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NECESSARY AND SUFFICIENT CONDITION FOR OSCILLATION AND ASYMPTOTIC BEHAVIOUR OF NONLINEAR NEUTRAL FIRST-ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. In this work, necessary and sufficient conditions for oscillations of the solutions of a class of nonlinear neutral first-order differential equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[x(t)+p(t)x(t-\tau)\big]+q(t)G\big(x(t-\sigma)\big)=0$$

are established under various ranges of the neutral coefficient p. Our main tools are Knaster-Tarski fixed point theorem and Banach's fixed point theorem. Finally, two illustrating examples are presented to show that feasibility and effectiveness of main results.

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1. Introduction

Consider a class of first-order nonlinear neutral delay differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[x(t)+p(t)x(t-\tau)\big]+q(t)G\big(x(t-\sigma)\big)=0, \tag{1.1}$$

where

$$\tau,\sigma\in\mathbb{R}_+=(0,+\infty),\ p\in PC([0,\infty),\mathbb{R}),\ q\in C(\mathbb{R}_+,\mathbb{R}_+),$$

and G is nondecreasing with

$$G\in C(\mathbb{R},\mathbb{R})\ \text{ with }\ uG(u)>0\ \text{ for }\ u\neq 0.$$

In [1], Ahmed et al. have studied the oscillation properties of a linear differential equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[r(t)(x(t)+p(t)x(t-\tau)\big]+q(t)x(t-\sigma)=0, \tag{E_1}$$

for the cases $p(t)\le -1$, $-1\le p(t)<0$ and $p(t)\equiv p\neq\pm 1$ and established sufficient conditions so that every solution of (E_1) is oscillates. In [2], Ahmed et al. considered the first order nonlinear neutral delay differential equations with variable coefficients of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[r(t)\big(a(t)x(t)+p(t)x(t-\tau)\big)\big]+q(t)G\big(x(t-\sigma)\big)=0, \tag{E_2}$$

and obtained some new sufficient conditions for the oscillation of all solutions of (E_2) by employing the Riccati transformation. In [4], Candan have obtained sufficient conditions for existence of nonoscillatory solutions of first order neutral differential Equations having both delay and advance terms (known as mixed equations) by using Banach contraction principle. In [8], Graef et al. considered (E_2) when $a(t)=1=r(t)$ and developed some sufficient conditions for the oscillation of all solutions of (E_2). Unlike the work in [1], [2] and [8] an attempt is made here to establish necessary and sufficient conditions for oscillations of (1.1) under various ranges of $p(t)$.

Recently, an increasing interest in obtaining sufficient conditions for oscillatory or non-oscillatory behavior of different classes of differential and functional differential equations has been manifested. In particular, investigation of neutral differential equations is important since they are encountered in many applications in science and technology and are used, for instance, to describe distributed networks with lossless transmission lines, in the study of vibrating masses attached to an elastic bar, as well as in some variational problems.

Definition 1.1. *By a solution of* (1.1) *we understand a function* $x \in C\big([-\rho,\infty),\mathbb{R}\big)$ *such that* $x(t) + p(t)x(t-\tau)$ *is once continuously differentiable and* (1.1) *is satisfied for* $t \ge 0$*, where* $\rho = \max\{\tau,\sigma\}$ *and* $\sup\{|x(t)| : t \ge t_0\} > 0$ *for every* $t_0 \ge 0$*. A solution of* (1.1) *is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory.*

2. Necessary and Sufficient Condition for Oscillation

In this section, sufficient and necessary conditions are obtained for oscillation of solutions of the equation (1.1). We need the following assumptions for our work in the sequel:

(A_1) there exists $\lambda > 0$ such that $G(u) + G(v) \ge \lambda G(u+v)$ for $u, v > 0$;
(A_2) $G(uv) = G(u)G(v)$ for $u, v \in \mathbb{R}$;
(A_3) $\int_{c_1}^{\pm c_2} \frac{d\eta}{G(\eta)} < \infty$, $c_1, c_2 > 0$;
(A_4) $\int_{\tau}^{\infty} Q(\eta)d\eta = \infty$, where $Q(t) = \min\big\{q(t), q(t-\tau)\big\}$;
(A_5) $\int_{\pm c}^{\pm\infty} \frac{d\eta}{G(\eta)} < \infty$, $c > 0$;
(A_6) $\int_{0}^{\infty} q(\eta)d\eta = \infty$.

Remark 2.1. *Assumption* (A_2) *implies* G *is a odd function. Indeed,* $G(1)G(1) = G(1)$ *and* $G(1) > 0$ *imply that* $G(1) = 1$*. Further,* $G(-1)G(-1) = G(1) = 1$ *implies that* $\big(G(-1)\big)^2 = 1$*. Since* $G(-1) < 0$*, we conclude that* $G(-1) = -1$*. Hence,*

$$G(-u) = G(-1)G(u) = -G(u).$$

On the other hand, $G(uv) = G(u)G(v)$ *for* $u > 0$ *and* $v > 0$ *and* $G(-u) = -G(u)$ *imply that* $G(xy) = G(x)G(y)$ *for every* $x, y \in \mathbb{R}$*.*

Remark 2.2. *We may note that if* $x(t)$ *is a solution of* (1.1)*, then* $y(t) = -x(t)$ *is also a solution of* (1.1) *provided that* G *satisfies* (A_2)*.*

Lemma 2.3. [7] *Let* $r, x, z \in C([0,\infty),\mathbb{R})$ *be such that* $z(t) = x(t) + p(t)x(t-\tau)$*,* $t \ge \tau > 0$*,* $x(t) > 0$ *for* $t \ge t_1 > \tau$*,* $\liminf_{t\to\infty} x(t) = 0$ *and* $\lim_{t\to\infty} z(t) = L$ *exists. Let* $p(t)$ *satisfy one of the following conditions:*

$$i)\ 0 \le p(t) \le p_3 < 1,\ ii)\ 1 < p_4 \le p(t) \le p_5 < \infty,\ iii)\ -\infty < -p_6 \le p(t) \le 0,$$

where $p > 0$*,* $3 \le i \le 6$*. Then* $L = 0$*.*

Remark 2.4. *If, in the above lemma,* $x(t) < 0$ *for* $t \ge \tau > 0$*,* $\limsup_{t\to\infty} x(t) = 0$ *and* $\lim_{t\to\infty} z(t) = L \in \mathbb{R}$ *exists, then* $L = 0$*.*

Theorem 2.5. *Let* $-1 < -p \le p(t) \le 0$*,* $t \in \mathbb{R}_+$ *and* $p > 0$*. Assume that* (A_2) *and* (A_3) *hold. Then every solution of the equation* (1.1) *oscillates if and only if* (A_6) *hold.*

Proof. Suppose for contrary that $x(t)$ is a nonoscillatory solution of equation (1.1). Then there exists $t_0 \ge \rho = \max\{\tau,\sigma\}$ such that $x(t) > 0$ or $x(t) < 0$ for $t \ge t_0$. Assume that $x(t) > 0$ for $t \ge t_0$. We set

$$z(t) = x(t) + p(t)x(t-\tau), \tag{2.1}$$

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it follows from (1.1) that

$$z'(t) = -q(t)G\big(x(t-\sigma)\big) \leq 0 \tag{2.2}$$

for $t \geq t_1 > t_0$. Consequently, $z(t)$ is monotonic on $[t_2, \infty)$, where $t_2 > t_1$. We have the following two possible cases.

Case 1. Let $z(t) > 0$ for $t_3 > t_2$. From (2.1), it follows that $z(t) \leq x(t)$ on $[t_3, \infty)$. Consequently, (2.2) becomes

$$z'(t) + q(t)G\big(z(t-\sigma)\big) \leq 0.$$

Because of nonincreasing $z(t)$, the last inequality becomes

$$\frac{z'(t)}{G(z(t))} + q(t) \leq 0.$$

Note that $\lim_{t\to\infty} z(t)$ exists. Integrating the last inequality from t_3 to t, we get

$$\int_{t_3}^{t} q(\eta)d\eta \leq -\int_{z(t_3)}^{z(t)} \frac{d\zeta}{G(\zeta)} < \infty, \text{ as } t \to \infty,$$

due to (A_3), we get a contradiction to (A_6).

Case 2. Let $z(t) < 0$ for $t_3 > t_2$. From (2.1), it is easy to verify that $x(t) < x(t-\tau)$ on $[t_3, \infty)$, $t_3 > t_2$. Further we can write

$$x(t) < x(t-\tau) < x(t-2\tau) < \cdots < y(t_3),$$

that is, $x(t)$ is bounded on $[t_3, \infty)$. Consequently, $z(t)$ is bounded and $\lim_{t\to\infty} z(t)$ exists. From (2.1), it follows that $z(t+\tau-\sigma) > p(t+\tau-\sigma)x(t-\sigma)$. Hence, (2.2) becomes

$$z'(t) + \frac{q(t)}{G(-p)}G\big(z(t+\tau-\sigma)\big) \leq 0, \tag{2.3}$$

due to (A_2). Since $z(t)$ is decreasing, then there exist $t_4 > t_3$ and $c > 0$ such that $z(t) \leq -c$ for $t \geq t_4$. Therefore, the inequality (2.3) can be viewed as

$$z'(t) + \frac{G(-c)}{G(-p)}q(t) \leq 0 \tag{2.4}$$

for $t \geq t_4$. Integrating (2.4) from t_4 to t, we obtain

$$\frac{G(-c)}{G(-p)}\int_{t_4}^{t} q(\eta)d\eta \leq -\big[z(\eta)\big]_{t_4}^{t} < \infty, \text{ as } t \to \infty,$$

which is a contradiction to (A_6).

If $x(t) < 0$ for $t \geq t_0$, then we set $y(t) = -x(t)$ for $t \geq t_0$ in (1.1). Using (A_2) and Remark 2.1, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[y(t) + p(t)y(t-\tau)\big] + q(t)G\big(y(t-\sigma)\big) = 0.$$

Then, proceeding as above, we find the same contradiction.

For the necessary Part, we suppose that (A_6) does not holds. Then there exist $t_1 > 0$ such that

$$\int_{t_1}^{\infty} q(\eta)d\eta < \frac{1-p}{15G(1)}, \quad t \geq t_1.$$

For $t_2 > t_1$, we let $Y = BC([t_2, \infty), \mathbb{R})$ be the space of all real valued bounded continuous functions defined on $[t_2, \infty)$. Clearly, Y is a Banach space with respect to sup norm defined by

$$||y|| = \sup\{|y(t)| : t \geq t_2\}.$$

Let $L = \{y \in Y : y(t) \geq 0,\ t \geq t_2\}$. Then, Y is a partially ordered Banach space (see for e.g. [7], p. 30). For $u, v \in Y$, we define $u \leq v$ if and only if $u - v \in L$. Let

$$S = \Big\{ u \in Y : \ \frac{1-p}{15} \leq u(t) \leq 1,\ t \geq t_2 \Big\}.$$

If $u_0(t) = \frac{1-p}{15}$, then $u_0 \in S$ and $u_0 = \text{g.l.b } S$. Further, if $\Phi \subset S^* \subset S$, then

$$S^* = \Big\{ u \in Y : \ \lambda \leq u(t) \leq \mu,\ \frac{1-p}{15} \leq \lambda,\ \mu \leq 1 \Big\}.$$

Let $v_0(t) = \mu_0,\ t \geq t_2$, where $\mu_0 = \sup\{\mu : \frac{1-p}{15} \leq \mu \leq 1\}$. Then $v_0 \in S$ and $v_0 = \text{l.u.b } S^*$. For $t_3 = t_2 + \rho$, define $T : S \to S$ by

$$(Tx)(t) = \begin{cases} (Tx)(t_3), & t \in [t_2, t_3], \\ -p(t)x(t-\tau) + \frac{1-p}{15} + \int_t^\infty q(\eta) G\big(x(\eta - \sigma)\big) d\eta, & t \geq t_3. \end{cases}$$

For every $x \in S$, $(Tx)(t) \geq \frac{1-p}{15}$ and

$$(Tx)(t) \leq p + G(1)\Big[\int_t^\infty q(\eta) d\eta\Big] + \frac{1-p}{15} < \frac{2+13p}{15} < 1$$

implies that $Tx \in S$. For $x_1, x_2 \in S$, it is easy to verify that $x_1 \leq x_2$ implies that $Tx_1 \leq Tx_2$. Hence by Knaster-Tarski fixed point theorem (see for e.g. [7], Theorem 1.7.3), T has a unique fixed point $x_0 \in S$. Hence x_0 is a positive solution of (1.1) such that $\liminf_{t\to\infty} x_0(t) > 0$. This completes the proof of the theorem. □

Theorem 2.6. *Let $-\infty < -p_1 \leq p(t) \leq -p_2 \leq -1$, $t \in \mathbb{R}_+$, $p_1, p_2 > 0$ and $\tau > \sigma$. Assume that (A_2) and (A_5) hold. Furthermore, assume that G is Lipschitzian on interval of the form $[a, b], 0 < a < b < \infty$. Then every solution of the equation (1.1) oscillates if and only if (A_6) holds.*

Proof. On the contrary, we proceed as in the proof of the Theorem 2.5 to obtain $z(t)$ is monotonic on $[t_2, \infty)$, $t_2 > t_1$. We claim that $z(t) < 0$ for $t \geq t_2$. If not, let $z(t) \geq 0$ for $t \geq t_2 > t_1$. Consequently,

$$x(t) \geq -p(t)x(t-\tau) \geq x(t-\tau) \geq x(t-2\tau) \geq x(t-3\tau) \geq \cdots \geq x(t_2)$$

implies that x is bounded from below by $b > 0$. Integrating (2.2) from t_2 to $t(> t_2)$, we obtain

$$z(t) - z(t_2) + \int_{t_2}^t q(\eta) G\big(x(\eta - \sigma)\big) d\eta = 0,$$

that is,

$$z(t) - z(t_2) + G(b) \int_{t_2}^t q(\eta) d\eta \leq 0.$$

Therefore,

$$z(t) \leq z(t_2) - G(b) \int_{t_2}^t q(\eta) d\eta \to -\infty,\ as\ t \to \infty,$$

due to (A_6), which is a contradiction because of boubded $z(t)$ (or $z(t) > 0$) on $[t_2, \infty)$. So, our claim holds. From (2.1), it follows that $z(t + \tau - \sigma) > p(t + \tau - \sigma)x(t - \sigma)$. Hence, (2.2) becomes

$$z'(t) + \frac{q(t)}{G(-p_1)} G\big(z(t + \tau - \sigma)\big) \leq 0. \tag{2.5}$$

Since z is decreasing on $[t_2, \infty)$, then

$$z'(t) + \frac{q(t)}{G(-p_1)} G\big(z(t)\big) \leq 0.$$

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Integrating the last inequality from t_2 to $t(> t_2)$, we get

$$\int_{t_2}^{t} \frac{z'(\eta)}{G\big(z(\eta)\big)} d\eta + \frac{1}{G(-p_1)} \int_{t_2}^{t} q(\eta) d\eta \geq 0,$$

that is,

$$\int_{t_2}^{t} q(\eta) d\eta \leq -G(-p_1) \int_{z(t_2)}^{z(t)} \frac{d\zeta}{G(\zeta)} < \infty, \quad \text{as } \ t \to \infty,$$

due to (A_5), we get a contradiction to (A_6). The case where x is eventually negative is very similar and we omit it here.

For the necessary part of the theorem, let (A_6) does't hold. Then it is possible to find a $t_1 > 0$ such that

$$\int_{t_1}^{\infty} q(\eta) d\eta < \frac{p_2 - 1}{2K},$$

where $K = \max\{K_1, K_2\}$ and K_1 is the Lipschitz constant of H on $[a, b]$ and $K_2 = G(b)$ such that

$$a = \frac{2\lambda p_2 - p_1(p_2 - 1)}{2p_1 p_2}$$

$$b = \frac{\lambda}{p_2 - 1}, \qquad \lambda > \frac{p_1(p_2 - 1)}{2p_2} > 0.$$

Let $Y = BC([t_2, \infty), \mathbb{R})$ be the space of real valued bounded continuous functions on $[t_2, \infty)$. Clearly, Y is a Banach space with respect to sup norm defined by

$$||y|| = \sup\{|y(t)| : t \geq t_2\}.$$

Define

$$S = \{u \in Y : \ a \leq u(t) \leq b, \ t \geq t_2\}.$$

It is easy to verify that S is a closed convex subspace of Y. Let $T : S \to S$ be such that

$$(Tx)(t) = \begin{cases} (Tx)(t_2 + \rho), & t \in [t_2, t_2 + \rho], \\ -\frac{x(t+\tau)}{p(t+\tau)} - \frac{\lambda}{p(t+\tau)} + \frac{1}{p(t+\tau)} \Big[\int_{t+\tau}^{\infty} q(\eta) G\big(x(\eta - \sigma)\big) d\eta \Big], & t \geq t_2 + \rho. \end{cases}$$

For every $x \in S$,

$$(Tx)(t) \leq -\frac{x(t+\tau)}{p(t+\tau)} - \frac{\lambda}{p(t+\tau)} \leq \frac{b + \lambda}{p_2} = \frac{\lambda}{p_2 - 1} = b$$

and

$$\begin{aligned}(Tx)(t) &\geq \frac{G(b)}{p(t+\tau)} \Big[\int_{t+\tau}^{\infty} q(\eta) d\eta \Big] - \frac{\lambda}{p(t+\tau)} \\ &\geq -\frac{1}{p_2} \frac{p_2 - 1}{2} + \frac{\lambda}{p_1} \\ &= \frac{2\lambda p_2 - p_1(p_2 - 1)}{2p_1 p_2} = a\end{aligned}$$

implies that $Tx \in S$. For $y_1, y_2 \in S$

$$\begin{aligned}|(Ty_1)(t) - (Ty_2)(t)| &\leq \frac{1}{|p(t+\tau)|} |y_1(t+\tau) - y_2(t+\tau)| \\ &+ \frac{K}{|p(t+\tau)|} \Big[\int_{t+\tau}^{\infty} q(\eta) |y_1(\eta - \sigma) - y_2(\eta - \sigma)| d\eta \Big],\end{aligned}$$

that is,

$$|(Ty_1)(t) - (Ty_2)(t)| \leq \frac{1}{p_2}||y_1 - y_2|| + \frac{K}{p_2}||y_1 - y_2||\Big[\int_{t+\tau}^{\infty} q(\eta)d\eta\Big]$$
$$< \Big(\frac{1}{p_2} + \frac{p_2 - 1}{2p_2}\Big)||y_1 - y_2||$$

implies that

$$||Ty_1 - Ty_2|| \leq \mu||y_1 - y_2||,$$

that is, T is a contraction, where $\mu = \Big(\frac{1}{p_2} + \frac{p_2-1}{2p_2}\Big) < 1$. Hence by the Banach's fixed point theorem, T has a unique fixed point which is a nonoscillatory solution of the equation (1.1) on $[a, b]$. Thus the proof of the theorem is complete. □

Theorem 2.7. *Let $-\infty < -p_1 \leq p(t) \leq -p_2 \leq -1$, $t \in \mathbb{R}_+$ and $p_1, p_2 > 0$. Assume that (A_2) holds. Furthermore, assume that G is Lipschitzian on intervals of the form $[a, b]$, $0 < a < b < \infty$. Then every bounded solutions of* (1.1) *oscillates if and only if* (A_6).

Proof. Suppose for contrary that $x(t)$ is bounded nonoscillatory solution of equation (1.1). We proceed as in the proof of the Theorem 2.5 to obtain $z(t)$ is monotonic on $[t_2, \infty)$, $t_2 > t_1$. Therefore the cases $z(t) > 0$ and $z(t) < 0$ for $t \geq t_2$ follows from the Theorem 2.6 and Theorem 2.5 respectively. Necessary part of the theorem follows from the Theorem 2.6. This completes the proof the theorem. □

Remark 2.8. *In Theorem 2.7, H could be linear, sublinear or superlinear.*

Theorem 2.9. *Let $0 \leq p(t) \leq p < \infty$, $t \in \mathbb{R}_+$ and $\tau \leq \sigma$. Assume that (A_1)–(A_4) holds. Then every solutions of* (1.1) *are oscillatory.*

Proof. On the contrary, we proceed as in the proof of the Theorem 2.5 to obtain $z(t)$ is monotonic on $[t_2, \infty)$, $t_2 > t_1$. Since $z(t) > 0$ for $t_2 > t_1$. From (2.2)

$$0 = z'(t) + q(t)G\big(x(t - \sigma)\big) + G(p)\left[z'(t - \tau) + q(t - \tau)G\big(x(t - \tau - \sigma)\big)\right]$$

for $t \geq t_2$ and because of (A_1), (A_2) and $z(t) \leq x(t) + px(t - \tau)$ we find that

$$0 \geq z'(t) + G(p)z'(t - \tau) + \lambda Q(t)\left[G\big(x(t - \sigma)\big) + G\big(rx(t - \tau - \sigma)\big)\right]$$
$$\geq z'(t) + G(p)z'(t - \tau) + \lambda Q(t)G\big(z(t - \sigma)\big).$$

Hence, there exists $t_3 > t_2$ such that

$$\frac{z'(t)}{G\big(z(t - \sigma)\big)} + G(p)\frac{z'(t - \tau)}{G\big(z(t - \sigma)\big)} + \lambda Q(t) \leq 0, \tag{2.6}$$

Since $z(t)$ is decreasing on $[t_3, \infty)$ and $\tau \leq \sigma$. Then the inequality (2.6) becomes

$$\frac{z'(t)}{G\big(z(t)\big)} + G(r)\frac{z'(t - \tau)}{G\big(z(t - \tau)\big)} + \lambda Q(t) \leq 0.$$

Note that $\lim_{t\to\infty} z(t)$ exists. Integrating the last inequality from t_3 to $t(> t_3)$, we get

$$\int_{t_3}^{t} \frac{z'(\eta)}{G\big(z(\eta)\big)}d\eta + G(p)\int_{t_3}^{t} \frac{z'(\eta - \tau)}{G\big(z(\eta - \tau)\big)}d\eta + \lambda\int_{t_3}^{t} Q(\eta)d\eta \leq 0,$$

that is,

$$\lambda\int_{t_3}^{t} Q(\eta)d\eta \leq -\Big[\int_{z(t_3)}^{z(t)} \frac{d\zeta}{G(\zeta)} + G(P)\int_{z(t_3-\tau)}^{z(t-\tau)} \frac{d\zeta}{G(\zeta)}\Big]$$
$$< \infty, \text{ as } t \to \infty,$$

due to (A_3), a contradiction to (A_4).

The case where x is eventually negative can be dealt similarly, and we omit the details here. This completes the proof. □

Theorem 2.10. *Let $0 \leq p(t) \leq p_3 < 1$, $t \in \mathbb{R}_+$. Let G be Lipschitzian on intervals of the form $[a,b]$, $0 < a < b < \infty$. Then every solution of (1.1) converges to zero as $t \to \infty$ if and only if (A_6) hold.*

Proof. Suppose that (A_6) holds. Let $x(t)$ be a solution of (1.1) on $[t_x, \infty]$, $t_x \geq 0$. Let the solution $x(t) > 0$ for $t \geq t_x$. Then proceeding as in Theorem 2.5, we have obtained (2.2) for $t \geq t_1 > t_0 + \sigma$, where $t_0 > \rho > t_x$. Since, $z(t) > 0$ for $t \geq t_2$. So, $\lim_{t\to\infty} z(t)$ exists. Consequently, $\lim_{t\to\infty} x(t)$ exists and $x(t)$ is bounded. We claim that $\liminf_{t\to\infty} x(t) = 0$. If not, then there exists $t_3 > t_2$ and $\alpha > 0$ such that $x(t-\sigma) \geq \alpha > 0$ for $t \geq t_3$. Ultimately,

$$\int_{t_3}^{t} Q(\eta) G\big(x(\eta-\sigma)\big) d\eta \geq G(\alpha) \int_{t_3}^{t} q(\eta) d\eta \to +\infty,$$

as $t \to \infty$, due to (A_6). On the other hand, we integrate (2.2) from t_3 to t to obtain

$$\big[z(\eta)\big]_{t_3}^{t} + \int_{t_3}^{t} q(\eta) G\big(x(\eta-\sigma)\big) d\eta = 0$$

and hence it follows that

$$\int_{t_3}^{t} q(\eta) G\big(x(\eta-\sigma)\big) d\eta = -\big[z(\eta)\big]_{t_3}^{t} < \infty,$$

as $t \to \infty$, we get a contradiction. So, our claim hold. Consequently, $\lim_{t\to\infty} z(t) = 0$ due to Lemma 2.3. As a result,

$$0 = \lim_{t\to\infty} z(t) = \limsup_{t\to\infty} \big[x(t) + p(t)x(t-\tau)\big] \geq \limsup_{t\to\infty} x(t)$$

implies that $\limsup_{t\to\infty} x(t) = 0$, that is, $\lim_{t\to\infty} x(t) = 0$. An equivalent procedure can be followed for $x(t) < 0$ for $t \geq t_x$ to show that $\lim_{t\to\infty} x(t) = 0$.

For the necessary part we suppose that (A_6) does't hold. Then it is possible to find a $t_1 > 0$ such that

$$\int_0^{\infty} q(\eta) d\eta < \frac{1-p_3}{10L},$$

where $L = \max\{L_1, H(1)\}$ and L_1 is the Lipschitz constant for H on $\left[\frac{2(1-p_3)}{5}, 1\right]$. For $t_2 > t_1$, we set $Y = BC([t_2, \infty), \mathbb{R})$, the space of real valued bounded continuous functions on $[t_2, \infty)$. Clearly, Y is a Banach space with respect to sup norm defined by

$$||y|| = \sup\{|y(t)| : t \geq t_2\}.$$

Let us define

$$S = \Big\{u \in Y : \ \frac{2}{5}(1-p_3) \leq u(t) \leq 1, \ t \geq t_2\Big\}.$$

Clearly, S is a closed and convex subspace of Y. Let $T : S \to S$ be defined by

$$(Tx)(t) = \begin{cases} (Tx)(t_2+\rho), & t \in [t_2, t_2+\rho], \\ -p(t)x(t-\tau) + \frac{2+3p_3}{5} + \int_t^{\infty} q(\eta) G\big(x(\eta-\sigma)\big) d\eta, & t \geq t_2+\rho. \end{cases}$$

For every $x \in S$,

$$(Tx)(t) \leq \frac{2+3p_3}{5} + G(1)\Big[\int_t^{\infty} q(\eta) d\eta\Big] < \frac{2+3p_3}{5} + \frac{1-p_3}{10} = \frac{1+p_3}{2} < 1$$

and

$$(Tx)(t) \geq -p(t)x(t-\tau) + \frac{2+3p_3}{5} \geq -p_3 + \frac{2+3p_3}{5} = \frac{2}{5}(1-p_3)$$

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implies that $Tx \in S$. Now, for $y_1, y_2 \in S$

$$|(Ty_1)(t) - (Ty_2)(t)| \leq |p(t)||y_1(t-\tau) - y_2(t-\tau)| + L_1 \int_t^\infty q(\eta)|y_1(\eta-\sigma) - y_2(\eta-\sigma)|d\eta,$$

that is,

$$|(Ty_1)(t) - (Ty_2)(t)| \leq p_3||y_1 - y_2|| + L_1||y_1 - y_2||\Big[\int_t^\infty q(\eta)d\eta\Big] < \Big(p_3 + \frac{1-p_3}{10}\Big)||y_1 - y_2||$$

implies that

$$||Ty_1 - Ty_2|| \leq \mu||y_1 - y_2||,$$

that is, T is a contraction mapping, where $\mu = p_3 + \frac{1-p_3}{10} = \frac{1+9p_3}{10} < 1$. Since S is complete and T is a contraction on S, then by the Banach's fixed point theorem T has a unique fixed point on $\big[\frac{2}{5}(1-p_3), 1\big]$. Hence $Tx = x$ and

$$x(t) = \begin{cases} x(t_2+\rho), & t \in [t_2, t_2+\rho], \\ -p(t)x(t-\tau) + \frac{2+3p_3}{5} + \int_t^\infty q(\eta)G\big(x(\eta-\sigma)\big)d\eta, & t \geq t_3+\rho \end{cases}$$

is a nonoscillatory solution of (1.1) on $\big[\frac{2}{5}(1-p_3), 1\big]$ such that $\lim_{t\to\infty} x(t) \neq 0$. This completes the proof of the theorem. □

Theorem 2.11. *Let $1 < p_4 \leq p(t) \leq p_5 < \infty$, $t \in \mathbb{R}_+$ and $p_4^2 > p_5$. Let G be Lipschitzian on intervals of the form $[a,b]$, $0 < a < b < \infty$. Then every solution of (1.1) converges to zero as $t \to \infty$ if and only if (A_6) hold.*

Proof. Sufficient part of the theorem follows from Theorem 2.10 and the necessary part of the theorem follows from the proof of the Theorem 2.6. But we need to mention the following:

$$\int_{t_1}^\infty q(\eta)d\eta < \frac{p_4-1}{2K},$$

where $K = \max\{K_1, K_2\}$ and K_1 is the Lipschitz constant of H on $[a,b]$, $K_2 = H(b)$ such that

$$a = \frac{2\lambda(p_4^2 - p_5) - p_5(p_4-1)}{2p_4^2 p_5}, \quad b = \frac{p_4 - 1 + 2\lambda}{2p_4} \quad \text{for} \quad \lambda > \frac{p_5(p_4-1)}{2(p_4^2 - p_5)} > 0,$$

and

$$(Tx)(t) = \begin{cases} Tx(t_2+\rho), & t \in [t_2, t_2+\rho], \\ -\frac{x(t+\tau)}{p(t+\tau)} + \frac{\lambda}{p(t+\tau)} + \frac{1}{p(t+\tau)}\Big[\int_{t+\tau}^\infty q(\eta)G\big(x(\eta-\sigma)\big)d\eta\Big], & t \geq t_2+\rho. \end{cases}$$

This completes the proof of the theorem. □

Remark 2.12. *In Theorem 2.10 and Theorem 2.11, G could be linear, sublinear or superlinear.*

3. Discussion and Examples

We could succeed to establish the necessary and sufficient conditions for oscillation of all solutions of (1.1) when $-\infty < p(t) \leq 0$. But, we failed to obtain the necessary and sufficient conditions for $0 \leq p(t) < \infty$. However, we established necessary and sufficient conditions for oscillations of solutions of (1.1) is either oscillates or converges to zero as $t \to \infty$ when $0 \leq p(t) < 1$ and $1 < p(t) < \infty$. Hence the undertaken problem is open for $0 \leq p(t) < \infty$. May be some other method is required to overcome the problem.

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Remark 3.1. *A prototype of the function G satisfying all the assumptions on G is*

$$\big(1+\alpha|u|^\beta\big)|u|^\gamma \mathrm{sgn}(u) \quad \text{for } u \in \mathbb{R},$$

where $\alpha \geq 1$ or $\alpha = 0$ and $\beta, \gamma > 0$ are reals. For verifying (A_1), we may take help of the well-known inequality (see [9, p. 292]*)*

$$u^p + v^p \geq h(p)(u+v)^p \quad \text{for } u, v > 0, \quad \text{where} \quad h(p) := \begin{cases} 1, & 0 \leq p \leq 1, \\ \dfrac{1}{2^{p-1}}, & p \geq 1. \end{cases}$$

Example 3.2. *Consider the differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - e^{\frac{3\pi}{2}} x\left(t - \frac{3\pi}{2}\right)\right] + 2e^{\pi} x(t-\pi) = 0, \tag{3.1}$$

where $\tau = \frac{3\pi}{2} > \sigma = \pi$. Clearly, all the conditions of Theorem 2.6 are satisfied. Hence, by Theorem 2.6 every solutions of (3.1) *oscillates. Indeed, $x(t) = e^t \sin(t)$ is such a solution of* (3.1)*.*

Example 3.3. *Consider the differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[x(t) + e^{-\pi} x(t-\pi)\Big] + 2e^{2t-6\pi}\big(x(t-2\pi)\big)^3 = 0, \tag{3.2}$$

where $0 < p(t) = e^{-\pi} < 1$ and $G(x) = x^3$. Clearly, all the conditions of Theorem 2.10 are satisfied. Hence, by Theorem 2.10 every solutions of (3.2) *converges to zero as $t \to \infty$. Indeed, $x(t) = e^{-t}$ is such a solution of* (3.2)*.*

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