ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2017, VOLUME 7, ISSUE 2, p.80-85

NECESSARY AND SUFFICIENT CONDITIONS FOR OSCILLATION OF SOLUTIONS OF NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this work, we establish necessary and sufficient conditions for oscillation of a class of second-order delay differential equations of the form:

$$\big(r(t)x'(t)\big)' + q(t)H\big(x(\sigma(t))\big) = 0, \ \ t \geq t_0,$$

under the assumptions $\int_0^\infty \frac{dt}{r(t)} = \infty$, when H is sublinear and superlinear. Finally, some illustrating examples are presented to show that feasibility and effectiveness of main results.

Mathematics Subject Classification (2010): 34C10, 34C15.
Key words: Oscillation, nonoscillation, nonlinear, sublinear, superlinear, delay, Lebesque's dominated convergence theorem.

Article history:
Received 7 December 2016
Accepted 25 September 2017

1. Introduction

Consider the nonlinear delay differential equations of the form

$$\big(r(t)x'(t)\big)' + q(t)H\big(x(\sigma(t))\big) = 0, \ \ t \geq t_0, \tag{1.1}$$

where $r, q, \sigma \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that $\sigma(t) \leq t$ with $\lim_{t\to\infty} \sigma(t) = \infty$ and $H \in C(\mathbb{R}, \mathbb{R})$ is nondecreasing and satisfying the property $uH(u) > 0$ for $u \neq 0$. The objective of this work is to establish the necessary and sufficient conditions for oscillation of solutions of (1.1) under the assumption

(A_1) $R(t) = \int_0^t \frac{ds}{r(s)} \to +\infty$ as $t \to \infty$.

The motivation of the present work has come from the work of [6]. In [6], Liu et al. have considered the the existence of oscillatory solutions of forced nonlinear delay differential equations of the form

$$\big[r(t)\Phi(x'(t))\big]' + \sum_{i=1}^{m} f_i\big(t, x(g_i(t))\big) = q(t).$$

and established a new sufficient condition for global existence of oscillatory solution by the Schauder-Tychonoff theorem. In this direction, we refer some related works ([1],[2], [4]–[7], [13]) to the readers and the references cited therein.

The delay differential equations find numerous applications in natural sciences and technology. Equations involving delay, and those involving advance and a combination of both arise in the models on lossless transmission lines in high speed computers which are used to interconnect switching circuits. The construction of these models using delays is complemented by the mathematical investigation of nonlinear equations. Moreover, the delay differential equations play an important role in modeling virtually every physical, technical, or biological process.

Definition 1.1. *By a solution of* (1.1) *we understand a function* $x \in C([t_0, \infty), \mathbb{R})$ *such that* $x(t)$ *and* $r(t)x'(t)$ *are once continuously differentiable and equation* (1.1)*) is satisfied for* $t \geq 0$*, where* $\sup\{|x(t)| : t \geq t_0\} > 0$ *for every* $t_0 \geq 0$*. A solution of* (1.1) *is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory.*

2. Main Results

This section deals with the necessary and sufficient conditions for oscillation of all solutions of (1.1). We need the following conditions for this work in the sequel.

(A_2) $H(uv) = H(u)H(v)$, $u, v \in \mathbb{R}$.

Remark 2.1. [8] *Assumption* (A_2) *implies that* $H(-u) = -H(u)$*. Indeed,* $H(1)H(1) = H(1)$ *and* $H(1) > 0$ *imply that* $H(1) = 1$*. Further,* $H(-1)H(-1) = H(1) = 1$ *implies that* $\big(H(-1)\big)^2 = 1$*. Since* $H(-1) < 0$*, we conclude that* $H(-1) = -1$*. Hence,*

$$H(-u) = H(-1)H(-u) = -H(-u).$$

On the other hand, $H(uv) = H(u)H(v)$ *for* $u > 0$ *and* $v > 0$ *and* $H(-u) = -H(u)$ *imply that* $H(xy) = H(x)H(y)$ *for every* $x, y \in \mathbb{R}$*.*

Remark 2.2. [8] *We may note that if* $x(t)$ *is a solution of* (1.1)*, then* $y(t) = -x(t)$ *is also a solution of* (1.1) *provided that* H *satisfies* (A_2)*.*

Theorem 2.3. *Assume that* (A_1) *and* (A_2) *hold. Furthermore assume that*

(A_3) *H is sublinear, that is,* $\frac{H(u)}{u^\beta} \geq \frac{H(v)}{v^\beta}$, $0 < u \leq v$, $\beta < 1$

hold. Then every solution of the equation (1.1) *oscillates if and only if*

(A_4) $\int_T^\infty q(t)H\big(CR(\sigma(t))\big)dt = +\infty$, $T > 0$ *for every* $C > 0$*.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1). So there exists $t_0 > 0$ such that $x(t) > 0$ or < 0 for $t \geq t_0$. Without loss of generality and because of (A_2), we may assume that $x(t) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1 > t_0$. From (1.1), it follows that

$$\big(r(t)x'(t)\big)' = -q(t)H\big(x(\sigma(t))\big) < 0, \tag{2.1}$$

hold for $t \geq t_1$. Hence there exists $t_2 > t_1$ such that $r(t)x'(t)$ is nonincreasing on $[t_2, \infty)$. We claim that $r(t)x'(t) > 0$ for $t \in [t_2, \infty)$. If $r(t)x'(t) \leq 0$ for $t \geq t_3$ then we can find $K > 0$ such that $r(t)x'(t) \leq -K$ for $t \geq t_3$. Integrating the relation $x'(t) \leq -\frac{K}{r(t)}$, $t \geq t_3$ from t_3 to $t(> t_3)$ and obtain $x(t) - x(t_3) \leq -K\int_{t_3}^t \frac{ds}{r(s)}$, that is, $x(t) \leq x(t_3) - K\left[\int_{t_3}^t \frac{ds}{r(s)}\right] \to -\infty$ as $t \to \infty$, a contradiction to the fact that $x(t)$ is a positive solution of the equation of (1.1). So our claim holds. We integrate (1.1) from $t(\geq t_3)$ to $+\infty$, we get

$$\big[r(s)x'(s)\big]_t^\infty + \int_t^\infty q(s)H\big(x(\sigma(s))\big)ds = 0.$$

Since, $\lim_{t\to\infty} r(t)x'(t)$ exists, then the above inequality becomes

$$\int_t^\infty q(s)H\big(x(\sigma(s))\big)ds \leq r(t)x'(t)$$

for $t \geq t_3$, therefore

$$x'(t) \geq \frac{1}{r(t)}\left[\int_t^\infty q(s)H\big(x(\sigma(s))\big)ds\right] \tag{2.2}$$

for $t \geq t_3$. Let $t_4 > t_3$ be such a point that

$$R(t) - R(t_4) \geq \frac{1}{2}R(t) \;\; for \;\; t \geq t_4.$$

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Integrating (2.2) from t_4 to $t(> t_4)$, we obtain

$$\begin{aligned}x(t) - x(t_4) &\geq \int_{t_4}^{t} \frac{1}{r(s)} \left[\int_{s}^{\infty} q(u)H\big(x(\sigma(u))\big)du\right] ds \\ &\geq \int_{t_4}^{t} \frac{1}{r(s)} \left[\int_{t}^{\infty} q(u)H\big(x(\sigma(u))\big)du\right] ds\end{aligned}$$

that is,

$$\begin{aligned}x(t) &\geq \int_{t_4}^{t} \frac{1}{r(s)} \left[\int_{t}^{\infty} q(u)H\big(x(\sigma(u))\big)du\right] ds \\ &\geq \frac{1}{2}R(t) \left[\int_{t}^{\infty} q(u)H\big(x(\sigma(u))\big)du\right] \end{aligned} \tag{2.3}$$

for $t \geq t_4$. Since, $r(t)x'(t)$ is nonincreasing on $[t_4, \infty)$, then there exists a constant $C > 0$ and $t_5 \geq t_4$ such that $r(t)x'(t) \leq C$ for $t \geq t_5$ and hence $x(t) \leq CR(t)$, $t \geq t_5$. Using the fact H is sublinear, we have

$$\begin{aligned}H\big(x(\sigma(t))\big) &= \frac{H\big(x(\sigma(t))\big)}{x^{\beta}\big(\sigma(t)\big)} x^{\beta}\big(\sigma(t)\big) \\ &\geq \frac{H\big(CR(\sigma(t))\big)}{C^{\beta}R^{\beta}\big(\sigma(t)\big)} x^{\beta}\big(\sigma(t)\big)\end{aligned}$$

and hence (2.3) reduces to

$$x(t) \geq \frac{R(t)}{2C^{\beta}} \left[\int_{t}^{\infty} q(u)H\big(CR(\sigma(u))\big)\frac{x^{\beta}\big(\sigma(u)\big)}{R^{\beta}\big(\sigma(u)\big)}du\right]$$

for $t \geq t_5$. If we define

$$w(t) = \frac{1}{2C^{\beta}} \left[\int_{t}^{\infty} q(u)H\big(CR(\sigma(u))\big)\frac{x^{\beta}\big(\sigma(u)\big)}{R^{\beta}\big(\sigma(u)\big)}du\right],$$

then $x(t) \geq R(t)w(t)$ for $t \geq t_5$. Now,

$$\begin{aligned}w'(t) &\leq -\frac{1}{2C^{\beta}} q(t)H\big(CR(\sigma(t))\big)\frac{x^{\beta}\big(\sigma(t)\big)}{R^{\beta}\big(\sigma(t)\big)} \\ &\leq -\frac{1}{2C^{\beta}} q(t)H\big(CR(\sigma(t))\big)w^{\beta}\big(\sigma(t)\big) \leq 0\end{aligned}$$

implies that $w(t)$ is nonincreasing on $[t_5, \infty)$ and $\lim_{t\to\infty} w(t)$ exists. It is easy to verify that

$$\begin{aligned}[w^{1-\beta}(t)]' &\leq -\frac{1}{2C^{\beta}}(1-\beta)q(t)H\big(CR(\sigma(t))\big)w^{-\beta}(t)w^{\beta}\big(\sigma(t)\big) \\ &\leq -\frac{1}{2C^{\beta}}\big(1-\beta\big)q(t)H\big(CR(\sigma(t))\big).\end{aligned}$$

Integrating the last inequality from t_5 to $t(> t_5)$, we obtain

$$\big[w^{1-\beta}(s)\big]_{t_5}^{t} \leq -\frac{1}{2}\big(1-\beta\big)C^{-\beta}\int_{t_5}^{t} q(s)H\big(CR(\sigma(s))\big)ds,$$

that is,

$$\begin{aligned}\frac{1}{2}\big(1-\beta\big)C^{-\beta}\int_{t_5}^{t} q(s)H\big(CR(\sigma(s))\big)ds &\leq -\big[w^{1-\beta}(s)\big]_{t_5}^{t} \\ &< \infty, \quad as \;\; t \to \infty,\end{aligned}$$

a contradiction to (A_4).

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Next, for the necessary part we suppose that (A_4) doesn't hold. So, for $C > 0$, let

$$\int_T^\infty q(t)H\big(CR(\sigma(t))\big)dt < \frac{C}{2}.$$

Let's consider

$$M = \big\{x : x \in C([t_0, +\infty), \mathbb{R}), x(t) = 0 \; for \; t \in [t_0, T] \; and \\ \frac{C}{2}[R(t) - R(T)] \leq x(t) \leq C[R(t) - R(T)]\big\}$$

and define $\Phi : M \to C([t_0, +\infty), \mathbb{R})$ such that

$$(\Phi x)(t) = \begin{cases} 0, & t \in [t_0, T) \\ \int_T^t \frac{1}{r(u)} \left[\frac{C}{2} + \int_u^\infty q(s)H\big(x(\sigma(s))\big)ds\right] du & t \geq T. \end{cases}$$

For every $x \in M$,

$$(\Phi x)(t) \geq \frac{C}{2} \int_T^t \frac{du}{r(u)} = \frac{C}{2}\left[R(t) - R(T)\right],$$

and the inequality $x(t) \leq CR(t)$ implies that

$$(\Phi x)(t) \leq C \int_T^t \frac{du}{r(u)} = C\left[R(t) - R(T)\right].$$

Thus, $(\Phi x)(t) \in M$. Let us define now the function $u_n : [t_0, +\infty) \to \mathbb{R}$ by the recursive formula

$$u_n(t) = (\Phi u_{n-1})(t), \qquad n \geq 1,$$

with the initial condition

$$u_0(t) = \begin{cases} 0, & t \in [t_0, T) \\ \frac{C}{2}[R(t) - R(T)], & t \geq T. \end{cases}$$

Inductively it is easily verified that

$$\frac{C}{2}[R(t) - R(T)] \leq u_{n-1}(t) \leq u_n(t) \leq C[R(t) - R(T)],$$

for $t \geq T$. Therefore for $t \geq t_0$, $\lim_{n\to+\infty} u_n(t)$ exists. Let $\lim_{n\to+\infty} u_n(t) = u(t)$ for $t \geq t_0$. By Lebesgue's dominated convergence theorem $u \in M$ and $(\Phi u)(t) = u(t)$, where $u(t)$ is a solution of the equation (1.1) on $[t_0, \infty)$ such that $u(t) > 0$. Hence, (A_4) is a necessary condition. This completes the proof of the theorem. ☐

Theorem 2.4. *Assume that (A_1), (A_2) hold and $r(t) \geq r(\sigma(t))$. Furthermore assume that*

(A_5) *H is superlinear, that is, $\frac{H(u)}{u^\beta} \geq \frac{H(v)}{v^\beta}$, $u \geq v > 0$, $\beta > 1$.*

Then every solution of the equation (1.1) is oscillatory if and only if

(A_6) $\int_0^\infty \frac{1}{r(t)} \left[\int_t^\infty q(s)ds\right] dt = +\infty.$

Proof. For sufficient part, we use the same type of argument as in the proof of the Theorem 2.3 for the case $r(t)x'(t) \leq 0$. Let's consider the case $r(t)x'(t) > 0$ for $t \geq t_3$. So there exists a constant $C > 0$ and $t_4 > t_3$ such that $x(\sigma(t)) \geq C$ for $t \geq t_4$. Consequently,

$$\begin{aligned} H\big(x(\sigma(t))\big) &= \frac{H\big(x(\sigma(t))\big)}{x^\beta\big(\sigma(t)\big)} x^\beta\big(\sigma(t)\big) \\ &\geq \frac{H(C)}{C^\beta} x^\beta\big(\sigma(t)\big), \; t \geq t_4. \end{aligned}$$

Therefore, (2.2) becomes

$$r(t)x'(t) \geq \int_t^{\infty} q(s)\frac{H(C)}{C^{\beta}}x^{\beta}\big(\sigma(s)\big)ds,$$

that is,

$$r\big(\sigma(t)\big)x'\big(\sigma(t)\big) \geq \left[\int_t^{\infty} q(s)x^{\beta}\big(\sigma(s)\big)ds\right]\frac{H(C)}{C^{\beta}},$$

implies that

$$\begin{aligned} x'\big(\sigma(t)\big) &\geq \frac{H(C)}{C^{\beta}r\big(\sigma(t)\big)}\left[\int_t^{\infty} q(s)ds\right]x^{\beta}\big(\sigma(t)\big) \\ &\geq \frac{H(C)}{C^{\beta}r(t)}\left[\int_t^{\infty} q(s)ds\right]x^{\beta}\big(\sigma(t)\big). \end{aligned}$$

Integrating the last inequality from t_4 to $+\infty$, we get

$$\frac{H(C)}{C^{\beta}}\int_{t_4}^{\infty}\frac{1}{r(t)}\left[\int_t^{\infty} q(s)ds\right]dt \leq \int_{t_4}^{\infty}\frac{x'\big(\sigma(t)\big)}{x^{\beta}\big(\sigma(t)\big)} < +\infty,$$

which is a contradiction to (A_6).

Next, we show that (A_6) is necessary. Assume that (A_6) fails to hold and let

$$H(C)\int_T^{\infty}\frac{1}{r(t)}\left[\int_t^{\infty} q(s)ds\right]dt \leq \frac{C}{2}, \ \ T \geq \sigma, \tag{2.4}$$

where $C > 0$ is a constant. Consider

$$\begin{aligned} M = \big\{x : x \in C([t_0, +\infty), \mathbb{R}), \, x(t) = \frac{C}{2} \ for \ t \in [t_0, T) \ and \\ \frac{C}{2} \leq x(t) \leq C, \ t \geq T\big\}, \end{aligned}$$

and let $\Phi : M \to C([t_0, +\infty), \mathbb{R})$ be defined by

$$(\Phi x)(t) = \begin{cases} \frac{C}{2}, & t \in [t_0, T) \\ \frac{C}{2} + \int_T^t \frac{1}{r(s)}\left[\int_s^{\infty} q(u)H\big(x(\sigma(u))\big)du\right]du, & t \geq T. \end{cases}$$

For every $x \in M$, $(\Phi x)(t) \geq \frac{C}{2}$. Using definition of the set M, definition of the mapping Φ and (2.4), we obtained $(\Phi x)(t) \leq C$. Therefore, $(\Phi x) \in M$. Analogously to the proof of the Theorem 2.3 we get that the mapping Φ has a fixed point $u \in M$, that is, $u(t) = (\Phi u)(t)$, $t \geq t_0$. It can be easily verified that $u(t)$ is a solution of (1.1), such that $\frac{C}{2} \leq u(t) \leq C$ for $t \geq T$, that is, $u(t)$ is a nonoscillatory solution of (1.1). Thus the proof of the theorem is complete. □

We conclude this section with the following examples to illustrate our main results:

Example 2.5. *Consider the delay differential equations*

$$\big(e^{-t}x'(t)\big)' + e^{t}x\big((t-2)\big)^{\frac{1}{3}} = 0, \tag{E_1}$$

where $r(t) = e^{-t}$, $q(t) = e^{t}$, $\sigma(t) = t-2$ *and* $H(x) = x^{\frac{1}{3}}$. *If we choose* $\beta = \frac{1}{2} < 1$, *then all the assumptions of the Theorem 2.3 holds. Hence by Theorem 2.3, every solution of* (E_1) *oscillates.*

Example 2.6. *Consider the delay differential equations*

$$\big(e^{-3t}x'(t)\big)' + e^{-2t}x\big((t-1)\big)^{3} = 0, \tag{E_2}$$

where $r(t) = e^{-3t}$, $q(t) = e^{-2t}$, $\sigma(t) = t-1$ *and* $H(x) = x^{3}$. *If we choose* $\beta = 2 > 1$, *then all the assumptions of the Theorem 2.4 holds. Hence by Theorem 2.4, every solution of* (E_2) *oscillates.*

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Acknowledgement: This work is supported by the Department of Science and Technology (DST), New Delhi, India, through the letter no. DST/INSPIRE Fellowship/2014/140, dated Sept. 15, 2014

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