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# ON THE $\mathcal{L}$-DUALITY OF A FINSLER SPACE WITH $(\alpha,\beta)$-METRIC $\frac{(\alpha+\beta)^2}{\alpha}$

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Abstract. The geometry of higher order Finsler spaces were studied in ([1], [6], [10]). The theory of higher order Lagrange and Hamilton spaces were discussed in ([8], [9], [11]). Some special problems concerning the $\mathcal{L}$-duality and classes of Finsler spaces were studied in ([3], [13]) . Various geometers such as ([2], [3], [4]) etc. have studied the $\mathcal{L}$-dual of Randers, Kropina and Matsumoto space. In this paper we have obtained the $\mathcal{L}$-dual of a special $(\alpha,\beta)$-metric $\frac{(\alpha+\beta)^2}{\alpha}$.

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## 1. Introduction

The concept of $\mathcal{L}$-duality between Lagrange and Finsler space was initiated by R. Miron [7] in 1987. Since then, many Finsler geometers have studied this topic.

One of the remarkable results obtained are the concrete $\mathcal{L}$-duals of Randers and Kropina metrics ([2], [3]). The importance of $\mathcal{L}$-duality is by far not limited to computing the dual of some Finsler fundamental functions but there are so many problems which have been solved by taking the $\mathcal{L}$-duals of Finsler spaces. The $\mathcal{L}$-duality between Finsler and Cartan spaces is used to study of the geometry of a Cartan space.

## 2. The Legendre transformation

The Finsler space $F^n = (M, F(x,y))$ is said to have an $(\alpha,\beta)$-metric if $F$ is a positively homogeneous function of degree one in two variables $\alpha = \sqrt{(a_{ij}(x)y^iy^j)}$ and $\beta = b_i(x)y^i$, where $\alpha^2 = a(y,y) = a_{ij}y^iy^j$, $y = y^i\frac{\partial}{\partial x^i}|_x \in T_xM$ is Riemannian metric, and $\beta = b_i(x)y^i$ is a 1-form on $\widetilde{TM} = TM \setminus \{0\}$.

A Finsler space with the fundamental function:

$$F(x,y) = \alpha(x,y) + \beta(x,y)$$

is called a *Randers space* [5].

A Finsler space having the fundamental function:

$$F(x,y) = \frac{\alpha^2(x,y)}{\beta(x,y)}$$

is called a *Kropina space* and one with

$$F(x,y) = \frac{\alpha^2(x,y)}{\alpha(x,y) - \beta(x,y)}$$

is called a *Matsumoto space.*

A Finsler space with the fundamental function:

$$F(x,y) = \frac{(\alpha(x,y) + \beta(x,y))^2}{\alpha(x,y)} \tag{2.1}$$

is called a *Finsler space with quadratic metric.*

**Definition 2.1.** [1]. A Cartan space $C^n$ is a pair $(M, H)$ which consists of a real $n$-dimensional $C^\infty$-manifold $M$ and a Hamiltonian function $H : T^*M \setminus \{0\} \to \Re$, where $(T^*M, \pi^*, M)$ is the cotangent bundle of $M$ such that $H(x,p)$ has the following properties:

1. It is two homogeneous with respect to $p_i \quad (i,j,k,\ldots = 1,2,\ldots,n)$.
2. The tensor field $g^{ij}(x,p) = \frac{1}{2}\frac{\partial^2 H}{\partial p_i \partial p_j}$ is nondegenerate.

Let $C^n = (M, K)$ be an $n$-dimensional Cartan space having the fundamental function $K(x,p)$. We can also consider Cartan spaces having the metric functions of the following forms

$$K(x,p) = \sqrt{a^{ij}(x)p_i p_j} + b^i(x)p_i$$

or

$$K(x,p) = \frac{a^{ij}p_i p_j}{b^i(x)p_i}$$

and we will again call these spaces Randers and Kropina spaces respectively on the cotangent bundle $T^*M$.

**Definition 2.2.** [1]. A regular Lagrangian $L(x,y)$ on a domain $D \subset TM$ is a real smooth function $L : D \to R$ and a regular Hamiltonian $H(x,p)$ on a domain $D^* \subset T^*M$ is a real smooth function $H : D^* \to R$.

Hence, the matrices with entries

$$g_{ab}(x,y) = \dot{\partial}_a \dot{\partial}_b L(x,y)$$

and

$$g^{*ab}(x,p) = \dot{\partial}^a \dot{\partial}^b H(x,p)$$

are everywhere nondegenerate on $D$ and $D^*$ respectively.

**Examples.** $(a)$ Every Finsler space $F^n = (M, F(x,y))$ is a Lagrange manifold with $L = \frac{1}{2}F^2$.

$(b)$ Every Cartan space $C^n = (M, \bar{F}(x,p))$ is a Hamilton manifold with $H = \frac{1}{2}\bar{F}^2$. (Here $\bar{F}$ is positively 1-homogeneous in $p_i$ and the tensor
$\bar{g}^{ab} = \frac{1}{2}\dot{\partial}_a\dot{\partial}_b\bar{F}^2$ is nondegenerate).

$(c)$ $(M, L)$ and $(M, H)$ with

$$L(x,y) = \frac{1}{2}a_{ij}(x)y^i y^j + b_i(x)y^i + c(x)$$



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and

$$H(x,p) = \frac{1}{2}\bar{a}^{ij}(x)p_ip_j + \bar{b}^i(x)p_i + \bar{c}(x)$$

are Lagrange and Hamilton manifolds respectively. (Here $a_{ij}(x), \bar{a}^{ij}$ are the fundamental tensors of Riemannian manifold, $b_i$ are components of covector field, $\bar{b}^i$ are the components of a vector field, $C$ and $\bar{C}$ are the smooth functions on $M$).

Let $L(x,y)$ be a regular Lagrangian on a domain $D \subset TM$ and let $H(x,p)$ be a regular Hamiltonian on a domain $D^* \subset T^*M$. If $L \in F(D)$ is a differential map, we can consider the fiber derivative of $L$, locally given by the diffeomorphism between the open set $U \subset D$ and $U^* \subset D^*$

$$\psi(x,y) = (x^i, \dot{\partial}_a L(x,y)),$$

which will be called the *Legendre transformation*.

It is easily seen that $L$ is a regular Lagrangian if and only if $\psi$ is a local diffeomorphism.

In the same manner if $H \in F(D^*)$ the fiber derivative is given locally by

$$\varphi(x,y) = (x^i, \dot{\partial}^a H(x,y)),$$

which is a local diffeomorphism if and only if $H$ is regular.

Let us consider a regular Lagrangian $L$. Then $\psi$ is a diffeomorphism between the open sets $U \subset D$ and $U^* \subset D^*$. We can define in this case the function:

$$H : U^* \to R,\ H(x,y) = p_a y^a - L(x,y), \tag{2.2}$$

where $y = (y^a)$ is the solution of the equations $y_a = \dot{\partial}_a L(x,y)$.

Also, if $H$ is a regular Hamiltonian on $M$, $\psi$ is a diffeomorphism between same open sets $U^* \subset D^*$ and $U \subset D$ and we can consider the function

$$L : U \to R,\ L(x,y) = p_a y^a - H(x,p), \tag{2.3}$$

$y = (y_a)$ is the solution of the equations

$$y^a = \dot{\partial}^a H(x,p).$$

The Hamiltonian given by (2.2) is the *Legendre transformation of the Lagrangian* $L$ and the Lagrangian given by (2.3) is called the *Legendre transformation of the Hamiltonian* H.

If $(M,K)$ is a Cartan space, then $(M,H)$ is a Hamilton manifold ([9], [12]), where $H(x,p) = \frac{1}{2}K^2(x,p)$ is 2-homogenous on a domain of $T^*M$. So we get the following transformation of $H$ on $U$:

$$L(x,y) = p_a y^a - H(x,p) = H(x,p). \tag{2.4}$$

**Theorem 2.3.** [12] *The scalar field given by* (2.4) *is a positively* 2*-homogeneous regular Lagrangian on* $U$.

*Therefore, we get Finsler metric* $F$ *of* $U$*, so that*

$$L = \frac{1}{2}F^2.$$

Thus for the Cartan space $(M,K)$ we always can locally associate a Finsler space $(M,F)$ which will be called the $\mathcal{L}$-dual of a Cartan space $(M, C_{|U^*})$ vice versa, we can associate, locally, a Cartan space to every Finsler space which will be called the $\mathcal{L}$ -dual of a Finsler space $(M, F_{|U})$.



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## 3. The $\mathcal{L}$-dual of a special Finsler space with metric $\frac{(\alpha+\beta)^2}{\alpha}$

In this case we put $\alpha^2 = y_i y^i,\ b^i = a^{ij} b_j,\ \beta = b_i y^i,\ \beta^* = b^i p_i,\ p^i = a^{ij} p_j,\ \alpha^{*2} = p_i p^i = a^{ij} p_i p_j$. we have $F = \frac{(\alpha+\beta)^2}{\alpha}$ and

$$p_i = \frac{1}{2}\dot{\partial}_i F^2 = \left[\frac{\partial}{\partial y^i}\left(\frac{(\alpha+\beta)^2}{\alpha}\right)\right] F,$$

$$p_i = \frac{(\alpha+\beta)}{\alpha^2}\left[(1-\frac{\beta}{\alpha})y_i + 2\alpha b_i\right] F. \tag{3.1}$$

Contracting (3.1) with $p^i$ and $b^i$ respectively, we get

$$\alpha^{*2} = \frac{(\alpha+\beta)}{\alpha^2}\left[(1-\frac{\beta}{\alpha})F^2 + 2\alpha\beta^*\right] F. \tag{3.2}$$

and

$$\beta^* = \frac{(\alpha+\beta)}{\alpha^2}\left[(1-\frac{\beta}{\alpha})\beta + 2\alpha b^2\right] F. \tag{3.3}$$

In [13], for a Finsler $(\alpha,\beta)$-metric $F$ on a Manifold $M$, one constructs a positive function $\phi = \phi(s)$ on $(-b_0; b_0)$ with $\phi(0) = 1$ and $F = \alpha\phi(s),\ \ s = \frac{\beta}{\alpha}$ , where $\alpha = \sqrt{a_{ij} y^i y^j}$ and $\beta = b_i y^i$ with $||\beta||_x < b_0, \forall x \in M$.

The function $\phi$ satisfies $\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0, (|s| \leq b_0)$.

This mertic is a $(\alpha,\beta)$-metric with $\phi = (1+s)^2$.

Using Shen's notation [14], $s = \frac{\beta}{\alpha}$ in (3.2) and (3.3), we get

$$\alpha^{*2} = \left[(1-s^2)(1+s)^2 F + 2(1+s)\beta^*\right] F, \tag{3.4}$$

and

$$\beta^* = \left[(1-s^2)s + 2(1+s)b^2\right] F, \tag{3.5}$$

Putting $1+s = t$, i.e $s = (t-1)$ in equations (3.4) and (3.5), we get

$$\alpha^{*2} = t[(2-t)t^2 F + 2\beta^*]F, \tag{3.6}$$

and

$$\beta^* = t[(2-t)(t-1) + 2b^2]F. \tag{3.7}$$

We consider two cases:
(1) $b^2 = 1$.
(2) $b^2 \neq 1$.

**Case 1.** For $b^2 = 1$ from (3.7), we get

$$\beta^* = t[(2-t)(t-1) + 2]F,$$

or

$$F = \frac{\beta^*}{t^2(3-t)}. \tag{3.8}$$

and by substitution of $F$ in (3.6), after some computations we get a cubic equation

$$t^3 - 6t^2 + 3t(3+k) - 8k = 0,$$

where

$$k = \frac{\beta^{*2}}{\alpha^{*2}}.$$

Using Mathematica for solving the above cubic equation, we get following roots of the above equation

$$t = 2 - \frac{-9+9k}{9(-1+k+\sqrt{k-2k^2+k^3})^{\frac{1}{3}}} + (-1+k+\sqrt{k-2k^2+k^3})^{\frac{1}{3}},$$

$$t = 2 + \frac{(1\pm i\sqrt{3})(-9+9k)}{18(-1+k+\sqrt{k-2k^2+k^3})^{\frac{1}{3}}} - \frac{1}{2}(1\pm i\sqrt{3})(-1+k+\sqrt{k-2k^2+k^3})^{\frac{1}{3}},$$

As our Finsler fundamental function is real, the dual Hamilton function is also real. So we choose real root.

$$t = 2 - \frac{-9+9k}{9(-1+k+\sqrt{k-2k^2+k^3})^{\frac{1}{3}}} + (-1+k+\sqrt{k-2k^2+k^3})^{\frac{1}{3}},$$

or

$$t = 2 - \frac{l}{(l+m)^{\frac{1}{3}}} + (l+m)^{\frac{1}{3}}, \tag{3.9}$$

where

$$l = -1+k = \frac{-\alpha^{*2}+\beta^{*2}}{\alpha^{*2}}, m = \sqrt{k-2k^2+k^3} = \frac{\beta^*(\alpha^{*2}-\beta^{*2})}{\alpha^{*3}}. \tag{3.10}$$

From (3.8) and (3.9), we get

$$F = \frac{\beta^*}{\left(1+\frac{l}{(l+m)^{\frac{1}{3}}}-(l+m)^{\frac{1}{3}}\right)\left(2-\frac{l}{(l+m)^{\frac{1}{3}}}+(l+m)^{\frac{1}{3}}\right)^2}$$

As we know that $H(x,p) = \frac{1}{2}F^2$ , hence we get

$$H(x,p) = \frac{\beta^{*2}}{2\left(1+\frac{l}{(l+m)^{\frac{1}{3}}}-(l+m)^{\frac{1}{3}}\right)^2\left(2-\frac{l}{(l+m)^{\frac{1}{3}}}+(l+m)^{\frac{1}{3}}\right)^4}, \tag{3.11}$$

Putting $\beta^* = b^i p_i$, in (3.11), we get

$$H(x,p) = \frac{(b^i p_i)^2}{2\left(1+\frac{l}{(l+m)^{\frac{1}{3}}}-(l+m)^{\frac{1}{3}}\right)^2\left(2-\frac{l}{(l+m)^{\frac{1}{3}}}+(l+m)^{\frac{1}{3}}\right)^4}. \tag{3.12}$$

**Case 2.** Next, we find $H(x,p)$ for $b^2 \neq 1$. From (3.7), we have

$$F = \frac{\beta^*}{t[(2-t)(t-1)+2b^2]}. \tag{3.13}$$

Using (3.13) in (3.6), we get

$$t^4 - 6t^3 + [9-4(b^2-1)+3k]t^2 + [12(b^2-1)-8k]t + [4(b^2-1)^2 - 4k(b^2-1)] = 0$$

Using Mathematica for solving the above quartic equation, we get four real roots, given in the following equation. As our Finsler fundamental function is real, the dual Hamilton function is also real.

$$t = \left(\frac{3}{2} \pm \delta_i\right), \ (i=1,2) \tag{3.14}$$

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where

$$
\begin{aligned}
\delta_1 &= \frac{a_1}{2} + \frac{1}{2}\sqrt{a_2 - \frac{a_3}{4a_1}},\\
\delta_2 &= \frac{a_1}{2} - \frac{1}{2}\sqrt{a_2 - \frac{a_3}{4a_1}},\\
a_1 &= \sqrt{\left[-4 + 4b^2 - 3k + \frac{1}{3}b_1 + \frac{2^{\frac{1}{3}}b_2}{[3(b_3+\sqrt{b_4})^{\frac{1}{3}}]} + \frac{[b_3+\sqrt{b_4}]^{\frac{1}{3}}}{(3.2^{\frac{1}{3}})}\right]},\\
a_2 &= \left[5 + 4b^2 + \frac{b_1}{3} - 3k - \frac{2^{\frac{1}{3}}b_2}{[3(b_3+\sqrt{b_4})^{\frac{1}{3}}]} - \frac{[b_3+\sqrt{b_4}]^{\frac{1}{3}}}{(3.2^{\frac{1}{3}})}\right],\\
a_3 &= (216 - 32b_5 - 24b_1),\\
b_1 &= (13 - 4b^2 + 3k),\\
b_2 &= (1 + 16b^2 + 64b^4 - 18k - 72b^2k + 9k^2),\\
b_3 &= (2 + 48b^2 + 384b^4 + 1024b^6 - 54k - 648b^2k - 1728b^4k + 270k^2\\
&\quad + 648b^2k^2 + 54k^3),\\
b_4 &= (1728b^2k^2 + 25920b^4k^2 + 82944b^6k^2 - 110592b^8k^2 - 1728k^3\\
&\quad - 57024b^2k^3 - 114048b^4k^3 + 359424b^6k^3 + 31104k^4 - 15552b^2k^4\\
&\quad - 388800b^4k^4 + 46656k^5 + 139968b^2k^5),\\
b_5 &= (-3 + 3b^2 - 2k).
\end{aligned}
$$

putting the value of t in (3.13), we get

$$F = \frac{\beta^*}{\left(\frac{3}{2} \pm \delta_i\right)\left[\left(\frac{1}{2} \pm \delta_i\right)^2 + 2b^2\right]},$$

Hence $H(x,p) = \frac{1}{2}F^2$ is given by

$$H(x,p) = \frac{\beta^{*2}}{2\left(\frac{3}{2} \pm \delta_i\right)^2\left[\left(\frac{1}{2} \pm \delta_i\right)^2 + 2b^2\right]^2}, \tag{3.15}$$

Putting $\beta^* = b^i p_i$, in (3.14), we get

$$H(x,p) = \frac{(b^i p_i)^2}{2\left(\frac{3}{2} \pm \delta_i\right)^2\left[\left(\frac{1}{2} \pm \delta_i\right)^2 + 2b^2\right]^2}, \tag{3.16}$$

Hence we have the following:

**Theorem 3.1.** *Let $(M, F = (\alpha+\beta)^2/\alpha)$ be a special Finsler space, where $\alpha^2 = a(y,y) = a_{ij}(x)y^iy^j$ is Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on $TM \setminus \{0\}$. Then, the $\mathcal{L}$-dual of $(M, F = (\alpha+\beta)^2/\alpha)$ is the space having the fundamental function on $T^*M$:*
*(1) If $b^2 = 1$, it is given by (3.12).*
*(2) If $b^2 \neq 1$, it is given by (3.16). where $b^2$ is the Riemannian length of $b_i$.*

# References

[1] P. L. Antonelli, *Hand Book of Finsler Geometry*, Kluwer Acad. Publ. FTPH, **58**, 1993.
[2] D. Hrimiuc and H. Shimada, On the $\mathcal{L}$-duality between Lagrange and Hamilton manifolds, *Nonlinear World*, **3**(1996), 613-641.
[3] D. Hrimiuc and H. Shimada, On some special problems concerning the $\mathcal{L}$-duality between Finsler and Cartan spaces, *Tensor, N. S.,* **58**(1997), 48-61.
[4] I. M. Masca, S. V. Sabau and H. Shimada, The $\mathcal{L}$-dual of a Matsumoto space, *Publ. Math. Debrecen,* **72**(2008), 227-242.
[5] M. Matsumoto, *Foundations of Finsler Geometry and Special Finsler Spaces*, Kaisheisha Press, Otsu, Japan, 1986.
[6] M. Matsumoto, *Finsler geometry in the 20th-century*, In: "Handbook of Finsler Geometry", Vol.I, II (Ed. by P. L. Antonelli), Kluwer Academic Publisher, Dordrecht/Boston/London, 2003, 557-966.
[7] R. Miron, Cartan spaces in a new point of view by considering them as duals of Finsler spaces, *Tensor, N. S.,* **46**(1987), 330-334.
[8] R, Miron, *The Geometry of Higher-order Lagrange Spaces*, Applications to Mechanics and Physics, Kluwer Acad. Publ. FTPH, **82**(1997).
[9] R. Miron, T*he Geometry of Higher-order Hamilton Spaces*, Applications to Hamiltonian Mechanics, Kluwer Acad. Publ. FTPH, **132** (2003).
[10] R. Miron, *The Geometry of Higher-order Finsler Spaces*, Applications to Hamiltonian Mechanics, Hadronic Press., Inc., USA, 1998.
[11] R. Miron and M. Anastasiei, *The Geometry of Lagrange Spaces*, Theory and Applications, Kluwer Acad. Publ. FTPH, **59** (1994).
[12] R. Miron, D. Hrimiuc, H. Shimada and S.V. Sabau, *The Geometry of Hamilton and Lagrange Spaces*, Kluwer Acad. Publ. FTPH, **118** (2001).
[13] S. V. Sabau, and H. Shimada, Classes of Finsler spaces with $(\alpha, \beta)$-metrics, *Rep. on Math. Phys.,* **47**(2001), 31-48.
[14] Z. Shen, On Landsberg $(\alpha, \beta)$-metrics, *http://www.math.iupiu.edu/∼ zshen/Research/papers*, 2006.

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