12] U. Krengel, *Ergodic Theorems*, Walter de Gruyter, Berlin, 1985.

[13] W. Lusky, On weighted spaces of harmonic and holomorphic functions, *J. London Math. Soc.* **51** (1995), no. 2, 309-320.

[14] P. Poggi-Corradini, Iteration of analytic self-maps of the disk: an overview, *Cubo Mathematical Journal* **6** (2004), no. 1, 73-80.

[15] J.H. Shapiro, *Composition Operators and Classical Function Theory*, Springer, 1993.

[16] E. Wolf, Differences of composition operators between weighted Banach spaces of holomorphic functions on the unit polydisk, *Results Math.* **51** (2008), no. 3-4, 361-372.

[17] E. Wolf, Bellwethers of composition operators for typical growth spaces, *Asian-European Journal of Mathematics* **7** (2014), 1450048, 13pp

[18] E. Wolf, Power bounded composition operators, *Comput. Methods. Funct. Theory* **12** (2012), no. 1, 105-117.

Mathematical Institute, University of Paderborn, D-33095 Paderborn, Germany
E-mail address: `lichte@math.uni-paderborn.de`