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A SPECIAL FAMILY OF SYMMETRIC SEMI-CLASSICAL FORMS OF CLASS THREE

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Abstract. A regular form (linear functional) u is said to be semi-classical, if it satisfies the distributional equation $(\Phi u)^{'} + \Psi u = 0$. Recently, all the symmetric semi-classical forms of class $s \leq 1$ and all the symmetric semi-classical forms of class $s = 2$ when $\Phi(0) = 0$ are determined. In this paper, by means of the quadratic decomposition, we carry out the complete description of the symmetric semi-classical forms of class $s = 3$, when $\Psi(0) = 0$. Essentially, four canonical cases appear.

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1. Introduction

Semi-classical orthogonal polynomials were introduced in [27]. They are a natural generalization of the classical polynomials. Maroni [19, 21] has worked on the linear form of moments and has given a unified theory of this kind of polynomials. A semi-classical form u satisfies the distributional equation $(\Phi u)^{'} + \Psi u = 0$ where $\Phi(x)$ is a monic polynomial and $\Psi(x)$ is a polynomial with $\deg(\Psi) \geq 1$. Since the system of Laguerre- Freud equations corresponding to the problem of determining all the semi-classical forms of class $s \geq 1$ becomes non-linear, the problem was only solved when $s = 1$ and for the symmetric case [2]. Thus, several authors use different processes in order to obtain semi-classical forms of class $s \geq 1$. For instance, we can mention the addition of either a Dirac mass or its derivative to semi-classical forms [3, 12, 15], the product and the division of a form by a polynomial [1, 6, 11, 17, 22, 23, 25, 26]. So, some examples of semi-classical forms are given in terms of classical ones. But, they are just few examples. The aim of this work is to approach the problem of determining all the symmetric semi-classical forms of class $s = 3$ when $\Psi(0) = 0$. The second section is devoted to the preliminary results and notations used in the sequel. In the third section, we find a relation between the symmetric semi-classical forms of class $s = 3$ and the classical forms (see theorem 3.2). Using this relation, we give, in Section 3, all the forms which we look for. Four canonical cases for the polynomial Φ arise: $\Phi(x) = x$, $\Phi(x) = x(x^2 - 1)$, $\Phi(x) = x(x^2 - 1)^2$ and $\Phi(x) = x(x^4 - 1)$. As it turned out, we obtained explicitly four nonsymmetric semi-classical forms of class $s = 1$.

2. Notations and preliminary results

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle v, f\rangle$ the action of $v \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(v)_n := \langle v, x^n\rangle , n \geq 0$, the moments of v. For any form v and any polynomial h let $Dv = v'$, hv, δ_0 and $x^{-1}v$ be the forms defined by: $\langle v', f\rangle := -\langle v, f'\rangle$, $\langle hv, f\rangle := \langle v, hf\rangle$, $\langle \delta_0, f\rangle := f(0)$, and $\langle x^{-1}v, f\rangle := \langle v, \theta_0 f\rangle$ where $\big(\theta_0 f\big)(x) = \dfrac{f(x)-f(0)}{x}$, $f \in \mathcal{P}$.

Then, it is straightforward to prove that for $v \in \mathcal{P}'$, we have

$$x^{-1}(xv) = v - (v)_0\delta_0 \ . \tag{2.1}$$

Let us define the operator $\sigma : \mathcal{P} \longrightarrow \mathcal{P}$ such that $(\sigma f)(x) = f(x^2)$. Then, we define the even part σv by $\langle \sigma v, f\rangle := \langle v, \sigma f\rangle$. Therefore, we have [9, 20]

$$f(x)(\sigma v) = \sigma(f(x^2)v) \ , \tag{2.2}$$

$$\sigma v^{'} = 2(\sigma x v)^{'} \ . \tag{2.3}$$

A form v is called regular if there exists a sequence of polynomials $\{S_n\}_{n\geq 0}$ ($\deg S_n \leq n$) such that $\langle v, S_nS_m\rangle = r_n\delta_{n,m}$, $r_n \neq 0$, $n \geq 0$. Then, $\deg S_n = n, n \geq 0$ and we can always suppose each S_n is monic. In such a case, the sequence $\{S_n\}_{n\geq 0}$ is unique. It is said to be the sequence of monic orthogonal polynomials with respect to v. In the sequel it will be denoted as MOPS. It is a very well known fact that the sequence $\{S_n\}_{n\geq 0}$ satisfies the recurrence relation (see, for instance, the monograph by Chihara [9])

$$\begin{aligned} &S_{n+2}(x) = (x - \xi_{n+1})S_{n+1}(x) - \rho_{n+1}S_n(x) \ , \quad n \geq 0 \ , \\ &S_1(x) = x - \xi_0 \ , \qquad S_0(x) = 1 \ . \end{aligned} \tag{2.4}$$

with $\big(\xi_n, \rho_{n+1}\big) \in \mathbb{C} \times (\mathbb{C} - \{0\})$, $n \geq 0$. By convention we set $\rho_0 = (v)_0 = 1$.

In this case, let $\{S_n^{(1)}\}_{n\geq 0}$ be the sequence of associated polynomials of first kind for the sequence $\{S_n\}_{n\geq 0}$ satisfying the three-term recurrence relation

$$\begin{aligned} &S_{n+2}^{(1)}(x) = (x - \xi_{n+2})S_{n+1}^{(1)}(x) - \rho_{n+2}S_n^{(1)}(x) \ , \quad n \geq 0 \ , \\ &S_1^{(1)}(x) = x - \xi_1, \qquad S_0^{(1)}(x) = 1 \ , \quad \big(S_{-1}^{(1)}(x) = 0\big) \ . \end{aligned} \tag{2.5}$$

Another important representation of $S_n^{(1)}(x)$ is, (see [10])

$$S_n^{(1)}(x) := \left\langle v, \frac{S_{n+1}(x) - S_{n+1}(\zeta)}{x - \zeta} \right\rangle = (v\theta_0 S_{n+1})(x) \ , \tag{2.6}$$

where the right-multiplication of a form by a polynomial is defined by

$$(vf)(x) := \left\langle v, \frac{xf(x) - \zeta f(\zeta)}{x - \zeta} \right\rangle = \sum_{m=0}^{n} \left(\sum_{j=m}^{n} a_j (v)_{j-m} \right) x^m \ , \quad f(x) = \sum_{j=0}^{n} a_j x^j \ . \tag{2.7}$$

Also, let $\{S_n(.,\mu)\}_{n\geq 0}$ be co-recursive polynomials for the sequence $\{S_n\}_{n\geq 0}$ satisfying [9]

$$S_n(x,\mu) = S_n(x) - \mu S_{n-1}^{(1)}(x) \ , \quad n \geq 0. \tag{2.8}$$

We recall that a form v is called symmetric if $(v)_{2n+1} = 0, n \geq 0$. The conditions $(v)_{2n+1} = 0, n \geq 0$ are equivalent to the fact that the corresponding MOPS $\{S_n\}_{n\geq 0}$ satisfies the recurrence relation (2.4) with $\xi_n = 0, n \geq 0$ [9].

Proposition 2.1. [9, 20] *If the form v is symmetric, then v is regular if and only if σv and $x\sigma v$ are both regular.*

Let v be a regular, normalized form (i.e. $(v)_0 = 1$) and $\{S_n\}_{n\geq 0}$ be the corresponding MOPS. For a $\lambda \in \mathbb{C} - \{0\}$, we can define a new symmetric form u as following:

$$x\sigma u = -\lambda v \ , \quad \sigma x u = 0 \ , \quad (u)_0 = 1 \ . \tag{2.9}$$

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From (2.1), we have

$$\sigma u = -\lambda x^{-1} v + \delta_0 \ . \tag{2.10}$$

When u is regular let $\{Z_n\}_{n\geq 0}$ be its corresponding sequence of polynomials satisfying the recurrence relation

$$\begin{aligned} &Z_{n+2}(x) = xZ_{n+1}(x) - \gamma_{n+1} Z_n(x) \ , \quad n \geq 0 \ , \\ &Z_1(x) = x \ , \quad Z_0(x) = 1 \ . \end{aligned} \tag{2.11}$$

Let us consider the quadratic decomposition of $\{Z_n\}_{n\geq 0}$ (see [19])

$$Z_{2n}(x) = P_n(x^2) \ , \quad Z_{2n+1}(x) = xR_n(x^2) \ . \tag{2.12}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are, respectively, orthogonal with respect to σu and $x\sigma u$. For instance, we have

$$\begin{aligned} &P_{n+2}(x) = (x - \beta^P_{n+1})P_{n+1}(x) - \gamma^P_{n+1} P_n(x) \ , \quad n \geq 0 \ , \\ &P_1(x) = x - \beta^P_0 \ , \quad P_0(x) = 1 \ , \end{aligned} \tag{2.13}$$

with

$$\beta^P_0 = \gamma_1, \quad \beta^P_{n+1} = \gamma_{2n+2} + \gamma_{2n+3}, \quad \gamma^P_{n+1} = \gamma_{2n+1}\gamma_{2n+2}, \quad n \geq 0. \tag{2.14}$$

From (2.9), we get

$$R_n(x) = S_n(x) \ , \quad n \geq 0 \ . \tag{2.15}$$

Then, we deduce the following result. The proof can be found in [14].

Proposition 2.2. *The form u defined by (2.9) is regular if and only if $S_n(0, \lambda) \neq 0$, $n \geq 0$. In this case, the corresponding MOPS $\{Z_n\}_{n\geq 0}$ satisfies (2.11) with*

$$\gamma_1 = -\lambda \ , \quad \gamma_{2n+2} = a_n \ , \quad \gamma_{2n+3} = \frac{\rho_{n+1}}{a_n} \ , \quad n \geq 0 \ , \tag{2.16}$$

where

$$a_n = -\frac{S_{n+1}(0, \lambda)}{S_n(0, \lambda)} \ , \quad n \geq 0 \ . \tag{2.17}$$

Remarks 2.3. (1). *u is regular if and only if $\lambda \neq \lambda_n$, $n \geq 0$ where*

$$\lambda_0 = 0 \ , \ \lambda_n = \frac{S_n(0)}{S^{(1)}_{n-1}(0)} \ , \ n \geq 1 \ .$$

(2). *If w is the symmetrized form associated with the form v (i.e. $(w)_{2n} = (v)_n$ and $(w)_{2n+1} = 0$, $n \geq 0$), then (2.9) is equivalent to $x^2 u = -\lambda w$. Notice that w is not necessarily a regular form in the problem under study. In* [1, 6], *the authors have solved it only when w is regular.*

Corollary 2.4. [24] *When the form v is symmetric, then u is regular for every $\lambda \neq 0$. Moreover,*

$$\begin{cases} \gamma_1 = -\gamma_2 = -\lambda \ , \\ \gamma_{4n+3} = -\gamma_{4n+4} = -\frac{1}{\lambda} \displaystyle\prod_{k=0}^{n} \frac{\rho_{2k+1}}{\rho_{2k}} \ , \\ \gamma_{4n+5} = -\gamma_{4n+6} = \lambda \rho_{2n+2} \displaystyle\prod_{k=0}^{n} \frac{\rho_{2k}}{\rho_{2k+1}} \ , \ n \geq 0 \ . \end{cases} \tag{2.18}$$

Proposition 2.5. *If we We suppose that the form v has the following integral representation:*

$$\langle v, f \rangle = \int_{-\infty}^{+\infty} V(x) f(x) dx \ , \ f \in \mathcal{P}, \ \textit{with} \ (v)_0 = \int_{-\infty}^{+\infty} V(x) dx = 1 \ ,$$

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where V is a locally integrable function with rapid decay and continuous at the point $x=0$, then

$$\langle \sigma u, f(x)\rangle = f(0)\Big\{1+\lambda P\int_{-\infty}^{+\infty}\frac{V(x)}{x}dx\Big\} - \lambda P\int_{-\infty}^{+\infty}\frac{V(x)}{x}f(x)dx\ , \tag{2.19}$$

$$\begin{aligned}\langle u, f(x)\rangle &= f(0)\Big\{1+\lambda P\int_{-\infty}^{+\infty}\frac{V(x)}{x}dx\Big\}+\\ &+\lambda P\int_{-\infty}^{+\infty}\frac{V(-x^2)}{|x|}f(ix)dx - \lambda P\int_{-\infty}^{+\infty}\frac{V(x^2)}{|x|}f(x)dx\ ,\end{aligned} \tag{2.20}$$

where

$$P\int_{-\infty}^{+\infty}\frac{V(x)}{x}f(x) = \lim_{\epsilon\to 0}\Big\{\int_{-\infty}^{-\varepsilon}\frac{V(x)}{x}f(x)dx + \int_{\epsilon}^{+\infty}\frac{V(x)}{x}f(x)dx\Big\}\ .$$

Proof. From (2.10), we can easily deduce (2.19).
Now, we decompose the polynomial f as follows

$$f(x) = f^e(x^2) + xf^o(x^2)\ .$$

Because u is symmetric, we get $\langle u, f(x)\rangle = \langle u, f^e(x^2)\rangle = \langle \sigma u, f^e(x)\rangle$.
Using (2.19) and taking into account that $f^e(0) = f(0)$, we obtain

$$\langle u, f\rangle = f(0)\Big\{1+\lambda P\int_{-\infty}^{+\infty}\frac{V(x)}{x}dx\Big\} - \lambda P\int_{-\infty}^{+\infty}\frac{V(x)}{x}f^e(x)dx\ .$$

Using the fact that $f^e(x) = \frac{f(\sqrt{x})+f(-\sqrt{x})}{2}$ and $f^e(-x) = \frac{f(i\sqrt{x})+f(-i\sqrt{x})}{2}$ for $x\geq 0$ and making the change of variables $t=\sqrt{x}$, we get the desired result (2.20). □

Definition 2.6. [21] *The regular form v is called a semi-classical form if exist two polynomials $\tilde{\Phi}$ and $\tilde{\Psi}$ such that*

$$(\tilde{\Phi}v)' + \tilde{\Psi}v = 0\ ,\quad deg(\tilde{\Psi})\geq 1\ ,\quad \tilde{\Phi}\ monic\ . \tag{2.21}$$

The corresponding MOPS $\{S_n\}_{n\geq 0}$ is called semi-classical.

Remarks 2.7. (1). *The semi-classical character is invariant by shifting. Indeed, the shifted linear form $\hat{v} = (h_{a^{-1}} o \tau_{-b})v\ ,\ \ a\in\mathbb{C}-\{0\}\ ,\ \ b\in\mathbb{C}$ satisfies*

$$(\hat{\Phi}\hat{v})' + \hat{\Psi}\hat{v} = 0\ , \tag{2.22}$$

with

$$\hat{\tilde{\Phi}}(x) = a^{-t}\tilde{\Phi}(ax+b)\ ,\quad \hat{\tilde{\Psi}}(x) = a^{1-t}\tilde{\Psi}(ax+b)\ ,\quad t=\deg(\tilde{\Phi})\ . \tag{2.23}$$

Where the linear forms $\tau_{-b}v$ $\big($ translation of v $\big)$ and $h_a v$ $\big($ dilatation of u $\big)$ are defined by

$$\langle \tau_b v, f\rangle := \langle v, \tau_{-b}f\rangle := \langle v, f(x+b)\rangle\ ,\quad \langle h_a v, f\rangle := \langle v, h_a f\rangle := \langle v, f(ax)\rangle\ ,\quad f\in\mathcal{P}\ .$$

The sequence $\{\hat{S}_n(x) = a^{-n}S_n(ax+b)\}_{n\geq 0}$ is orthogonal with respect to $\hat{v}$ and fulfils (2.4) with

$$\hat{\xi}_n = \frac{\xi_n - b}{a}\ ,\quad \hat{\rho}_{n+1} = \frac{\rho_{n+1}}{a^2}\ ,\quad n\geq 0\ . \tag{2.24}$$

(2). *The semi-classical form v satisfying (2.21) is of class $\tilde{s} = \max(\deg(\tilde{\Psi})-1, \deg(\tilde{\Phi})-2)$ if and only if*

$$\prod_{c\in\mathcal{Z}}\Big\{\big|\tilde{\Phi}'(c)+\tilde{\Psi}(c)\big| + \big|\langle v, \theta_c^2\tilde{\Phi}+\theta_c\tilde{\Psi}\rangle\big|\Big\}\neq 0\ , \tag{2.25}$$

where $\mathcal{Z}$ denotes the set of zeros of $\tilde{\Phi}$ [21].

(3). *When $\tilde{s}=0$, the form v is usually called classical (Hermite, Laguerre, Bessel and Jacobi)* [18].

We can state characterizations of semi-classical orthogonal sequences. $\{S_n\}_{n\geq 0}$ is semi-classical of class $\tilde{s}$ if and only if one of the following statements holds:

(a) The formal Stieltjes function of v, namely

$$S(v)(z) = -\sum_{n\geq 0} \frac{(v)_n}{z^{n+1}} \tag{2.26}$$

satisfies a linear non-homogeneous first order differential equation [7, 21]

$$\tilde{\Phi}(z)S'(v)(z) = \tilde{C}_0(z)S(v)(z) + \tilde{D}_0(z)\ , \tag{2.27}$$

where

$$\tilde{C}_0(x) = -\tilde{\Phi}'(x) - \tilde{\Psi}(x)\ , \tag{2.28}$$

and

$$\tilde{D}_0(x) = -(v\theta_0\tilde{\Phi})'(x) - (v\theta_0\tilde{\Psi})(x)\ , \tag{2.29}$$

with $\tilde{\Phi}$ and $\tilde{\Psi}$ are the same polynomials as in (2.21).

(b) $\{S_n\}_{n\geq 0}$ fulfills the following structure recurrence relation (written in a compact form):

$$\tilde{\Phi}(x)S'_{n+1}(x) = \frac{\tilde{C}_{n+1}(x)-\tilde{C}_0(x)}{2}S_{n+1}(x) - \rho_{n+1}\tilde{D}_{n+1}(x)S_n(x)\ ,\ n\geq 0\ , \tag{2.30}$$

where (for $n\geq 0$)

$$\begin{cases} \tilde{C}_{n+1}(x) = -\tilde{C}_n(x) + 2(x-\xi_n)\tilde{D}_n(x)\ , \\ \rho_{n+1}\tilde{D}_{n+1}(x) = -\tilde{\Phi}(x) + \rho_n\tilde{D}_{n-1}(x) - (x-\xi_n)\tilde{C}_n(x) + \\ \qquad +(x-\xi_n)^2\tilde{D}_n(x)\ , \end{cases} \tag{2.31}$$

$\tilde{\Phi}$, $\tilde{\Psi}$, $\tilde{C}_0$ and $\tilde{D}_0$ are the same polynomials introduced in (a); ξ_n , ρ_n are the coefficients of the three term recurrence relation (2.4). Notice that $\tilde{D}_{-1}(x)=0$, $\deg(\tilde{C}_n)\leq \tilde{s}+1$ and $\deg(\tilde{D}_n)\leq \tilde{s}$, $n\geq 0$. [21]

In the sequel the form v will be supposed semi-classical of class $\tilde{s}$ satisfies (2.21) and (2.25).

Proposition 2.8. [24] *If v is a semi-classical form and satisfies (2.21), then for every $\lambda\in\mathbb{C}-\{0\}$ such that $S_n(0,\lambda)\neq 0, n\geq 0$, the form u defined by (2.9) is regular and semi-classical. It satisfies*

$$(\Phi u)' + \Psi u = 0\ , \tag{2.32}$$

where

$$\Phi(x) = x\tilde{\Phi}(x^2)\ ,\quad \Psi(x) = 2x^2\tilde{\Psi}(x^2)\ , \tag{2.33}$$

and u is of class $s\leq 2\tilde{s}+3$.

Proposition 2.9. [24] *The class of u depends only on the zero $x=0$ of Φ.*

According to proposition 2.8, the form u is also semi-classical and its MOPS $\{Z_n\}_{n\geq 0}$ satisfies a structure relation. In general, $\{Z_n\}_{n\geq 0}$ fulfils

$$\Phi(x)Z'_{n+1}(x) = \frac{C_{n+1}(x)-C_0(x)}{2}Z_{n+1}(x) - \gamma_{n+1}D_{n+1}(x)Z_n(x)\ \ , n\geq 0, \tag{2.34}$$

with

$$\begin{cases} \gamma_{n+1}D_{n+1}(x) = -\Phi(x) + \gamma_n D_{n-1}(x) - xC_n(x) + x^2D_n(x), \\ C_{n+1}(x) = -C_n(x) + 2xD_n(x), \quad , n\geq 0, \end{cases} \tag{2.35}$$

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where

$$\begin{cases} C_0(x) = -\Phi'(x) - \Psi(x) = -\tilde{\Phi}(x^2) + 2x^2\tilde{C}_0(x^2) \ , \\ D_0(x) = -(u\theta_0\Phi)'(x) - (u\theta_0\Psi)(x) = -2\lambda x\tilde{D}_0(x^2) + 2x\tilde{C}_0(x^2) \ . \end{cases} \tag{2.36}$$

We are going to establish the expression of C_n and D_n , $n \geq 0$ in terms of those of the sequence $\{S_n\}_{n\geq 0}$.

Proposition 2.10. [24] *The sequence $\{Z_n\}_{n\geq 0}$ fulfils (2.30) with (for $n \geq 0$)*

$$\begin{cases} C_{2n+1}(x) = \tilde{\Phi}(x^2) + 2x^2\tilde{C}_n(x^2) + 4\gamma_{2n+1}x^2\tilde{D}_n(x^2) \ , \\ D_{2n+1}(x) = 2x^3\tilde{D}_n(x^2) \ . \end{cases} \tag{2.37}$$

$$\begin{cases} C_{2n+2}(x) = -\tilde{\Phi}(x^2) + 2x^2\tilde{C}_{n+1}(x^2) + 4x^2\gamma_{2n+2}\tilde{D}_n(x^2) \ , \\ D_{2n+2}(x) = 2x\gamma_{2n+2}\tilde{D}_n(x^2) + 2x\gamma_{2n+3}\tilde{D}_{n+1}(x^2) + 2x\tilde{C}_{n+1}(x^2) \ . \end{cases} \tag{2.38}$$

$C_0(x)$ and $D_0(x)$ are given by (2.36) and γ_{n+1} by (2.16).

Proposition 2.11. *We have*

$$2x^2\Phi(x)P'_{n+1}(x^2) = \Big(\tfrac{x}{2}\big(C_{2n+2}(x) - C_0(x)\big) - \gamma_{2n+2}D_{2n+2}(x)\Big)P_{n+1}(x^2) -$$

$$-\gamma^P_{n+1}D_{2n+2}(x)P_n(x^2) \ , \quad n \geq 0 \ . \tag{2.39}$$

Proof. In the relation (2.34), replace n by $2n+1$ and then multiply it by x, so that $x\Phi(x)Z'_{2n+2}(x) = x\frac{C_{2n+2}(x)-C_0(x)}{2}Z_{2n+2}(x) - \gamma_{2n+2}xD_{2n+2}(x)Z_{2n+1}(x)$, but $xZ_{2n+1}(x) = Z_{2n+2}(x) + \gamma_{2n+1}Z_{2n}(x)$ according to (2.11), then

$$x\Phi(x)Z'_{2n+2}(x) = \Big(x\frac{C_{2n+2}(x)-C_0(x)}{2} - \gamma_{2n+2}D_{2n+2}(x)\Big)Z_{2n+2}(x) -$$

$$-\gamma_{2n+2}\gamma_{2n+1}D_{2n+2}(x)Z_{2n}(x).$$

Finally, from (2.12) and (2.14) we get (2.39). □

3. Symmetric semi-classical forms of class $s = 3$ when $\Psi(0) = 0$

We begin recalling an important result in our work [2].

Proposition 3.1. [2] *Let u be a symmetric semi-classical form of class s, satisfying (2.32), then if s is even, Φ is even and Ψ is odd; if s is odd, Φ is odd and Ψ is even.*

In the sequel, we will assume that u is a symmetric semi-classical form of class $s = 3$ satisfying (2.32) with $\Psi(0) = 0$. Then, according to proposition 3.1 and (2.25), u satisfies (2.32) with

$$\begin{cases} \Phi(x) = a_5x^5 + a_3x^3 + a_1x \ , \\ \Psi(x) = b_4x^4 + b_2x^2 \ , \\ a_1(|a_5| + |b_4|) \neq 0 \ . \end{cases} \tag{3.1}$$

In this particular case, it is possible to characterize the involved semi-classical forms of class $s = 3$:

Theorem 3.2. *The following statements are equivalent*

(a) u is a symmetric semi-classical normalized form of class $s = 3$ satisfying (2.32) with $\Psi(0) = 0$.

(b) There exist a classical normalized form v and

$(\tilde{a}_2\ ,\ \tilde{a}_1\ ,\ \tilde{a}_0\ ,\ \tilde{b}_1\ ,\ \tilde{b}_0) \in \mathbb{C}^5$ *such that:*

(3.2) $$x\sigma u = -\lambda v, \quad \lambda = -(u)_2\ ,$$

(3.3) $$\begin{cases} \left((\tilde{a}_2x^2 + \tilde{a}_1x + \tilde{a}_0)v\right)' + (\tilde{b}_1x + \tilde{b}_0)v = 0\ , \\ |\tilde{a}_2| + |\tilde{b}_1| \neq 0\ . \end{cases}$$

(3.4) $$\tilde{a}_0 \neq 0.$$

Proof. (a)⇒(b) We have u is symmetric, then v is regular according to (3.2) and proposition 2.1. From (2.32) and (3.1), we obtain

(3.5) $$\Big((a_5x^5 + a_3x^3 + a_1x)u\Big)' + (b_4x^4 + b_2x^2)u = 0\ .$$

On account of (2.2) and (2.3), it follows that

$$\Big((a_5x^2 + a_3x + a_1)v\Big)' + \frac{1}{2}(b_4x + b_2)v = 0\ .$$

Hence, we have

$$\Big((\tilde{a}_2x^2 + \tilde{a}_1x + \tilde{a}_0)v\Big)' + (\tilde{b}_1x + \tilde{b}_0)v = 0\ ,$$

with

(3.6) $$\begin{cases} \tilde{a}_2 = a_5\ , \quad \tilde{a}_1 = a_3\ , \quad \tilde{a}_0 = a_1\ , \\ \tilde{b}_1 = \frac{1}{2}b_4\ , \quad \tilde{b}_0 = \frac{1}{2}b_2\ . \end{cases}$$

Finally, we have $|\Phi'(0) + \Psi(0)| + |\langle u, \theta_0^2\Phi + \theta_0\Psi\rangle| = a_1 \neq 0$ according to (2.25). Then, using (3.6), we obtain (3.4).

(b)⇒(a). From (3.3), v satisfies (2.21) with $\tilde{\Phi}(x) = \tilde{a}_2x^2 + \tilde{a}_1x + \tilde{a}_0$ and $\tilde{\Psi}(x) = \tilde{b}_1x + \tilde{b}_0$. Then, thanks to proposition 2.8 and (2.25), we deduce that u satisfies (2.32) with

(3.7) $$\Phi(x) = \tilde{a}_2x^5 + \tilde{a}_1x^3 + \tilde{a}_0x\ , \quad \Psi(x) = 2(\tilde{b}_1x^4 + \tilde{b}_0x^2)\ .$$

By virtue of (3.4) and proposition 2.9, it is not possible to simplify the equation (3.7) and the class of u is $s = 3$. □

Determination of the canonical cases

From (2.22), proposition 3.1 and theorem 3.2., we distinguish four canonical cases for Φ:

$$\Phi(x) = x\ , \quad \Phi(x) = x(x^2-1)\ , \quad \Phi(x) = x(x^2-1)^2\ , \quad \Phi(x) = x(x^4-1)$$

associated with the four canonical cases for $\tilde{\Phi}$:

$$\tilde{\Phi}(x) = 1\ , \quad \tilde{\Phi}(x) = x-1\ , \quad \tilde{\Phi}(x) = (x-1)^2\ , \quad \tilde{\Phi}(x) = x^2-1\ .$$

First case: $\Phi(x) = x$**.** In this case , $\tilde{\Phi}(x) = 1$.

Thus, v is the Hermite form. We have [8, 21]

(3.8) $$\xi_n = 0\ , \quad \rho_{n+1} = \frac{1}{2}(n+1)\ , \quad n \geq 0\ ,$$

(3.9) $$\tilde{\Phi}(x) = 1\ , \quad \tilde{\Psi}(x) = 2x\ ,$$

(3.10) $$\tilde{C}_n(x) = -2x\ , \quad \tilde{D}_n(x) = -2\ , \quad n \geq 0\ .$$

In accordance with corollary 2.4 and (2.18), u is regular for every $\lambda \neq 0$ and we have

$$\begin{cases} \gamma_1 = -\gamma_2 = -\lambda \ , \\ \gamma_{4n+3} = -\gamma_{4n+4} = -\frac{1}{\lambda}\frac{\Gamma(2n+2)}{2^{2n+1}\Gamma^2(n+1)} \ , \\ \gamma_{4n+5} = -\gamma_{4n+6} = \lambda\frac{2^{2n+1}\Gamma(n+1)\Gamma(n+2)}{\Gamma(2n+2)} \ , \ n \geq 0 \ . \end{cases} \tag{3.11}$$

For (2.33) and (2.36), we have

$$\Phi(x) = x \ , \quad \Psi(x) = 4x^4 \ , \quad C_0(x) = -4x^4 - 1 \ , \quad D_0(x) = -4x^3 + 4\lambda x \ . \tag{3.12}$$

According to proposition 2.10, (3.10) and (3.11), we have, for $n \geq 0$,

$$\begin{cases} C_{2n+1}(x) = -4x^4 - 8\gamma_{2n+1}x^2 + 1 \ , \quad C_{2n+2}(x) = -4x^4 - 8\gamma_{2n+2}x^2 - 1 \ , \\ D_{2n+1}(x) = -4x^3 \ , \quad D_{2n+2}(x) = -4x^3 - 4(\gamma_{2n+1} + \gamma_{2n+3})x \ , \end{cases} \tag{3.13}$$

where γ_n, $n \geq 1$ are given by (3.11).

The form v has the following integral representation [21]

$$\langle v, f\rangle = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-x^2} f(x)dx \ , \ f \in \mathcal{P} \ . \tag{3.14}$$

Therefore, for $\lambda \neq 0$ and $f \in \mathcal{P}$, (2.20) becomes

$$\langle u, f\rangle = f(0) - \frac{\lambda}{\sqrt{\pi}} P\int_{-\infty}^{+\infty} \frac{e^{-x^4}}{|x|}\Big(f(x) - f(ix)\Big)dx \ . \tag{3.15}$$

Next, the focus will be put on σu: the even part of u.

The form u verifies the functional equation

$$(xu)' + 4x^4 u = 0.$$

Applying the operator σ for the above equation and using (2.2) and (2.3), we deduce that σu is a semi-classical form and satisfies the functional equation

$$(\Phi^P(x)\sigma u)' + \Psi^P(x)\sigma u = 0 \ , \tag{3.16}$$

where

$$\Phi^P(x) = x \ , \quad \Psi^P(x) = 2x^2 \ .$$

Here, we have $(\Phi^P)'(0) + \Psi^P(0) = 1$. Then, the class of σu is equal to 1.

From (2.14), the coefficients $\{\beta_n^P, \gamma_{n+1}^P\}_{n\geq 0}$ of $\{P_n\}_{n\geq 0}$ are given by

$$\beta_0^P = \gamma_1 \ , \quad \beta_{n+1}^P = \gamma_{2n+2} + \gamma_{2n+3} \ , \quad \gamma_{n+1}^P = \gamma_{2n+1}\gamma_{2n+2} \ , \quad n \geq 0 \ ,$$

where γ_n , $n \geq 1$ are given by (55).

According to proposition 2.11 and (3.13) where $x^2 \to x$, the sequence $\{P_n\}_{n\geq 0}$ satisfies the following differential recurrence equation (for $n \geq 0$)

$$\Phi^P(x)P'_{n+1}(x) = \frac{C_{n+1}^P(x) - C_0^P(x)}{2}P_{n+1}(x) - \gamma_{n+1}^P D_{n+1}^P(x)P_n(x) \ , \tag{3.17}$$

with

$C_0^P(x) = -2x^2 - 1$, $\frac{C_{n+1}^P(x) - C_0^P(x)}{2} = 4\gamma_{2n+2}(\gamma_{2n+1} + \gamma_{2n+3})$, $n \geq 0$

$D_0^P(x) = -2(x - \lambda)$. $D_{n+1}^P(x) = -4x - 4(\gamma_{2n+1} + \gamma_{2n+3})$, $n \geq 0$.

Finally, we give the integral representation of σu.

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From (2.19) and (3.14), we get

$$\langle \sigma u, f(x)\rangle = f(0) - \frac{\lambda}{\sqrt{\pi}} P \int_{-\infty}^{+\infty} \frac{e^{-x^2}}{x} f(x)dx \, .$$

In the three other cases, we are going to proceed with the same stages and techniques.

Second case: $\Phi(x) = x(x^2-1)$**.** In this case, $\tilde{\Phi}(x) = x - 1$.
Thus, $v = \tau_1 \mathcal{L}(\alpha)$ where $\mathcal{L}(\alpha)$ is the classical Laguerre form.
Here [18, 21]

$$\xi_n = 2n + \alpha + 2 \, , \quad \rho_{n+1} = (n+1)(n+\alpha+1) \, , \quad n \geq 0 \, , \tag{3.18}$$

the regularity condition is $\alpha \neq -n$, $n \geq 1$.

$$\tilde{\Phi}(x) = x - 1 \, , \quad \tilde{\Psi}(x) = x - \alpha - 2 \, , \tag{3.19}$$

$$\tilde{C}_n(x) = -x + 2n + \alpha + 1, \quad \tilde{D}_n(x) = -1 \, , \quad n \geq 0 \, . \tag{3.20}$$

Using the relation (2.11) in [9], we get

$$S_n(0) = (-1)^n \sum_{k=0}^{n} \frac{\Gamma(n+1)\Gamma(n+\alpha+1)}{\Gamma(n-k+1)\Gamma(\alpha+k+1)\Gamma(k+1)} \, , \; n \geq 0 \, . \tag{3.21}$$

From (2.6) and the relation (2.11) in [9], we obtain for $n \geq 0$,

$$S_n^{(1)}(0) = (-1)^{n+1} \sum_{k=0}^{n+1} \frac{\Gamma(n+2)\Gamma(n+\alpha+2)}{\Gamma(n-k+2)\Gamma(\alpha+k+1)\Gamma(k+1)} \Omega_k \, , \tag{3.22}$$

where

$$\Omega_0 = 0 \, , \quad \Omega_n = -\sum_{k=0}^{n-1} (-1)^k \frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+1)} \, , \quad n \geq 1 \, .$$

By virtue of (2.8), (3.21) and (3.22), we deduce

$$S_n(0,\lambda) = (-1)^n \Gamma(n+\alpha+1)\Gamma(n+1) d_n(\alpha) \, , \; n \geq 0 \, , \tag{3.23}$$

where

$$d_n(\alpha) = \sum_{k=0}^{n} \frac{1 - \lambda \Omega_k}{\Gamma(n-k+1)\Gamma(\alpha+k+1)\Gamma(k+1)} \, , \; n \geq 0 \, .$$

The regularity conditions are $\alpha \neq -n$, $d_n(\alpha) \neq 0$, $n \geq 1$.
(2.17) and (3.23) give

$$a_n = (n+\alpha+1)(n+1) \frac{d_{n+1}(\alpha)}{d_n(\alpha)} \, , \; n \geq 0 \, .$$

Then, with (2.16), we get

$$\begin{cases} \gamma_1 = -\lambda \, , \\ \gamma_{2n+3} = \frac{d_n(\alpha)}{d_{n+1}(\alpha)} \, , \\ \gamma_{2n+2} = (n+\alpha+1)(n+1) \frac{d_{n+1}(\alpha)}{d_n(\alpha)} \, , \; n \geq 0 \, . \end{cases} \tag{3.24}$$

Taking into account that the form v is semi-classical and by virtue of proposition 2.8, the form u is also semi-classical. It satisfies (2.32) with

$$\Phi(x) = x(x^2-1) \, , \; \Psi(x) = 2x^2(x^2 - \alpha - 2) \, . \tag{3.25}$$

Now, we are going to give the elements of the structure relation of the sequence $\{Z_n\}_{n\geq 0}$

$$\left\{\begin{array}{l} C_0(x) = -2x^4 + (2\alpha+1)x^2 + 1\ , \quad C_1(x) = -2x^4 + (2\alpha+4\lambda+3)x^2 - 1\ , \\ C_{2n+3}(x) = -2x^4 + (4n+2\alpha+7-4\frac{d_n(\alpha)}{d_{n+1}(\alpha)})x^2 - 1\ , \quad n\geq 0\ , \\ C_{2n+2}(x) = -2x^4 + (4n+2\alpha+5-4(n+\alpha+1)(n+1)\frac{d_{n+1}(\alpha)}{d_n(\alpha)})x^2 + 1\ , \ n\geq 0\ , \\ D_0(x) = 2x(-x^2+\alpha+\lambda+1), \quad D_{2n+1}(x) = -2x^3\ , \quad n\geq 0\ , \\ D_{2n+2}(x) = -2x^3 + \varpi_n x\ , \quad n\geq 0\ , \end{array}\right. \tag{3.26}$$

where

$$\varpi_n = 2(2n+\alpha+3-\frac{d_n(\alpha)}{d_{n+1}(\alpha)} - (n+\alpha+1)(n+1)\frac{d_{n+1}(\alpha)}{d_n(\alpha)})\ , \quad n\geq 0\ .$$

The form v has the following integral representation [21]

$$\langle v, f\rangle = \frac{1}{\Gamma(\alpha+1)}\int_1^{+\infty}(x-1)^\alpha e^{1-x} f(x)dx\ ,\ \mathcal{R}(\alpha) > -1\ ,\ f\in\mathcal{P}\ . \tag{3.27}$$

Then, using (2.20), we obtain the following integral representation of u

$$\langle u, f\rangle = \frac{\lambda}{\Gamma(\alpha+1)}\int_{-\infty}^{-1}\frac{(x^2-1)^\alpha e^{1-x^2}}{x}f(x)dx - \frac{\lambda}{\Gamma(\alpha+1)}\int_1^{+\infty}\frac{(x^2-1)^\alpha e^{1-x^2}}{x}f(x)dx +$$

$$+f(0)\{1+\frac{\lambda}{\Gamma(\alpha+1)}\int_1^{+\infty}\frac{(x-1)^\alpha e^{1-x}}{x}dx\}\ ,\ \mathcal{R}(\alpha) > -1\ ,\ f\in\mathcal{P}\ . \tag{3.28}$$

The form σu is a semi-classical form and satisfies the functional equation (3.21) with

$$\Phi^P(x) = x(x-1)\ , \quad \Psi^P(x) = x(x-\alpha-2)\ .$$

From (2.14), the coefficients $\{\beta_n^P\ ,\ \gamma_{n+1}^P\}_{n\geq 0}$ of $\{P_n\}_{n\geq 0}$ are given by

$$\beta_0^P = \gamma_1\ , \quad \beta_{n+1}^P = \gamma_{2n+2} + \gamma_{2n+3}\ , \quad \gamma_{n+1}^P = \gamma_{2n+1}\gamma_{2n+2}\ , \quad n\geq 0\ ,$$

where γ_n , $n\geq 1$ are given by (3.24).

From proposition 2.11 and (3.26), the sequences $\{P_n\}_{n\geq 0}$ satisfies (3.17) with

$\frac{C_{n+1}^P(x)-C_0^P(x)}{2} = 2(n+1)x - (n+1)(n+\alpha+1)\frac{d_{n+1}(\alpha)}{d_n(\alpha)}\varpi_n\ , \quad n\geq 0\ ,$

$D_{n+1}^P(x) = -2x + \varpi_n\ , \quad n\geq 0\ ,$

$C_0^P(x) = -x^2 + \alpha x + 1\ , \quad D_0^P(x) = -x + \lambda + \alpha + 1\ .$

Finally, we give the integral representation of σu.

From (2.19) and (3.27), we get

$$\langle \sigma u, f(x)\rangle = f(0)\{1+\frac{\lambda}{\Gamma(\alpha+1)}\int_1^{+\infty}\frac{(x-1)^\alpha e^{1-x}}{x}dx\} - \frac{\lambda}{\Gamma(\alpha+1)}\int_1^{+\infty}\frac{(x-1)^\alpha e^{1-x}}{x}f(x)dx\ .$$

Third case: $\Phi(x) = x(x^2-1)^2$**.** In this case, $\tilde{\Phi}(x) = (x-1)^2$.

Thus, $v = \tau_1\mathcal{B}(\alpha)$ where $\mathcal{B}(\alpha)$ is the classical Bessel form.

Here [18, 21]

$$\left\{\begin{array}{l} \xi_0 = 1-\frac{1}{\alpha}\ ,\ \xi_{n+1} = 1+\frac{1-\alpha}{(n+\alpha)(n+\alpha+1)}\ , \\ \rho_{n+1} = -\frac{(n+1)(n+2\alpha-1)}{(2n+2\alpha-1)(n+\alpha)^2(2n+2\alpha+1)}\ , \quad n\geq 0\ , \end{array}\right. \tag{3.29}$$

the regularity condition is $\alpha\neq -\frac{n}{2}$, $n\geq 0$.

$$\tilde{\Phi}(x) = (x-1)^2\ , \quad \tilde{\Psi}(x) = -2(\alpha x + 1 - \alpha)\ , \tag{3.30}$$

$$\tilde{C}_n(x) = 2(n+\alpha-1)(x-1) + 2\frac{\alpha-1}{n+\alpha-1}\ , \quad \tilde{D}_n(x) = 2n+2\alpha-1\ , \quad n\geq 0\ , \tag{3.31}$$

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By applying the same process as we did to obtain (3.23) and using the above results, we get after some straightforward calculations

$$S_n(0,\lambda) = 2^n \frac{\Gamma(n+1)}{\Gamma(2n++2\alpha-1)} e_n(\alpha) \ , \ n \geq 0 \ , \tag{3.32}$$

where

$$e_n(\alpha) = \sum_{k=0}^{n} (-1)^k \frac{(1-\lambda\Lambda_k)\Gamma(n+k+2\alpha-1)}{2^k \Gamma(n-k+1)\Gamma(k+1)} \ , \ n \geq 0 \ ,$$

and

$$\Lambda_0 = 0 \ , \quad \Lambda_n = -\sum_{k=0}^{n-1} 2^k \frac{\Gamma(2\alpha)}{\Gamma(k+2\alpha)} \ , \quad n \geq 1 \ .$$

The regularity conditions are $e_n(\alpha) \neq 0$, $\alpha \neq -\frac{n}{2}$, $n \geq 0$.
From (2.17) and (3.32) we get

$$a_n = -\frac{n+1}{(n+\alpha)(2n+2\alpha-1)} \frac{e_{n+1}(\alpha)}{e_n(\alpha)} \ , \ n \geq 0 \ . \tag{3.33}$$

Using (2.16) and (3.33), we obtain

$$\begin{cases} \gamma_1 = -\lambda \ , \\ \gamma_{2n+3} = \frac{n+2\alpha-1}{(n+\alpha)(2n+2\alpha+1)} \frac{e_n(\alpha)}{e_{n+1}(\alpha)} \ , \\ \gamma_{2n+2} = -\frac{n+1}{(n+\alpha)(2n+2\alpha-1)} \frac{e_{n+1}(\alpha)}{e_n(\alpha)} \ , \ n \geq 0 \ . \end{cases} \tag{3.34}$$

According to proposition 2.8, the form u is also semi-classical. It satisfies (2.32) with

$$\Phi(x) = x(x^2-1)^2 \ , \ \Psi(x) = -4x^2(\alpha x^2 + 1 - \alpha) \ . \tag{3.35}$$

Now, we are going to give the elements of the structure relation of the sequence $\{Z_n\}_{n\geq 0}$

$$\begin{cases} C_0(x) = (4\alpha-5)x^4 + 2(5-2\alpha)x^2 - 1 \ , \\ C_1(x) = (4\alpha-3)x^4 + (6-4\alpha-4\lambda(\alpha-1))x^2 + 1 \ , \\ C_{2n+3}(x) = (4n+4\alpha+1)x^4 + 2\left(2\frac{n+2\alpha-1}{n+\alpha}\frac{e_n(\alpha)}{e_{n+1}(\alpha)} + 2\frac{\alpha-1}{n+\alpha} - 2n - 2\alpha - 1\right)x^2 + 1 \ , \\ C_{2n+2}(x) = (4n+4\alpha-1)x^4 + 2\left(-2\frac{n+1}{n+\alpha}\frac{e_{n+1}(\alpha)}{e_n(\alpha)} + 2\frac{\alpha-1}{n+\alpha} - 2n - 2\alpha + 1\right)x^2 - 1 \ , \\ D_0(x) = 2x\big(2(\alpha-1)x^2 + 2(2-\alpha) + \lambda(1-2\alpha)\big) \ , \\ D_{2n+1}(x) = 2(2n+2\alpha-1)x^3 \ , \\ D_{2n+2}(x) = 2x\big(2(n+\alpha)x^2 + \varsigma_n\big) \ , \end{cases} \tag{3.36}$$

where

$$\varsigma_n = -\frac{n+1}{n+\alpha}\frac{e_{n+1}(\alpha)}{e_n(\alpha)} + \frac{n+2\alpha-1}{n+\alpha}\frac{e_n(\alpha)}{e_{n+1}(\alpha)} + 2\frac{\alpha-1}{n+\alpha} - 2(n+\alpha) \ , \quad n \geq 0 \ .$$

The form v has the following integral representation [21]

$$\langle v, f\rangle = J(\alpha)^{-1} \int_1^{+\infty} (x-1)^{2\alpha-2} e^{-\frac{2}{x-1}} \left(\int_{x-1}^{+\infty} \zeta^{-2\alpha} e^{\frac{2}{\zeta}} s(\zeta) d\zeta \right) f(x) dx \ , \ \mathcal{R}(\alpha) > -1 \ , \ f \in \mathcal{P} \ , \tag{3.37}$$

with

$$J(\alpha) = 4 \int_0^{+\infty} t^{3-8\alpha} e^{\frac{2}{t^4}-t} sin\, t \left(\int_0^{t^4} x^{2\alpha-2} e^{-\frac{2}{x}} dx \right) dt \ ,$$

$$s(x) = \begin{cases} 0, & x \leq 0 \ , \\ e^{-x^{\frac{1}{4}}} sin\, x^{\frac{1}{4}} \ , & x > 0 \ . \end{cases}$$

Then, using (2.20), we obtain the following integral representation of u

$$
\begin{aligned}
\langle u, f\rangle = f(0)\Big\{1+\lambda J(\alpha)^{-1}\int_1^{+\infty}\frac{(x-1)^{2\alpha-2}e^{-\frac{2}{x-1}}}{x}\Big(\int_{x-1}^{+\infty}\zeta^{-2\alpha}e^{\frac{2}{\zeta}}s(\zeta)d\zeta\Big)dx\Big\}+ \\
+\lambda J(\alpha)^{-1}\int_{-\infty}^{-1}\frac{(x^2-1)^{2\alpha-2}e^{-\frac{2}{x^2-1}}}{x}\Big(\int_{x^2-1}^{+\infty}\zeta^{-2\alpha}e^{\frac{2}{\zeta}}s(\zeta)d\zeta\Big)f(x)dx- \\
-\lambda J(\alpha)^{-1}\int_1^{+\infty}\frac{(x^2-1)^{2\alpha-2}e^{-\frac{2}{x^2-1}}}{x}\Big(\int_{x^2-1}^{+\infty}\zeta^{-2\alpha}e^{\frac{2}{\zeta}}s(\zeta)d\zeta\Big)f(x)dx\,.
\end{aligned}
\tag{3.38}
$$

The form σu is a semi-classical form and satisfies (2.21) with

$$\Phi^P(x)=x(x-1)^2\ ,\quad \Psi^P(x)=-2x(\alpha x+1-\alpha)\,.$$

From (2.14) the coefficients $\{\beta_n^P\ ,\ \gamma_{n+1}^P\}_{n\geq 0}$ of $\{P_n\}_{n\geq 0}$ are given by

$$\beta_0^P=\gamma_1\ ,\quad \beta_{n+1}^P=\gamma_{2n+2}+\gamma_{2n+3}\ ,\quad \gamma_{n+1}^P=\gamma_{2n+1}\gamma_{2n+2}\ ,\quad n\geq 0,$$

where γ_n , $n\geq 1$ are given by (3.34).

The sequence $\{P_n\}_{n\geq 0}$ satisfies (3.17) with (for $n\geq 0$)

$$
\begin{aligned}
\tfrac{C_{n+1}^P(x)-C_0^P(x)}{2} &= 2(n+1)x^2+2\Big(\tfrac{\alpha-1}{n+\alpha}-\tfrac{n+1}{n+\alpha}\tfrac{e_{n+1}(\alpha)}{e_n(\alpha)}+2\tfrac{n+1}{2n+2\alpha-1}\tfrac{e_{n+1}(\alpha)}{e_n(\alpha)}-n-2\Big)x+ \\
&+2\tfrac{n+1}{(n+\alpha)(2n+2\alpha-1)}\tfrac{e_{n+1}(\alpha)}{e_n(\alpha)}\varsigma_n\ ,
\end{aligned}
$$

$D_{n+1}^P(x)=2\big(2(n+\alpha)x+\varsigma_n\big)$,

$C_0^P(x)=(2\alpha-3)x^2+2(3-\alpha)x-1$,

$D_0^P(x)=2(\alpha-1)x+\lambda(1-2\alpha)+2(2-\alpha)$.

Finally, we give the integral representation of σu.

From (2.19) and (3.37), we have

$$
\begin{aligned}
\langle\sigma u, f(x)\rangle = f(0)\Big\{1+\lambda J(\alpha)^{-1}\int_1^{+\infty}\frac{(x-1)^{2\alpha-2}e^{-\frac{2}{x-1}}}{x}\Big(\int_{x-1}^{+\infty}\zeta^{-2\alpha}e^{\frac{2}{\zeta}}s(\zeta)d\zeta\Big)dx\Big\}- \\
-\lambda J(\alpha)^{-1}\int_1^{+\infty}\frac{(x-1)^{2\alpha-2}e^{-\frac{2}{x-1}}}{x}\Big(\int_{x-1}^{+\infty}\zeta^{-2\alpha}e^{\frac{2}{\zeta}}s(\zeta)d\zeta\Big)f(x)dx\,.
\end{aligned}
$$

Fourth case: $\Phi(x)=x(x^4-1)$. In this case, we have $\tilde\Phi(x)=x^2-1$.

Thus, $v=\mathcal{J}(\alpha,\beta)$ where $\mathcal{J}(\alpha,\beta)$ is the classical Jacobi form.

Here [18, 21]

$$
\begin{cases}
\xi_0=\frac{\alpha-\beta}{\alpha+\beta+2}\ ,\ \xi_{n+1}=\frac{\alpha^2-\beta^2}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+4)}\ ,\ n\geq 0, \\
\rho_{n+1}=4\frac{(n+1)(n+\alpha+\beta+1)(n+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+3)}\ ,\quad n\geq 0\,.
\end{cases}
\tag{3.39}
$$

the regularity conditions are $\alpha\ ,\ \beta\neq -n\ ,\quad \alpha+\beta\neq -n-1\ ,\ n\geq 1$.

$$\tilde\Phi(x)=x^2-1\ ,\quad \tilde\Psi(x)=-(\alpha+\beta+2)x+\alpha-\beta\ ,\tag{3.40}$$

$$\tilde C_n(x)=(2n+\alpha+\beta)x-\frac{\alpha^2-\beta^2}{2n+\alpha+\beta}\ ,\quad \tilde D_n(x)=2n+\alpha+\beta+1\ ,\quad n\geq 0\ ,\tag{3.41}$$

By applying the same process as we did to obtain (3.23) and using the above results, we get after some straightforward calculations

$$S_n(0,\lambda)=2^n\frac{\Gamma(n+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+1)}f_n(\alpha,\beta)\ ,\ n\geq 0\ ,\tag{3.42}$$

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with

$$f_n(\alpha,\beta)=\sum_{k=0}^{n}\frac{(-1)^k\Gamma(n+k+\alpha+\beta+1)}{2^k\Gamma(k+1)\Gamma(n-k+1)\Gamma(k+\beta+1)}(1-\lambda\Theta_k)\ ,\ n\geq 0\ ,$$

and (for $n\geq 1$)

$$\Theta_0=0\ ,\quad \Theta_n=-\sum_{k=0}^{n-1}\frac{(-1)^k\Gamma(n+1)}{(k+1)\Gamma(n-k)}\sum_{\mu=0}^{k}\frac{2^{\mu-1}\Gamma(\alpha+\beta+2)}{\Gamma(\mu+1)\Gamma(k-\mu+1)\Gamma(\mu+\alpha+\beta+2)}F_{k,\mu}(\alpha,\beta)\ ,$$

where

$$F_{n,k}(\alpha,\beta)=(-1)^{n-k}\frac{\Gamma(k+\alpha+1)}{\Gamma(\alpha+1)}+(-1)^k\frac{\Gamma(k+\beta+1)}{\Gamma(\beta+1)}\ .$$

The regularity conditions are $\alpha\ ,\ \beta\neq -n\ ,\ \alpha+\beta\neq -n-1\ ,\ f_n(\alpha,\beta)\neq 0\ ,\ n\geq 1$.
(2.17) and (3.42) give

$$a_n=-2\frac{(n+1)(n+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}\frac{f_{n+1}(\alpha,\beta)}{f_n(\alpha,\beta)}\ ,\ n\geq 0\ .$$

Then, with (2.16), we get

$$\begin{cases}\gamma_1=-\lambda\ ,\\ \gamma_{2n+3}=-2\frac{(n+\alpha+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+3)}\frac{f_n(\alpha,\beta)}{f_{n+1}(\alpha,\beta)}\ ,\\ \gamma_{2n+2}=-2\frac{(n+1)(n+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}\frac{f_{n+1}(\alpha,\beta)}{f_n(\alpha,\beta)}\ ,\ n\geq 0\ .\end{cases}\tag{3.43}$$

Remark 3.3. *If $\alpha=\beta$, (3.43) becomes*

$$\begin{cases}\gamma_1=-\gamma_2=-\lambda\\ \gamma_{4n+3}=-\gamma_{4n+4}=\dfrac{2}{\lambda}\dfrac{\Gamma(1+\alpha)\Gamma(n+\frac{3}{2})\Gamma(n+\alpha+\frac{3}{2})}{(4n+2\alpha+3)\sqrt{\pi}\Gamma(\frac{3}{2}+\alpha)\Gamma(n+1)\Gamma(n+\alpha+1)},\\ \gamma_{4n+5}=-\gamma_{4n+6}=-2\lambda\dfrac{\sqrt{\pi}\Gamma(\frac{3}{2}+\alpha)\Gamma(n+2)\Gamma(n+\alpha+2)}{(4n+2\alpha+5)\Gamma(1+\alpha)\Gamma(n+\frac{3}{2})\Gamma(n+\alpha+\frac{3}{2})}.\end{cases}$$

Taking into account that the form v is semi-classical and by virtue of proposition 2.8, the form u is also semi-classical. It satisfies (2.32) with

$$\Phi(x)=x(x^4-1)\ ,\ \Psi(x)=2x^2\big(-(\alpha+\beta+2)x^2+\alpha-\beta\big)\ .\tag{3.44}$$

Now, we are going to give the elements of the structure relation of the sequence $\{Z_n\}_{n\geq 0}$

$$\begin{cases}C_0(x)=(2\alpha+2\beta-1)x^4+2(\beta-\alpha)x^2+1\ ,\\ C_1(x)=(2\alpha+2\beta+1)x^4+2(-2\lambda(\alpha+\beta+1)+\beta-\alpha)x^2-1\ ,\\ C_{2n+3}(x)=(4n+2\alpha+2\beta+5)x^4-\frac{2}{2n+\alpha+\beta+2}\Bigg(\alpha^2-\beta^2+\\ \qquad +4(n+\alpha+1)(n+\alpha+\beta+1)\frac{f_n(\alpha,\beta)}{f_{n+1}(\alpha,\beta)}\Bigg)x^2-1\ ,\\ C_{2n+2}(x)=(4n+2\alpha+2\beta+3)x^4-\frac{2}{2n+\alpha+\beta+2}\Bigg(\alpha^2-\beta^2+\\ \qquad +4(n+1)(n+\beta+1)\frac{f_{n+1}(\alpha,\beta)}{f_n(\alpha,\beta)}\Bigg)x^2+1\ ,\\ D_0(x)=D_0(x)=2x\big((\alpha+\beta)x^2+\beta-\alpha-\lambda(\alpha+\beta+1)\big)\ ,\\ D_{2n+1}(x)=2(2n+\alpha+\beta+1)x^3\ ,\\ D_{2n+2}(x)=2(2n+\alpha+\beta+2)x^3+\upsilon_n x\ ,\end{cases}\tag{3.45}$$

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where(for $n \geq 0$)

$$\upsilon_n = -\frac{2}{2n+\alpha+\beta+2}\left(\alpha^2-\beta^2+2(n+1)(n+\beta+1)\frac{f_{n+1}(\alpha,\beta)}{f_n(\alpha,\beta)}+2(n+\alpha+1)(n+\alpha+\beta+1)\frac{f_n(\alpha,\beta)}{f_{n+1}(\alpha,\beta)}\right).$$

The form v has the following integral representation [21]

$$\langle v, f\rangle = \mathcal{A}\int_{-1}^{1}(1-x)^{\beta}(x+1)^{\alpha}f(x)dx\ ,\ \mathcal{R}(\alpha)\ ,\ \mathcal{R}(\beta) > -1\ ,\ f\in\mathcal{P}\ , \tag{3.46}$$

with

$$\mathcal{A} = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}\ .$$

Then, using (2.20), we obtain

$$\begin{aligned}\langle u, f\rangle &= f(0)\{1+\lambda\mathcal{A}P\int_{-1}^{1}\frac{(1-x)^{\beta}(x+1)^{\alpha}}{x}dx\}+\\ &+\lambda\mathcal{A}P\int_{-1}^{1}\frac{(1+x^2)^{\beta}(1-x^2)^{\alpha}}{x}f(ix)dx - \lambda\mathcal{A}P\int_{-1}^{1}\frac{(1-x^2)^{\beta}(x^2+1)^{\alpha}}{x}f(x)dx\ .\end{aligned} \tag{3.47}$$

σu is a semi-classical form and satisfies (3.21) with

$$\Phi^P(x) = x(x^2-1)\ ,\quad \Psi^P(x) = x(-(\alpha+\beta+2)x+\alpha-\beta)\ .$$

From (2.14) the coefficients $\{\beta_n^P,\gamma_{n+1}^P\}_{n\geq 0}$ of $\{P_n\}_{n\geq 0}$ are given by

$$\beta_0^P = \gamma_1\ ,\quad \beta_{n+1}^P = \gamma_{2n+2}+\gamma_{2n+3}\ ,\quad \gamma_{n+1}^P = \gamma_{2n+1}\gamma_{2n+2}\ ,\quad n\geq 0,$$

where γ_n , $n \geq 1$ are given by (3.43).

The sequence $\{P_n\}_{n\geq 0}$ satisfies (3.17) with (for $n \geq 0$)

$\frac{C_{n+1}^P(x)-C_0^P(x)}{2} = 2(n+1)x^2 + \left(\beta-\alpha-\frac{\alpha^2-\beta^2}{2n+\alpha+\beta+2}\right)x + 2\frac{(n+1)(n+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}\frac{f_{n+1}(\alpha,\beta)}{f_n(\alpha,\beta)}\upsilon_n\ ,$

$D_{n+1}^P(x) = 2(2n+\alpha+\beta+2)x+\upsilon_n\ ,$

$C_0^P(x) = (\alpha+\beta-1)x^2+(\beta-\alpha)x+1\ ,$

$D_0^P(x) = (\alpha+\beta)(x-\lambda)+\beta-\alpha-\lambda\ .$

Finally, we give the integral representation of σu.

From (2.19) and (3.46), we get

$$\langle\sigma u, f(x)\rangle = f(0)\{1+\lambda\mathcal{A}P\int_{-1}^{1}\frac{(1-x)^{\beta}(x+1)^{\alpha}}{x}dx\} - \lambda\mathcal{A}P\int_{-1}^{1}\frac{(1+x)^{\alpha}(1-x)^{\beta}}{x}f(x)dx\ . \tag{3.48}$$

Remarks 3.4. (1). *If $\beta = \alpha - 1$, then from (3.48), we get*

$$\begin{aligned}\langle\sigma u, x^{2n+2}-x^{2n+3}\rangle &= -\lambda\mathcal{A}\int_{-1}^{1}(1+x)^{\alpha}(1-x)^{\alpha-1}(x^{2n+1}-x^{2n+2})dx\\ &= -\lambda\mathcal{A}\int_{-1}^{1}(1-x^2)^{\alpha}x^{2n+1}dx\\ &= 0\ ,\quad n\geq 0\ .\end{aligned}$$

Hence, if $\lambda = -1$, we obtain

$$(\sigma u)_{2n+1} = (\sigma u)_{2n}\ ,\quad n\geq 0\ .$$

Therefore, from the Lemma 2.2 in [1], *σu verifies (2.13) and (3.16) with* [8, 16]

$\beta_n^P = (-1)^n\ ,\quad n\geq 0\ ,$

$\gamma_{2n+1}^P = -\frac{2(n+\alpha)(2n+2\alpha-1)}{(4n+2\alpha-1)(4n+2\alpha+1)}\ ,\quad \gamma_{2n+2}^P = -\frac{2(n+1)(2n+1)}{(4n+2\alpha+1)(4n+2\alpha+3)}\ ,\quad n\geq 0\ ,$

$\Phi^P(x) = x^3-x\ ,\quad \Psi^P(x) = -(2\alpha+1)x^2+x\ .$

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(2). *Theorem 3.2 is the main result of our paper. From it, we carry out the complete description of the symmetric semi-classical forms of class* $s = 3$*, when* $\Psi(0) = 0$*. Unfortunately, the case when* $\Psi(0) \neq 0$ *is not covered by this theorem and the description of these forms remains open. Some illustrative examples of symmetric semi-classical forms of class* $s = 3$ *related to the last case are given in* [4, 5].

(3). *In* [13], *Marcellán et al. made use of this approach to provide a full description of all symmetric semi-classical forms of class* $s = 3$.

(4). *The above four canonical cases, can be determined by symmetrization of the semi-classical form of class one and verify (2.21) with* $\tilde{\Phi}(0) = 0$ *and* $\tilde{\Phi}'(0) + 2\tilde{\Psi}(0) \neq 0$. [4]

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