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# OSCILLATION PROPERTIES OF A CLASS OF SECOND ORDER NEUTRAL DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENTS

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Abstract. In this work, we investigate the oscillation and nonoscillation of a class of second order neutral differential equations with piecewise constant arguments of the form:

$$((r(t)(y(t)+p(t)y(t-1))')'+q(t)y([t-1])=f(t),$$

where [ ] denotes the greatest integer function.

**Keywords:** Neutral differential equation, oscillation, nonoscillation, piecewise constant argument.

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## 1. Introduction

Consider the second order neutral differential equation with piecewise constant argument of the form

$$\frac{d}{dt}(r(t)\frac{d}{dt}(y(t)+p(t)y(t-1)))+q(t)y([t-1])=0 \tag{1.1}$$

or

$$\frac{d}{dt}(r(t)\frac{d}{dt}(y(t)+p(t)y(t-1)))+q(t)y([t-1])=f(t), \tag{1.2}$$

where $r\in C((0,\infty),\mathbb{R}_+); q,p,f\in C((0,\infty),\mathbb{R}\backslash\{0\})$ and [ ] denotes the greatest integer function.

The objective of this work is to study the oscillatory and nonoscillatory behaviour of solutions of (1.1) and (1.2) for any $|p(t)|<\infty$.

A function $y:\mathbb{R}\to\mathbb{R}$ is a solution of (1.1)/(1.2), if the following conditions hold:

**i):** $y$ is continuous on $\mathbb{R}$;
**ii):** $r(t)\frac{d}{dt}(y(t)+p(t)y(t-1))$ there exists and it is also continuous on $\mathbb{R}$;
**iii):** $\frac{d}{dt}(r(t)\frac{d}{dt}(y(t)+p(t)y(t-1)))$ exists on $\mathbb{R}$, except possibly at the points $t=n, n\in\mathbb{Z}=\{...,-1,0,1,...\}$, where one sided second derivative exist;
**iv):** $y$ satisfies (1.1)/(1.2) on every interval $(n,n+1)$, for all $n\in\mathbb{Z}$.

As is customary, a solution of (1.1)/ (1.2) is oscillatory if it has arbitrarily large zeros. Otherwise, it is called nonoscillatory.

If an initial function $y_0\in C([-1,0],\mathbb{R})\bigcap C^2((-1,0),\mathbb{R})$ is given, then the existence and uniqueness of the solution of (1.1) follows by the method of steps. Let

$$y(t)=y_0(t), -1\le t\le 0,$$

where $y(t)$ is the unique solution of (1.1). If $y(n) = C_n, n = 0, 1, 2, ....$, then for $t \in (n, n+1)$

$$\frac{d}{dt}(r(t)\frac{d}{dt}(y(t) + p(t)y(t-1))) + q(t)C_{n-1} = 0. \tag{1.3}$$

Integrating (1.3) from $n$ to $t$, we obtain

$$r(t)\frac{d}{dt}(y(t) + p(t)y(t-1)) - B_n + C_{n-1}\int_n^t q(s)ds = 0, \tag{1.4}$$

where $B_n = [r(t)\frac{d}{dt}(y(t) + p(t)y(t-1))]_{t=n}, n \in \mathbb{Z}$. By using the continuity at $t = n+1$, (1.4) becomes

$$B_{n+1} - B_n + C_{n-1}\int_n^{n+1} q(s)ds = 0. \tag{1.5}$$

Integrating (1.4) from $n$ to $t$,we get

$$y(t) + p(t)y(t-1) - B_n\int_n^t \frac{ds}{r(s)} + C_{n-1}\int_n^t \frac{1}{r(s)}\int_n^s q(u)duds - y(n) - p(n)y(n-1) = 0. \tag{1.6}$$

Using the continuity at $t = n+1$, (1.6) becomes

$$C_{n+1} + p_{n+1}C_n - C_n - p_nC_{n-1} - B_nv_n^{(1)} + C_{n-1}v_n^{(2)} = 0, \tag{1.7}$$

where $p(n) = p_n$, $v_n^{(1)} = \int_n^{n+1}\frac{ds}{r(s)}$ and $v_n^{(2)} = \int_n^{n+1}\frac{1}{r(s)}\int_n^s q(u)duds$. Consequently, (1.7) simplifies as

$$B_n = \frac{1}{v_n^{(1)}}[C_{n+1} + (p_{n+1} - 1)C_n - (p_n - v_n^{(2)})C_{n-1}], n \in \mathbb{Z}$$

and hence (1.5) resolves as

$$C_{n+2} + [p_{n+2} - 1 - \frac{v_{n+1}^{(1)}}{v_n^{(1)}}]C_{n+1} - [p_{n+1} - v_{n+1}^{(2)} + \frac{p_{n+1}-1}{v_n^{(1)}}v_{n+1}^{(1)}]C_n + v_{n+1}^{(1)}[\frac{p_n - v_n^{(2)}}{v_n^{(1)}} + \\ + \int_n^{n+1} q(s)ds]C_{n-1} = 0,$$

for $n \in \mathbb{Z}$. In general the above equation takes the form

$$C_{n+3} + F_nC_{n+2} + G_nC_{n+1} + H_nC_n = 0, n \in \mathbb{Z}, \tag{1.8}$$

where

$$F_n = p_{n+3} - 1 - \frac{v_{n+2}^{(1)}}{v_{n+1}^{(1)}}, n \in \mathbb{Z},$$

$$G_n = -p_{n+2} + v_{n+2}^{(2)} - \frac{p_{n+2}-1}{v_{n+1}^{(1)}}v_{n+2}^{(1)}, n \in \mathbb{Z}$$

and

$$H_n = v_{n+2}^{(1)}[\frac{p_{n+1} - v_{n+1}^{(2)}}{v_{n+1}^{(1)}} + \int_{n+1}^{n+2} q(s)ds], n \in \mathbb{Z}.$$

Similarly for (1.2), we have the following difference equation

$$C_{n+3} + F_nC_{n+2} + G_nC_{n+1} + H_nC_n = K_n, n \in \mathbb{Z}, \tag{1.9}$$

where

$$K_n = -\frac{v_{n+2}^{(1)}}{v_{n+1}^{(1)}}F_{n+1}^{(2)} + v_{n+2}^{(1)}F_{n+1}^{(1)} + F_{n+2}^{(2)},$$

$$F_n^{(1)} = \int_n^{n+1} f(s)ds, \quad F_n^{(2)} = \int_n^{n+1}\frac{1}{r(s)}\int_n^s f(u)duds.$$

If $r(t) \equiv 1, p(t) = p$ and $q(t) = q$, then (1.8) becomes

$$C_{n+3} + (p-2)C_{n+2} + (1 - 2p + \frac{q}{2})C_{n+1} + (p + \frac{q}{2})C_n = 0, \tag{1.10}$$

for $n \in \mathbb{Z}$. (1.8) is a resulting equation of (1.1), which is a third order difference equation. Hence to study (1.1), it is enough to study (1.8).

Differential equations with piecewise constant arguments (DEPCA) describe hybrid dynamical systems (a combination of continuous and discrete systems) and, therefore contains the properties of both differential and difference equations. DEPCA may also have applications in certain biomedical models [1]. In [2], [3] and the references cited therein there have been a lot of results concerning differential equations with piecewise constant arguments. The purpose of this work is to investigate the oscillatory and nonoscillatory behaviour of solutions of (1.1)/(1.2) with the help of (1.8) and (1.9). It is known that there is no such work deal with (1.1)/(1.2). However, the works associated with the characteristic equations of difference equations are available in the literature (see for e.g [4], [5], [8]-[12]).

## 2. Preliminaries

In [6] and [7], Parhi and Tripathy have discussed the oscillation and nonoscillation of third order difference equations of the form

$$y_{n+3} + \alpha_n y_{n+2} + \beta_n y_{n+1} + \gamma_n y_n = 0 \tag{2.1}$$

and

$$y_{n+3} + \alpha y_{n+2} + \beta y_{n+1} + \gamma y_n = 0, \tag{2.2}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\gamma \neq 0$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are real valued sequences defined on $\mathbb{N}(n_0) = \{n_0, n_0+1, n_0+2, ....\}, n_0 \geq 0$.

A nontrivial solution $\{y_n\}$ of (2.1) is said to be oscillatory, if for every positive integer $N$ there exists $n \geq N$ such that $y_n y_{n+1} \leq 0$. Otherwise, the solution is nonoscillatory. In other words, a solution $\{y_n\}$ is oscillatory if it is neither eventually positive nor eventually negative. Equation (2.1) is said to be oscillatory if all its solutions are oscillatory and strongly nonoscillatory if all its solutions are nonoscillatory. A solution $\{y_n\}, n \geq n_0 \geq 0$ of (2.1) has a generalized zero at $r > n_0$ if either $y_r = 0$ or $y_{r-1}y_r < 0$. In other words, a generalized zero of a solution is either an actual zero or where the solution changes its sign.

In the following, we state some of the main results of [6] and [7] which will be useful for our next discussion.

**Proposition 2.1.** *Let $\gamma > 0$. If $G^2 + 4H^3 > 0$ or $G = 0$ and $H = 0$, then (2.2) is oscillatory. If $G^2 + 4H^3 \leq 0$, then (2.2) admits an oscillatory solution, where*

$$G = \gamma - \frac{\alpha\beta}{3} + \frac{2\alpha^3}{27}, H = \frac{1}{3}(\beta - \frac{\alpha^2}{3}).$$

**Corollary 2.2.** *Let $\gamma > 0$. If one of the following cases*

**(i):** $3\beta > \alpha^2$;

**(ii):** $\beta \leq 0,\ \alpha \geq 0,\ \gamma - \frac{\alpha\beta}{3} + \frac{2\alpha^3}{27} - \frac{2}{3\sqrt{3}}(\frac{\alpha^2}{3} - \beta)^{\frac{3}{2}} > 0$;

**(iii):** $\beta \geq 0,\ \alpha \leq 0,\ 3\beta \leq \alpha^2, \gamma - \frac{\alpha\beta}{3} + \frac{2\alpha^3}{27} - \frac{2}{3\sqrt{3}}(\frac{\alpha^2}{3} - \beta)^{\frac{3}{2}} > 0$;

**(iv):** $3\beta = \alpha^2, \gamma = \frac{\alpha\beta}{3} - \frac{2\alpha^3}{27}$

*holds, then (2.2) is oscillatory.*

**Remark 2.3.** We may notice that $\gamma > 0,\ 3\beta = \alpha^2$ and $\gamma = \frac{\alpha\beta}{3} - \frac{2\alpha^3}{27}$ imply that $\gamma > 0$ and $\beta > 0$. If $\alpha \geq 0,\ \beta \geq 0$ and $\gamma \geq 0$ such that $\alpha + \beta + \gamma > 0$, then (2.2) is oscillatory.

**Theorem 2.4.** *Let $\gamma < 0$ and $\alpha > 0$. Assume that one of the following conditions*



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**(i):** $\gamma - \frac{\alpha\beta}{3} + \frac{2\alpha^3}{27} < \frac{2}{3\sqrt{3}}(\frac{\alpha^2}{3} - \beta)^{\frac{3}{2}},\ \beta < \frac{\alpha^2}{3}$;
**(ii):** $0 < \gamma - \frac{\alpha\beta}{3} + \frac{2\alpha^3}{27} < \frac{2}{3\sqrt{3}}(\frac{\alpha^2}{3} - \beta)^{\frac{3}{2}},\ 0 \leq \beta < \frac{\alpha^2}{3}$;
**(iii):** $\gamma - \frac{\alpha\beta}{3} + \frac{2\alpha^3}{27} = 0,\ 0 \leq \beta < \frac{2\alpha^2}{9}$
*holds. Then* (2.2) *admits one nonoscillatory solution.*

**Theorem 2.5.** *Let* $\gamma < 0,\ \alpha > 0$ *and* $\beta < \frac{\alpha^2}{3}$. *If*

$$\frac{2}{3\sqrt{3}}(\frac{\alpha^2}{3} - \beta)^{\frac{3}{2}} = \gamma - \frac{\alpha\beta}{3} + \frac{2\alpha^3}{27} > 0,$$

*then* (2.2) *admits two nonoscillatory solutions.*

**Theorem 2.6.** *Let* $\gamma < 0$ *and* $\alpha < 0$. *If one of the following conditions*
**(i):** $0 < \frac{\alpha\beta}{3} - \gamma - \frac{2\alpha^3}{27} < \frac{2}{3\sqrt{3}}(\frac{\alpha^2}{3} - \beta)^{\frac{3}{2}},\ \frac{\gamma}{\alpha} \leq \beta < \frac{\alpha^2}{3}$;
**(ii):** $0 < \frac{\alpha\beta}{3} - \gamma - \frac{2\alpha^3}{27} = \frac{2}{3\sqrt{3}}(\frac{\alpha^2}{3} - \beta)^{\frac{3}{2}},\ \frac{\gamma}{\alpha} \leq \beta < \frac{\alpha^2}{3}$
*holds, then* (2.2) *is strongly nonoscillatory.*

**Theorem 2.7.** *Let* $\gamma < 0,\ \alpha < 0$ *and* $0 < \frac{\alpha\beta}{3} - \gamma - \frac{2\alpha^3}{27} < \frac{2}{3\sqrt{3}}(\frac{\alpha^2}{3} - \beta)^{\frac{3}{2}}$. *If* $\beta < \frac{\gamma}{\alpha} < \frac{\alpha^2}{3}$ *or* $\beta < \frac{\alpha^2}{3} \leq \frac{\gamma}{\alpha}$ *holds, then* (2.2) *admits two oscillatory solutions.*

**Theorem 2.8.** *Suppose that* $\gamma < 0,\ \alpha < 0$ *and*

$$0 < \frac{\alpha\beta}{3} - \gamma - \frac{2\alpha^3}{27} = \frac{2}{3\sqrt{3}}\left(\frac{\alpha^2}{3} - \beta\right)^{\frac{3}{2}}$$

*hold. If* $\beta < \frac{\gamma}{\alpha} < \frac{\alpha^2}{3}$ *or* $\beta < \frac{\alpha^2}{3} \leq \frac{\gamma}{\alpha}$ *holds, then* (2.2) *admits two oscillatory solutions.*

**Theorem 2.9.** *Let* $\gamma_n > 0,\ \beta_n < 0$, *and* $\alpha_n < 0$, *for* $n \geq 0$. *If*

$$\alpha_{n+1}(\alpha_{n-1}\gamma_n - \gamma_n - \beta_n\beta_{n-1}) \geq \beta_{n-1}(\beta_{n+1} - \gamma_{n+1} - \alpha_n\alpha_{n+1})$$

*and*

$$\gamma_{n+1}\beta_{n-1} \leq \alpha_{n+1}(\beta_n\beta_{n-1} - \gamma_n\alpha_{n-1})$$

*holds for large* $n$, *then* (2.1) *is oscillatory.*

**Theorem 2.10.** *Suppose that* $\gamma_n > 0,\ \beta_n > 0$, *and* $\alpha_n < 0$, *for* $n \geq 0$. *If* $\inf\limits_{n\geq 0} \alpha_n = l < 0,\ \liminf\limits_{n\longrightarrow\infty} \beta_n = m > 0$ *and* $\liminf\limits_{n\longrightarrow\infty} \gamma_n = s > 0$ *such that*

$$\frac{2m^3}{27s^2} - \frac{ml}{3s^2} + \frac{1}{s} - \frac{2}{3\sqrt{3}}\left(\frac{m^2}{3s^2} - \frac{l}{s}\right)^{\frac{3}{2}} > 0,$$

*then* (2.1) *is oscillatory.*

**Theorem 2.11.** *If* $\gamma_n < 0,\ \beta_n < 0$, *and* $\alpha_n < 0$, *for* $n \geq 0$, *then* (2.1) *admits two oscillatory solutions.*

**Theorem 2.12.** *Let* $\gamma_n < 0,\ \beta_n > 0$, *and* $\alpha_n > 0$, *for* $n \geq 0$. *Then* (2.1) *admits a nonoscillatory solution.*

**Theorem 2.13.** *Let* $\gamma_n \geq 0,\ \beta_n \geq 0$, *and* $\alpha_n < 0$, *for* $n \geq 0$. *If* $\liminf\limits_{n\longrightarrow\infty} \beta_n = m \geq 0$ *and*

$$\limsup_{n\longrightarrow\infty} \beta_n > \limsup_{n\longrightarrow\infty} \alpha_{n-1}\left(\alpha_n - \frac{m}{\alpha_{n+1}}\right)$$

*hold, then* (2.1) *is oscillatory.*

**Theorem 2.14.** *Let $\gamma_n \geq 0$, $\beta_n < 0$, and $\alpha_n > 0$, for $n \geq 0$. If $\liminf_{n\longrightarrow\infty} \gamma_n = s \geq 0$ and*

$$\limsup_{n\longrightarrow\infty} \gamma_n > \limsup_{n\longrightarrow\infty} \frac{\beta_{n-1}}{\alpha_{n-1}} \left(\beta_n - \frac{s\alpha_n}{\beta_{n+1}}\right)$$

*hold, then* (2.1) *is oscillatory.*

**Theorem 2.15.** *Assume that $\gamma_n \geq 0$, $\beta_n > 0$, and $\alpha_n \leq 0$, for $n \geq 0$. If $4m > l^2$, then* (2.1) *is oscillatory, where $m = \liminf_{n\longrightarrow\infty} \beta_n$ and $l = \liminf_{n\longrightarrow\infty} \alpha_n$.*

**Theorem 2.16.** *Suppose that $\gamma_n \geq 0$, $\beta_n > 0$, and $\alpha_n \leq 0$, for $n \geq 0$. If $l^2 > 3m$ and*

$$s - \frac{lm}{3} + \frac{2l^3}{27} - \frac{2}{3\sqrt{3}} \left(\frac{l^2}{3} - m\right)^{\frac{3}{2}} > 0$$

*hold, then* (2.1) *is oscillatory, where $s = \liminf_{n\longrightarrow\infty} \gamma_n$, $m = \liminf_{n\longrightarrow\infty} \beta_n$ and $l = \liminf_{n\longrightarrow\infty} \alpha_n$.*

**Theorem 2.17.** *Let $\gamma_n > 0$, $\beta_n < 0$, and $\alpha_n > 0$, for $n \geq 0$. Assume that $l$, $m$, and $s$ hold as in Theorem 2.16. If*

$$s - \frac{lm}{3} + \frac{2l^3}{27} - \frac{2}{3\sqrt{3}} \left(\frac{l^2}{3} - m\right)^{\frac{3}{2}} > 0,$$

*then* (2.1) *is oscillatory.*

**Theorem 2.18.** *If $\alpha_n \geq 0$, $\beta_n \geq 0$, and $\gamma_n \geq 0$, for $n \geq 0$ such that $\alpha_n + \beta_n + \gamma_n > 0$, than* (2.1) *is oscillatory.*

**Theorem 2.19.** *If $\gamma_n \geq 0$, $\beta_n \geq 0$, $\alpha_n < 0$, and*

$$\frac{\gamma_{n+1}}{\alpha_{n+1}\alpha_{n-1}} > \frac{\beta_{n+1}}{\alpha_{n+1}} + \frac{\beta_n}{\alpha_{n-1}} - \alpha_n$$

*hold for large $n$, then* (2.1) *is oscillatory.*

**Theorem 2.20.** *If $\gamma_n \geq 0$, $\beta_n < 0$, $\alpha_n \geq 0$ and*

$$\beta_n > \frac{\alpha_n\gamma_{n+1}}{\beta_{n+1}} + \frac{\gamma_n\alpha_{n-1}}{\beta_{n-1}}$$

*hold for large $n$, then* (2.1) *is oscillatory.*

**Theorem 2.21.** *Let $\gamma_n \geq 0$, $-1 \leq \beta_n < 0$ and $\alpha_n \geq 1$. Let $f_n = g_{n+2} - g_{n+1}$, where for each $n \geq 1$ there exists $m > n$ such that $g_n g_m < 0$. If*

$$\sum_{n=1}^{\infty} [g^+_{n+3} + (\alpha_n - 1)g^+_{n+2} + (1 + \beta_n)g^+_{n+1} + \gamma_n g^+_n] = \infty$$

*and*

$$\sum_{n=1}^{\infty} [g^-_{n+3} + (\alpha_n - 1)g^-_{n+2} + (1 + \beta_n)g^-_{n+1} + \gamma_n g^-_n] = \infty,$$

*then every solution of the equation*

$$y_{n+3} + \alpha_n y_{n+2} + \beta_n y_{n+1} + \gamma_n y_n = f_n \tag{2.3}$$

*oscillates, where $f_n$ is a sequence of real numbers, $g^+_n = \max\{g_n, 0\}$ and $g^-_n = \max\{0, -g_n\}$.*

**Theorem 2.22.** *Let $\gamma_n \geq 0$, $\beta_n \geq 0$ and $-1 \leq \alpha_n < 0$. Let $f_n = g_{n+3} - g_{n+2}$, where for each $n \geq 1$ there exists $m > n$ such that $g_n g_m < 0$. If*

$$\sum_{n=1}^{\infty} [(1 + \alpha_n)g^+_{n+2} + \beta_n g^+_{n+1} + \gamma_n g^+_n] = \infty$$

*and*

$$\sum_{n=1}^{\infty}[(1+\alpha_n)g_{n+2}^{-}+\beta_n g_{n+1}^{-}+\gamma_n g_n^{-}]=\infty,$$

*then* (2.3) *is oscillatory.*

**Theorem 2.23.** *Let* $-1\le\gamma_n<0$, $\beta_n\ge 1$ *and* $\alpha_n\ge 0$. *Let* $f_n=g_{n+1}-g_n$, *where for each* $n\ge 1$, *there exists* $m>n$ *such that* $g_ng_m<0$. *If*

$$\sum_{n=1}^{\infty}[g_{n+3}^{+}+\alpha_n g_{n+2}^{+}+(\beta_n-1)g_{n+1}^{+}+(1+\gamma_n)g_n^{+}]=\infty$$

*and*

$$\sum_{n=1}^{\infty}[g_{n+3}^{-}+\alpha_n g_{n+2}^{-}+(\beta_n-1)g_{n+1}^{-}+(1+\gamma_n)g_n^{-}]=\infty,$$

*then* (2.3) *is oscillatory.*

**Theorem 2.24.** *If*

$$\sum_{n=1}^{\infty}n[|\alpha_n+2|+|\beta_n-1|+|\gamma_n|]<\infty,$$

*then* (2.1) *admits a bounded nonoscillatory solution.*

## 3. Main Results

In the section, we investigate the oscillation and nonoscillation of (1.1) through (2.1) and (2.2). For $r(t)\equiv 1$, $p(t)=p$ and $q(t)=q$, (1.1) reduces to

$$(y(t)+py(t-1))''+qy([t-1])=0 \tag{3.1}$$

which then converts to (1.10) in the interval $(n,n+1)$, for $n\in\mathbb{N}$. We may note that a solution $\{C_n\}$ of (1.10) is said to be oscillatory if the terms $C_n$ of the sequence $\{C_n\}$ are not eventually of a fixed sign. Otherwise, the solution $\{C_n\}$ is called nonoscillatory. It is easy to see that the characteristic equation of (1.10) is given by

$$\lambda^3+(p-2)\lambda^2+(1-2p+\frac{q}{2})\lambda+(p+\frac{q}{2})=0. \tag{3.2}$$

It is easy to verify the following result;

**Lemma 3.1.** *Every solution of* (1.10) *is oscillatory if and only if (3.2) has no positive real roots.*

**Proposition 3.2.** *Let* $2p+q>0$. *Then* (1.10) *is oscillatory if and only if* $G^2+4H^3>0$ *or* $G=0$ *and* $H=0$, *where*

$$G=\frac{1}{54}(4p^3+12p^2+12p+45q-9pq+4),\ H=\frac{1}{18}(3q-2p^2-4p-2).$$

**Proof** The proof follows from Proposition 2.1.

**Corollary 3.3.** *Every solution of* (1.10) *is oscillatory if and only if one of the following sets of conditions is satisfied:*

**(i):** $2p+q>0, 3q>2p^2+4p+2$;
**(ii):** $2p+q>0, 4p\ge\max\{8,2+q\}$, $4p^3+12p^2+12p+45q-9pq+4-\sqrt{2}(2p^2+4p+2-3q)^{\frac{3}{2}}>0$;
**(iii):** $2p+q>0, 4p\le\min\{8,2+q\}, p^2+5p+1\ge 0$, $4p^3+12p^2+12p+45q-9pq+4-\sqrt{2}(2p^2+4p+2-3q)^{\frac{3}{2}}>0$;
**(iv):** $2p^2+4p+2=3q$, $p^3-15p^2-33p-17=0$.



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**Proof** By Lemma 3.1, (1.10) is oscillatory if and only if (3.2) has no positive real roots. Upon the choice of negative and complex roots, we have the four possibilities due to Cor. 2.2. Hence, the corollary is proved.

**Corollary 3.4.** *Assume that Cor.3.3 (iv) holds. If* $p \geq 2$ *and* $q \geq 6$*, then* (1.10) *is oscillatory.*

**Proof** When Cor.3.3 (iv) holds, and because of Remark 2.3, it follows that

$$p-2 \geq 0,\ 1-2p+\frac{q}{2} \geq 0,\ p+\frac{q}{2} \geq 0,$$

that is, if and only if

$$p \geq 2,\ q \geq 6,\ p+\frac{q}{2} \geq 5,$$

and also,

$$p-2+1-2p+\frac{q}{2}+p+\frac{q}{2} \geq 5.$$

Hence by Remark 2.3, (1.10) is oscillatory.

**Proposition 3.5.** *If* $y(n)$ *is an oscillatory solution of* (1.10)/ (1.8)/ (1.9)*, then* $y(t)$ *is an oscillatory solution of* (1.1)/ (1.2)*.*

**Proof** Let $\{C_n\}$ be an oscillatory solution of (1.10). Then $\{C_n\}$ has either generalized zeros or zeros, for $n \in \mathbb{N}$. Since $C_n = y(n)$, $n = 0, 1, 2, 3....$ and because of continuity of the solution, then $y(t)$ is an oscillatory solution of (3.1), where we consider the infinite number of zeros associated with $y(n)$. Hence, the proposition is proved.

**Theorem 3.6.** *Assume that Proposition 3.2 holds. Then* (3.1) *admits three oscillatory solutions.*

**Proof** The proof of the theorem follows from Lemma 3.1 and Proposition 3.5. Hence the details are omitted.

**Theorem 3.7.** *Assume that any one of four sets of conditions given in Corollary 3.3 or conditions of Corollary 3.4 holds. Then* (3.1) *admits three oscillatory solutions.*

**Proof** Due to Corollary 3.3, (1.10) is oscillatory upon the choice of negative and complex roots of (3.2) and hence by Proposition 3.5, (3.1) admits three oscillatory solutions. A similar conclusion by Corollary 3.4.

**Proposition 3.8.** *If* $y(t)$ *is a nonoscillatory solution of* (3.1)*, then* $y(n)$ *is a nonoscillatoty solution of* (1.10)*.*

**Proof** Let $y(t)$ be a nonoscillatory solution of (3.1) on $[T, \infty)$, that is, $y(t) > 0$ or $< 0$, for $t \geq T$. Then using the same type of arguments as in (1.4) to (1.8), we find that $C(n) = y(n)$ is a solution of (1.10), and because of continuity $y(n) > 0$ or $< 0$. This completes the proof of the proposition.

**Proposition 3.9.** *Let* $y(t)$ *be a nontrivial solution of* (3.1) *and* $y(n)$ *be a nontrivial solution of* (1.10)*. If* $y(n)$ *is nonoscillatory, then* $y(t)$ *is either nonoscillatory or bounded nonoscillatory.*

**Proof** Let $y(t)$ be a nontrivial solution of (3.1) on $[N_y, \infty)$, where $N_y \geq 1$ is an integer. Suppose there exists $n_1 > N_y$ such that $y(n) > 0$, for $n \geq n_1$. Hence $y(t) > 0$ for $t \in [n_1, n_1 + 1)$. Let $y(n_1) > 0$. Setting for $z(t) = y(t) + py(t-1)$, it follows from (3.1) that

$$z''(t) = -qy(n_1) \leq 0, t \in [n_1+1, n_1+2),$$

where we consider case $p \geq 0$ and $q < 0$. Thus $z'(t)$ is nonincreasing on $[n_1+1, n_1+2)$. If $z'(t) < 0$,for $t \in [n_1+1, n_1+2)$, then $z(t) < z(n_1+1)$. Because of continuity $z(n_1+2) \leq z(n_1+1)$ implies that $z(t)$ is nonincreasing on $[n_1+1, n_1+2)$. There for $z(t)$ is nonoscillatory on $[n_1+1, n_1+2)$ and hence $z(t) > 0$ or $< 0$. Ultimately, because of continuity $z(n_1+1) > 0$ or $< 0$ if and only if

$$y(n_1+1) + py(n_1) > 0 \ \ or \ \ y(n_1+1) + py(n_1) < 0,$$

that is, if and only if $y(n_1+1)>0$. As a result, $z(n_1+2)>0$ if and only if $y(n_1+2)>0$ and $y(t)$ is nonoscillatory for $t\in[n_1+1,n_1+2)$.

Next, we suppose that $z'(t)>0$, for $t\in[n_1+1,n_1+2)$. Then $z(t)$ is nondecreasing and using the preceding argument, we can show that $y(t)$ is nonoscillatory on $[n_1+1,n_1+2)$. Continuing in this way, it follows that $y(t)$ is nonoscillatory on the intervals of the form $[n_1+2,n_1+3),[n_1+3,n_1+4)$,... Therefore, $y(t)$ is nonoscillatory on $[N_y,\infty)$. The argument is similar when $y(n_1)<0$.

Assume that $p<0$ and $q>0$. Using the same type of reasoning as above, we have that $z(t)>0$ or $<0$, for $t\in[n_1+1,n_1+2)$. If $z(t)>0$, for $t\in[n_1+1,n_1+2)$, then because of continuity $z(n_1+1)>0$ if and only if $y(n_1+1)+py(n_1)>0$, that is, if and only if $y(n_1+1)>-py(n_1)>0$. Consequently, $z(n_1+2)>0$ implies that $y(n_1+2)>0$. Hance, $y(t)$ is nonoscillatory on $[n_1+1,n_1+2)$. If $z(t)<0$, for $t\in[n_1+1,n_1+2)$, then because of continuity $z(n_1+1)<0$ and $z(n_1+2)<0$. As a result, $y(n_1+1)<-py(n_1)<\infty$ and $y(n_1+2)<-py(n_{1+1})<\infty$. So $y(t)$ is nonoscillatory or bounded on $[n_1+1,n_1+2)$. We can use the above fact for intervals of the form $[n_1+2,n_1+3),[n_1+3,n_1+4)$,... and to conclude that $y(t)$ is nonoscillatory or bounded on $[N_y,\infty)$. The following two cases

$$(i)p\geq 0,q<0;(ii)p<0,q<0$$

can similarly be dealt with. This completes the proof of the proposition.

**Theorem 3.10.** *Let $2p+q<0$ and $p>2$. Assume that one of the following three sets of conditions*

**(i):** $\frac{1}{2}(2p+q)-\frac{1}{6}(p-2)(2-4p+q)+\frac{2}{27}(p-2)^3<\frac{2}{3\sqrt{3}}[\frac{1}{3}(p-2)^2-\frac{1}{2\sqrt{2}}(2-4p+q)^{\frac{3}{2}}]$, $3q<2p^2+4p+2$;

**(ii):** $0<\frac{1}{2}(2p+q)-\frac{1}{6}(p-2)(2-4p+q)+\frac{2}{27}(p-2)^3<\frac{2}{3\sqrt{3}}[\frac{1}{3}(p-2)^2-\frac{1}{2\sqrt{2}}(2-4p+q)^{\frac{3}{2}}]$, $0\leq 3(2-4p+q)<2(p-2)^2$;

**(iii):** $\frac{1}{2}(2p+q)-\frac{1}{6}(p-2)(2-4p+q)+\frac{2}{27}(p-2)^3=0$, $9(2-4p+q)<4(p-2)^2$,

*holds. Then* (3.1) *admits a solution which is either nonoscillatory or bounded nonoscillatory.*

**Proof** (1.10) admits a nonoscillatory if and only if (3.2) has a positive root, that is, if and only if $p+\frac{q}{2}<0$. Consequently, (1.10) has a nonoscillatory solution due to Theorem 2.4. Hence $y(t)$ is either nonoscillatory or bounded by Proposition 3.9.

**Theorem 3.11.** *Let $2p+q<0$, $p>2$ and $3q<2p^2+4p+2$. If*

$$0<\frac{1}{2}(2p+q)-\frac{1}{6}(p-2)(2-4p+q)+\frac{2}{27}(p-2)^3=\frac{2}{3\sqrt{3}}[\frac{1}{3}(p-2)^2-\frac{1}{2\sqrt{2}}(2-4p+q)^{\frac{3}{2}}]$$

*holds, then* (3.1) *admits two nonoscillatory solutions.*

**Proof** By Theorem 2.5, it follows that (1.10) admits two nonoscillatory solutions if and only if (3.2) has two positive roots. Hence, (3.1) has two nonoscillatory solutions due to Proposition 3.9.

**Theorem 3.12.** *Let $2p+q<0$ and $p<2$. Assume that one of the following set of conditions*

$$(i)\ \ (2p^2+4p-3q+2)^{\frac{3}{2}}\geq-\frac{1}{3\sqrt{2}}(4p^3+12p^2+12p+45q-9pq+4)>0,$$

$$(ii)\ \ 6p+3q\geq 3(p-2)(2-4p+q)>2(p-2)^3$$

*is true. Then* (3.1) *is strongly nonoscillatory.*

**Proof** By Theorem 2.6, it follows that (1.8) admits three nonoscillatory solutions if and only if (3.2) has three positive roots. Consequently, (3.1) is strongly nonoscillatory due to Proposition 3.9.

**Theorem 3.13.** *Let $2p+q<0$ and $p<2$. Assume that*

$$(2p^2+4p-3q+2)^{\frac{3}{2}}\geq-\frac{1}{3\sqrt{2}}(4p^3+12p^2+12p+45q-9pq+4)>0,$$

*holds. If $3(p-2)(2-4p+q)>6p+3q>2(p-2)^3$ or $3(p-2)(2-4p+q)>2(p-2)^3\geq 6p+3q$ holds true, then* (3.1) *admits two oscillatory solutions.*

**Proof** By Theorems 2.7 and 2.8, it follows that (1.10) admits two oscillatory solutions if and only if (3.2) has two imaginary roots. Hence (3.1) has two oscillatory solutions due to Proposition 3.5.

**Proposition 3.14.** *Let $|p(t)| < \infty$, $t \in [0, \infty)$. Let $y(t)$ be a nontrivial solution of (1.1) and $y(n)$ be a nontrivial solution of (1.8). If $y(n)$ is nonoscillatory, then $y(t)$ is either nonoscillatory or bounded nonoscillatory.*

**Proof** The proof of the proposition can be followed from Proposition 3.9. Hence the details are omitted.

**Theorem 3.15.** *Assume that $F_n < 0$, $G_n < 0$ and $H_n > 0$, for all $n \geq 0$. If*

$$F_{n+1}(F_{n-1}H_n - H_n - G_nG_{n-1}) \geq G_{n-1}(G_{n+1} - H_{n+1} - F_nF_{n+1})$$

*and*

$$H_{n+1}G_{n-1} \leq F_{n+1}(G_nG_{n-1} - H_nF_{n-1})$$

*hold for large $n$, then (1.1) admits three oscillatory solutions.*

**Proof** Using the conditions, it follows that (1.8) is oscillatory due to Theorem 2.9. Hence by Proposition 3.5, (1.1) admits three oscillatory solutions.

**Theorem 3.16.** *Suppose that $F_n < 0$, $G_n > 0$ and $H_n > 0$, for all $n \geq 0$. If $\inf_{n\geq 0} F_n = \alpha < 0$, $\liminf_{n\longrightarrow\infty} G_n = \beta > 0$ and $\liminf_{n\longrightarrow\infty} H_n = \gamma > 0$ such that*

$$\frac{2\beta^3}{27\gamma^3} - \frac{\alpha\beta}{3\gamma^2} + \frac{1}{\gamma} - \frac{2}{3\sqrt{3}}\left(\frac{\beta^2}{3\gamma^2} - \frac{\alpha}{\gamma}\right)^{\frac{3}{2}} > 0,$$

*then (1.1) admits three oscillatory solutions.*

**Proof** The proof of the theorem follows from the proof of the Theorem 2.10. and Proposition 3.5. Hence the details are omitted.

**Theorem 3.17.** *Let $F_n < 0$, $G_n \geq 0$ and $H_n \geq 0$, for all $n \geq 0$. If*

$$\liminf_{n\longrightarrow\infty} G_n = \beta \geq 0$$

*and*

$$\limsup_{n\longrightarrow\infty} G_n > \limsup_{n\longrightarrow\infty} F_{n-1}\left(F_n - \frac{\beta}{F_{n+1}}\right)$$

*holds for large $n$, then (1.1) admits three oscillatory solutions.*

**Proof** The proof of the theorem can be followed from Theorem 2.13 and Proposition 3.5. Hence the details are omitted.

**Theorem 3.18.** *Let $F_n > 0$, $G_n < 0$ and $H_n \geq 0$, for all $n \geq 0$. If*

$$\liminf_{n\longrightarrow\infty} H_n = \gamma \geq 0$$

*and*

$$\limsup_{n\longrightarrow\infty} H_n > \limsup_{n\longrightarrow\infty} \frac{G_{n-1}}{F_{n-1}}\left(G_n - \frac{\gamma F_n}{G_{n+1}}\right),$$

*holds for large $n$, then (1.1) admits three oscillatory solutions.*

**Proof** The proof of the theorem follows from the proof of Theorem 2.14 and Proposition 3.5. Hence the details are omitted.

**Theorem 3.19.** *Assume that $F_n \leq 0$, $G_n > 0$ and $H_n \geq 0$, for all $n \geq 0$. Let $\beta = \liminf_{n\longrightarrow\infty} G_n$ and $\alpha = \liminf_{n\longrightarrow\infty} F_n$ be such that $4\beta > \alpha^2$. Then (1.1) admits three oscillatory solutions.*

**Proof** The proof of the theorem can be followed from Theorem 2.15 and Proposition 3.5. This completes the proof of the theorem.

**Theorem 3.20.** *Suppose that* $F_n \leq 0$, $G_n > 0$ *and* $H_n \geq 0$, *for all* $n \geq 0$. *Let* $\alpha = \liminf_{n\longrightarrow\infty} F_n$, $\beta = \liminf_{n\longrightarrow\infty} G_n$ *and* $\gamma = \liminf_{n\longrightarrow\infty} H_n$ *be such that* $\alpha^2 > 3\beta$ *and*

$$\gamma - \frac{\alpha\beta}{3} + \frac{2\alpha^3}{27} - \frac{2}{3\sqrt{3}}\left(\frac{\alpha^2}{3} - \beta\right)^{\frac{3}{2}} > 0$$

*holds, then* (1.1) *admits three oscillatory solutions.*

**Proof** The proof of the theorem follows from the proof of Theorem 2.16 and Proposition 3.5. Hence, the theorem is proved.

**Theorem 3.21.** *Let* $F_n > 0$, $G_n < 0$ *and* $H_n > 0$, *for all* $n \geq 0$. *If all conditions of Theorem 3.21 hold, then* (1.1) *admits three oscillatory solutions.*

**Proof** The proof of the theorem follows from Theorem 2.17 and Proposition 3.5. Hence the proof of the theorem is complete.

**Theorem 3.22.** *Assume that any one of the following conditions:*

**(i):** $F_n \geq 0$, $G_n < 0$ *and* $H_n \geq 0$, *for all* $n \geq 0$ *such that*

$$G_n > \frac{F_n H_{n+1}}{G_{n+1}} + \frac{H_n F_{n-1}}{G_{n-1}}, \quad for\ large\ n;$$

**(ii):** $F_n < 0$, $G_n \geq 0$ *and* $H_n \geq 0$, *for all* $n \geq 0$ *such that*

$$\frac{H_{n+1}}{F_{n+1}F_{n-1}} > \frac{G_{n+1}}{F_{n+1}} + \frac{G_n}{F_{n-1}} - F_n, \quad for\ large\ n$$

*holds, then* (1.1) *admits three oscillatory solutions.*

**Proof** The proof of the theorem can be followed from Theorems 2.19, 2.20 and Proposition 3.5. Hence, the proof of the theorem is complete.

**Theorem 3.23.** *If* $F_n \geq 0$, $G_n \geq 0$ *and* $H_n \geq 0$, *for all* $n \geq 0$ *such that* $F_n + G_n + H_n > 0$, (1.1) *admits three oscillatory solutions.*

**Proof** The proof of the theorem can be followed from Theorem 2.18, and Proposition 3.5. Thus the proof of the theorem is complete.

**Theorem 3.24.** *If* $F_n < 0$, $G_n < 0$ *and* $H_n < 0$, *for all* $n \geq 0$. *Then* (1.1) *admits two oscillatory solutions.*

**Proof** The proof of the theorem follows from Theorem 2.11 and Proposition 3.5. Therefore, the theorem is proved.

**Theorem 3.25.** *Let* $H_n \geq 0$, $-1 \leq G_n < 0$ *and* $F_n \geq 1$. *Let* $K_n = g_{n+2} - g_{n+1}$, *where for each* $n \geq 1$ *there exists* $m > n$ *such that* $g_n g_m < 0$. *If*

$$\sum_{n=1}^{\infty}[g_{n+3}^{+} + (F_n - 1)g_{n+2}^{+} + (1 + G_n)g_{n+1}^{+} + H_n g_n^{+}] = \infty$$

*and*

$$\sum_{n=1}^{\infty}[g_{n+3}^{-} + (F_n - 1)g_{n+2}^{-} + (1 + G_n)g_{n+1}^{-} + H_n g_n^{-}] = \infty,$$

*then* (1.2) *admits three oscillatory solutions.*

**Proof** Using the given conditions, it follows from Theorem 2.21 that (1.9) is oscillatory. Hence, by Proposition 3.5, (1.2) admits three oscillatory solutions. Thus the proof of the theorem is complete.

**Theorem 3.26.** *Let $H_n \geq 0$, $G_n \geq 0$ and $-1 \leq F_n < 0$. Let $K_n = g_{n+3} - g_{n+2}$, where for each $n \geq 1$ there exists $m > n$ such that $g_n g_m < 0$. If*

$$\sum_{n=1}^{\infty}[(1+F_n)g_{n+2}^{+} + G_n g_{n+1}^{+} + H_n g_n^{+}] = \infty$$

*and*

$$\sum_{n=1}^{\infty}[(1+F_n)g_{n+2}^{-} + (1+G_n)g_{n+1}^{-} + H_n g_n^{-}] = \infty,$$

*then* (1.2) *admits three oscillatory solutions.*

**Proof** Using the given conditions, it follows from Theorem 2.22 that (1.9) is oscillatory. Therefore, by Proposition 3.5, (1.2) admits three oscillatory solutions. This completes the proof of the theorem.

**Theorem 3.27.** *Let $-1 \leq H_n < 0$, $G_n \geq 1$ and $F_n \geq 0$. Let $K_n = g_{n+1} - g_n$, where for each $n \geq 1$ there exists $m > n$ such that $g_n g_m < 0$. If*

$$\sum_{n=1}^{\infty}[g_{n+3}^{+} + F_n g_{n+2}^{+} + (G_n - 1)g_{n+1}^{+} + (1+H_n)g_n^{+}] = \infty$$

*and*

$$\sum_{n=1}^{\infty}[g_{n+3}^{-} + F_n g_{n+2}^{-} + (G_n - 1)g_{n+1}^{-} + (1+H_n)g_n^{-}] = \infty,$$

*then* (1.2) *admits three oscillatory solutions.*

**Proof** Because of given conditions, (1.9) is oscillatory due to Theorem 2.23. Hence by Proposition 3.5, (1.2) admits three oscillatory solutions. So the proof of the theorem is complete.

**Theorem 3.28.** *Let $F_n > 0$, $G_n > 0$ and $H_n < 0$, for all $n \geq 0$. Then* (1.1) *admits a nontrivial solution, which is either nonoscillatory or bounded nonoscillatory.*

**Proof** The proof of the theorem follows from Theorem 2.12 and Proposition 3.14. Hence the proof of the theorem is complete.

**Theorem 3.29.** *If*

$$\sum_{n=1}^{\infty} n[|F_n + 2| + |G_n - 1| + |H_n|] < \infty,$$

*then* (1.1) *has a bounded nonoscillatory solution.*

**Proof** Due to Theorem 2.24, (1.8) admits a bounded nonoscillatory solution. Hence by Proposition 3.14, (1.1) has a bounded nonoscillatory solution. This completes the proof of the theorem.

## 4. Discussion and Examples

The study of (1.1)/ (1.2) depends on the behaviour of solutions of (1.8)/ (1.9). Hence, we are able to predict the oscillatory and nonoscillatory characters of (1.1)/ (1.2) subject to its corresponding discrete equations (1.8)/ (1.9) due to Propositions 3.5, 3.9 and 3.14. Because the solutions of DEPCA are hybrid in nature, then it is important to know the number of linearly independent solutions of (3.1) but, we could partially succeed to keep our view subject to the linearly independent solutions of (1.10) only. We note that it is very difficult to predict the linearly independent solutions of any kind of neutral delay differential equations without piecewise constant arguments. In case of (1.1) and (1.2), we have discussed the existence of oscillatory and nonoscillatory solutions with respect to the difference equations (1.8) and (1.9) respectively. We conclude this section with the following examples to illustrate our main results:

**Example 4.1.** Consider

$$(y(t) + y(t-1))'' + 5y([t-1]) = 0, t > 1. \tag{4.1}$$

The corresponding difference equation of (4.1) is given by

$$C_{n+3} - C_{n+2} + \frac{3}{2}C_{n+1} + \frac{7}{2}C_n = 0. \tag{4.2}$$

Clearly, $G > 0$ and $H > 0$. Hence by Proposition 3.2, the solution space of (4.2) becomes

$$\{\{(-1)^n\}, \{(3.5)^{\frac{n}{2}} cosn\theta\}, \{(3.5)^{\frac{n}{2}} sinn\theta\}\},$$

where $\theta = tan^{-1}(\frac{\sqrt{10}}{2})$. Therefore, Theorem 3.6 implies that (4.1) admits three oscillatory solutions, viz.,

$$\{\{(-1)^{[t]}\}, \{(3.5)^{\frac{[t]}{2}} cos[t]\theta\}, \{(3.5)^{\frac{[t]}{2}} sin[t]\theta\}\}.$$

**Example 4.2.** Consider

$$(y(t) - 4y(t-1))'' + 2y([t-1]) = 0, t > 1. \tag{4.3}$$

The corresponding difference equation of (4.3) is given by

$$C_{n+3} - 6C_{n+2} + 10C_{n+1} - 3C_n = 0. \tag{4.4}$$

By Theorem 3.2, (4.4) is strongly nonoscillatory and the solution space of (4.4) becomes

$$\left\{\{3^n\}, \left\{\left(\frac{3+\sqrt{5}}{2}\right)^n\right\}, \left\{\left(\frac{3-\sqrt{5}}{2}\right)^n\right\}\right\}.$$

Therefore, Proposition 3.9 implies that (4.3) admits three nonoscillatory solutions, viz.,

$$\left(3^{[t]}, \left(\frac{3+\sqrt{5}}{2}\right)^{[t]}, \left(\frac{3-\sqrt{5}}{2}\right)^{[t]}\right).$$

**Example 4.3.** Consider

$$(e^{-t}(y(t) + e^{-t}y(t-1))')' + e^{-t}y([t-1]) = 0, t > 1. \tag{4.5}$$

The corresponding difference equation of (4.5) is given by (1.8), where $F_n = (e^{-(n+3)} - 1 - e)$, $G_n = (2e - 2 - e^{-(n+2)} - e^{-(n+1)})$ and $H_n = (1 + e^{-(n+2)})$, and it is easy to verify that $F_n < 0$, $G_n > 0$ and $H_n > 0$, for $n \in \mathbb{Z}_+$. Clearly, $\lim_{n\to\infty} H_n = 1$, $\lim_{n\to\infty} G_n = 2e - 2$ and $\inf_{n>1} F_n = -1 - e$. It is easy to see that all conditions of Theorem 3.16 are satisfied and hence (4.5) admits three oscillatory solutions.

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## References

[1] S. Busenberg, K. L. Cooke; *Models of vertically transmitted disese with sequential continuous dynamics* in "Nonlinear Phenomena in Mathemetical Sciences" (V. Lakshmikantham Ed.), New York, Academic Press, 1982, pp. 179-189.

[2] K. L. Cooke, J. Winer; Retarded differential equations with piecewise constant delays, *J. Math. Anal. Appl.*, **99**(1984), pp. 256-297.

[3] K. L. Cooke, J. Winter; A survey of Differential Equations with piecewise continuous Argument, *Lecture Notes in Mathematics*, Springer-Verleg, 1991, 1475: 1-5.

[4] S. N. Elaydi; *An Introduction to Difference Equations,* Springer, New York, 1999.

[5] E. C. Partheniadis; Stability and oscillation of neutral delay differantial equations with piecewise constant argument, *Diff. Integ. Eq.*, **4**(1988), pp. 459-472.

[6] N. Parhi, A. K. Tripathy; On oscillatory third order difference equations, *J. Differ. Equ. Appl.*, **6**(2000), pp. 53-74.

[7] N. Parhi, A. K. Tripathy; On the behaviour of solutions of a class of third order difference equations, *J. Differ. Equ. Appl.*, **8**(2002), pp. 415-426.
[8] J. H. Shen, I. P. Stavroulakis; Oscillatory and nonoscillatory delay equations with piecewise constant argument, *J. Math. Anal. Appl.*, **248**(2000), pp. 385-401.
[9] G. Q. Wang; Periodic solutions of a neutral differential equation with piecewise constant arguments, *J. Math. Anal. Appl.*, **326**(2007), pp. 736-747.
[10] P. D. Xiong; Almost periodic solutions of neutral differential difference equations withpiecewise constant arguments, *Acta Math. Sin.*, **18**(2002), pp. 263-276.
[11] B. G. Zhang, N. Parhi; Oscillatory and nonoscillatory propoties of first order differential equations with piecewise constant deviating arguments, *J. Math. Anal. Appl.*, **139**(1989), pp. 23-35.
[12] R. K. Zhuang, R. Yuan; The existence of pseudoalmost periodic solutions of third order neutral differential equations with piecewise constant argument, *Acta Math. Sin.*, **29**(2013), pp. 943-958.

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