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CONTINUITY OF SUPERPOSITION OPERATORS ON DOUBLE SEQUENCES SPACES OF MADDOX $C_{r0}(p)$

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ABSTRACT. Sağır and Güngör [15] defined the superposition operator P_g where $g : \mathbb{N}^2 \times \mathbb{R} \to \mathbb{R}$ by $P_g((x_{ks})) = g(k, s, x_{ks})$ for all real double sequences (x_{ks}). Chew & Lee [4] and Petranuarat & Kemprasit [12] characterized $P_g : c_0 \to l_1$ and $P_g : c_0 \to l_q$ where $1 \leq q < \infty$, respectively. Sağır and Güngör [16] gave the necessary and sufficient conditions for the continuity of the superposition operator P_g acting from the double sequences space C_{r0} into $\mathcal{L}_p$ where $1 \leq p < \infty$. In this study, we have generalized P_g acting from the double sequences space of Maddox $C_{r0}(p)$ into $\mathcal{L}(q)$ where $p = (p_{ks})$ and $q = (q_{ks})$ are bounded double sequences of positive numbers. The main aim of this study is to give the necessary and sufficient conditions for the continuity of $P_g : C_{r0}(p) \to \mathcal{L}(q)$.

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1. INTRODUCTION

Let $\mathbb{R}$ be the set of all real numbers, $\mathbb{N}$ be the set of all natural numbers, $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ and Ω denotes the space of all real double sequences which is the vector space with coordinatewise addition and scalar multiplication. Let any sequence $x = (x_{ks}) \in \Omega$. If for any $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $l \in \mathbb{R}$ such that $|x_{ks} - l| < \varepsilon$ for all $k, s \geq N$, then we call that the double sequence $x = (x_{ks})$ is convergent in the sense of Pringsheim and denoted by $p - \lim x_{ks} = l$. If the double sequence $x = (x_{ks})$ converges in the sense of Pringsheim and, in addition, the limits that $\lim_k x_{ks}$ and $\lim_s x_{ks}$ exist, then it is called regularly convergent and denoted by $r - \lim x_{ks}$. The space $C_{r0}(p)$ is defined by

$$C_{r0}(p) = \left\{ x = (x_{ks}) \in \Omega : r - \lim_{k,s\to\infty} |x_{ks}|^{p_{ks}} = 0 \right\}$$

where $p = (p_{ks})$ is a bounded sequence of positive numbers and $\|.\|_{C_{r0}(p)} : C_{r0}(p) \to \mathbb{R}$ is defined by

$$\|x\|_{C_{r0}(p)} = \sup_{k,s\in\mathbb{N}} |x_{ks}|^{\frac{p_{ks}}{M_1}} .$$

where $M_1 = \max\left\{1, \sup_{k,s\in\mathbb{N}} p_{ks}\right\}$. The Maddox space $M_u(p)$ is defined by

$$M_u(p) := \left\{ x = (x_{ks}) \in \Omega : \sup_{k,s\in\mathbb{N}} |x_{ks}|^{p_{ks}} < \infty \right\}$$

where $p=(p_{ks})$ is a bounded sequence of positive numbers. The function $\|.\|_{M_u(p)} : M_u(p) \to \mathbb{R}$ is defined by

$$\|x\|_{M_u(p)} = \sup |x_{ks}|^{\frac{p_{ks}}{M_1}}$$

where $M_1 = \max\left\{1, \sup_{k,s\in\mathbb{N}} p_{ks}\right\}$. The Maddox space $\mathcal{L}(q)$ is defined by

$$\mathcal{L}(q) = \left\{ x=(x_{ks}) \in \Omega : \sum_{k,s=1}^{\infty} |x_{ks}|^{q_{ks}} < \infty \right\}$$

where $q=(q_{ks})$ is a bounded sequence of positive numbers. Let $\|.\|_{\mathcal{L}(q)} : \mathcal{L}(q) \to \mathbb{R}$ is defined by

$$\|x\|_{\mathcal{L}(q)} = \sum_{k,s=1}^{\infty} |x_{ks}|^{\frac{q_{ks}}{M_2}}$$

where $M_2 = \max\left\{1, \sup_{k,s\in\mathbb{N}} q_{ks}\right\}$. Let $X \in \{C_{r0}(p), M_u(p), \mathcal{L}(q)\}$, then we can see easily show that the following properties hold:

$$\begin{aligned} \|x\|_X &\geq 0 \\ \|x\|_X &= 0 \Leftrightarrow x=0 \\ \|x\|_X &= \|-x\|_X \\ \|x+y\|_X &\leq \|x\|_X + \|y\|_X \end{aligned} \tag{1.1}$$

for all $x, y \in X$. If we take $d : X \times X \to \mathbb{R}$ defined by $d(x,y) = \|x-y\|_X$, then it follows from the above properties that d is a metric on X. The space $\mathcal{L}_p$ is defined by

$$\mathcal{L}_p := \left\{ x=(x_{ks}) \in \Omega : \sum_{k,s=1}^{\infty} |x_{ks}|^p < \infty \right\}$$

where $1 \leq p < \infty$. $\mathcal{L}_p$ is a Banach space with the norm $\|x\|_p = \left(\sum_{k,s=1}^{\infty} |x_{ks}|^p \right)^{\frac{1}{p}}$. It is known that $\mathcal{L}_1 \subset C_{r0}(p) \subset M_u(p)$ and $\mathcal{L}(q) \subset M_u(q)$. The sequence e^{ks} is defined as

$$e_{ij}^{ks} = \begin{cases} 1, & (k,s)=(i,j) \\ 0, & \text{otherwise} \end{cases}.$$

If we consider the sequence s_{nm} defined by $s_{nm} = \sum_{k=1}^{n}\sum_{s=1}^{m} x_{ks}$ $(n, m \in \mathbb{N})$, then the pair of $((x_{nm}), (s_{nm}))$ is called a double series. Also (x_{nm}) is called the general term of the series and (s_{nm}) is called the sequence of partial sums. Let v be convergence notions, i.e., in the sense of Pringsheim or regularly convergent. If the sequence of partial sums (s_{nm}) is convergent to a real number s in v-sense, i.e.

$$v - \lim_{n,m} \sum_{k=1}^{n}\sum_{s=1}^{m} x_{ks} = s,$$

then the series $((x_{nm}), (s_{nm}))$ is called $v-$convergent and the sum of the series equals to s. It's denoted by

$$\sum_{k,s=1}^{\infty} x_{ks} = s.$$

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It is known that if the series is $v-$convergent, then the $v-$limit of the general term of the series equals to zero. The remaining term of the series $\sum_{k=1}^{\infty}\sum_{s=1}^{\infty} x_{ks}$ is defined by

$$R_{nm} = \sum_{k=1}^{n-1}\sum_{s=m}^{\infty} x_{ks} + \sum_{k=n}^{\infty}\sum_{s=1}^{m-1} x_{ks} + \sum_{k=n}^{\infty}\sum_{s=m}^{\infty} x_{ks}. \tag{1.2}$$

We will denote the formula (1.2) briefly with

$$\sum_{\max\{k,s\}\geq N} x_{ks}$$

for $n = m = N$. It is known that if the series is $v-$convergent, then the $v-$limit of the remaining term of the series is zero. For more details on double sequences and series, one can referee [1],[2],[3],[8],[10],[11],[14],[18] and the references therein.

We extend the definition of superposition operator for the double sequences spaces as follows. Let X, Y be two double sequences spaces. A superposition operator P_g on X is a mapping from X into Ω defined by $P_g(x) = (g(k,s,x_{ks}))_{k,s=1}^{\infty}$ where the function $g : \mathbb{N}^2 \times \mathbb{R} \to \mathbb{R}$ satisfies
(1) $g(k,s,0) = 0$ for all $k,s \in \mathbb{N}$.
If $P_g(x) \in Y$ for all $x \in X$, we say that P_g acts from X into Y and write $P_g : X \to Y$ [15]. Moreover, we shall assume the additionally some of the following conditions:
(2) $g(k,s,.)$ is continuous for all $k,s \in \mathbb{N}$.
($2'$) $g(k,s,.)$ is bounded on every bounded subset of $\mathbb{R}$ for all $k,s \in \mathbb{N}$.
It is obvious that if the function $g(k,s,.)$ satisfies the propety (2), then g satisfies ($2'$).

Continuity of the superposition operators on sequences spaces are discussed by some authors [4], [5], [7], [9], [12], [13],[17]. In [4], Chew and Lee gave necessary and sufficient conditions for the continuity of the superposition operator acting from the sequences space c_0 into l_1. In [12], Petranuarat and Kemprasit characterized necessary and sufficient conditions for continuity of the superposition operator acting from the sequences space c_0 into l_q with $1 \leq q < \infty$. Sağır and Güngör [16] gave necessary and sufficient conditions for the continuity of the superposition operator acting from the double sequences space C_{r0} into $\mathcal{L}_q$ with $1 \leq q < \infty$.

In this paper, we characterize the superposition operator acting from the double sequences space of Maddox $C_{r0}(p)$ into $\mathcal{L}_1$ under the hypothesis that the function $g(k,s,.)$ satisfies ($2'$). We discuss the continuity of the superposition operator P_g by using the methods in [4], [12]. Then by using the methods developed in [12], we generalize our works as the superposition operator acting from the space $C_{r0}(p)$ into $\mathcal{L}(q)$ without assuming that the function $g(k,s,.)$ satisfies ($2'$).

2. SUPERPOSITION OPERATORS OF $C_{r0}(p)$ INTO $\mathcal{L}_1$

Theorem 2.1. *Assume $g : \mathbb{N}^2 \times \mathbb{R} \to \mathbb{R}$ satisfies ($2'$). Then $P_g : C_{r0}(p) \to \mathcal{L}_1$ if and only if there exist $\alpha > 0$ and $(c_{ks})_{k,s=1}^{\infty} \in \mathcal{L}_1$ such that*

$$|g(k,s,t)| \leq c_{ks} \text{ whenever } |t| \leq \alpha$$

for all $k,s \in \mathbb{N}$.

Proof. Assume that there exist $\alpha > 0$ and $(c_{ks})_{k,s=1}^{\infty} \in \mathcal{L}_1$ such that $|g(k,s,t)| \leq c_{ks}$ whenever $|t| \leq \alpha$ for all $k,s \in \mathbb{N}$. Let $x = (x_{ks}) \in C_{r0}(p)$. Hence $p - \lim |x_{ks}|^{p_{ks}} = 0$ and the limits that $\lim_{k\to\infty} |x_{ks}|^{p_{ks}}$ and

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$\lim_{s\to\infty} |x_{ks}|^{p_{ks}}$ exist. Therefore there exists $N \in \mathbb{N}$ such that $|x_{ks}| \le \alpha$ for all $k, s \in \mathbb{N}$ with $\max\{k, s\} \ge N$. Then, we find

$$\sum_{\max\{k,s\}\ge N} |g(k, s, x_{ks})| \le \sum_{\max\{k,s\}\ge N} c_{ks} \le \sum_{k,s=1}^{\infty} |c_{ks}| < \infty.$$

So, we get $P_g(x) = g(k, s, x_{ks}) \in \mathcal{L}_1$.

Conversely, suppose that $P_g : C_{r0}(p) \to \mathcal{L}_1$. The sets $A(\alpha)$ and $B(k, s, \alpha)$ are defined as

$$A(\alpha) = \left\{t \in \mathbb{R} : |t|^{\frac{p_{ks}}{M_1}} \le \min\left\{\alpha^{\frac{1}{M_1}}, \alpha^{\frac{p_{ks}}{M_1}}\right\}\right\}$$

and

$$B(k, s, \alpha) = \sup\{|g(k, s, t)| : t \in A(\alpha)\}$$

for all $k, s \in \mathbb{N}$ and $\alpha > 0$. So, we see that $|g(k, s, t)| \le B(k, s, \alpha)$ whenever $|t| \le \alpha$. We will show that there is $\alpha_1 > 0$ such that $(B(k, s, \alpha_1))_{k,s=1}^{\infty} \in \mathcal{L}_1$. Assume the contrary, that is, $\sum_{k,s=1}^{\infty} B(k, s, \alpha) = \infty$ for all $\alpha > 0$. Therefore $\sum_{k,s=1}^{\infty} B\left(k, s, \frac{1}{i} + \frac{1}{j}\right) = \infty$ for each $i, j \in \mathbb{N}$. Then there exist two sequences of positive integers $n_0 = 0 < n_1 < n_2 < \cdots < n_i < \cdots$ and $m_0 = 0 < m_1 < m_2 < \cdots < m_j < \cdots$ such that

$$\sum_{k=n_{i-1}+1}^{n_i} \sum_{s=m_{j-1}+1}^{m_j} B\left(k, s, \frac{1}{i} + \frac{1}{j}\right) > 1 \tag{2.1}$$

for each $i, j \in \mathbb{N}$. Let $i, j \in \mathbb{N}$ be fixed. Since g satisfies $(2')$, we see that $B\left(k, s, \frac{1}{i} + \frac{1}{j}\right) < \infty$ for all $i, j \in \mathbb{N}$ with $n_{i-1} + 1 \le k \le n_i$ and $m_{j-1} + 1 \le s \le m_j$. Then, there exists $x_{ks} \in A\left(\frac{1}{i} + \frac{1}{j}\right)$ such that

$$B\left(k, s, \frac{1}{i} + \frac{1}{j}\right) < |g(k, s, x_{ks})| + 2^{-(i+j)} \tag{2.2}$$

for each $k, s \in \mathbb{N}$ satisfying $n_{i-1} + 1 \le k \le n_i$ and $m_{j-1} + 1 \le s \le m_j$. So, we find

$$\begin{aligned} r^2 &< \sum_{i=1}^{r}\sum_{j=1}^{r}\left(\sum_{k=n_{i-1}+1}^{n_i} \sum_{s=m_{j-1}+1}^{m_j} B\left(k, s, \frac{1}{i} + \frac{1}{j}\right)\right) \\ &< \sum_{k=1}^{n_r}\sum_{s=1}^{m_r} |g(k, s, x_{ks})| + \sum_{k=1}^{n_r}\sum_{s=1}^{m_r} 2^{-(i+j)} \\ &< \sum_{k=1}^{n_r}\sum_{s=1}^{m_r} |g(k, s, x_{ks})| + \sum_{k=1}^{\infty}\sum_{s=1}^{\infty} 2^{-(i+j)} \end{aligned}$$

by using (2.1) and (2.2). Therefore we obtain that

$$\sum_{k=1}^{\infty}\sum_{s=1}^{\infty} |g(k, s, x_{ks})| = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\sum_{k=n_{i-1}+1}^{n_i} \sum_{s=m_{j-1}+1}^{m_j} |g(k, s, x_{ks})|\right) = \infty.$$

Hence we get $g(k, s, x_{ks}) \notin \mathcal{L}_1$. Since $x_{ks} \in A\left(\frac{1}{i} + \frac{1}{j}\right)$ whenever $n_{i-1} + 1 \le k \le n_i$ and $m_{j-1} + 1 \le s \le m_j$, we find $|x_{ks}|^{p_{ks}} \le \frac{1}{i} + \frac{1}{j}$. Hence, we obtain $x = (x_{ks}) \in C_{r0}(p)$. This contradicts the assumption that $P_g : C_{r0}(p) \to \mathcal{L}_1$. Then there exists $\alpha_1 > 0$ such that $(B(k, s, \alpha_1))_{k,s=1}^{\infty} \in \mathcal{L}_1$. If we put $c_{ks} = B(k, s, \alpha_1)$ for all $k, s \in \mathbb{N}$, this completes the proof. □

Theorem 2.2. *If $P_g : C_{r0}(p) \to \mathcal{L}_1$, then P_g is continuous on $C_{r0}(p)$ if and only if $g(k, s, .)$ is continuous on $\mathbb{R}$ for all $k, s \in \mathbb{N}$.*

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Proof. Suppose that P_g is continuous on $C_{r0}(p)$. Let $k, s \in \mathbb{N}$, $t_0 \in \mathbb{R}$ and $\varepsilon > 0$. Since P_g is continuous at $t_0 e^{(nm)} \in C_{r0}(p)$, there exists $\delta > 0$ such that

$$\left\| z - t_0 e^{(nm)} \right\|_{C_{r0}(p)} < \delta \text{ implies } \left\| P_g(z) - P_g\left(t_0 e^{(nm)}\right) \right\|_1 < \varepsilon \tag{2.3}$$

for all $z = (z_{ks}) \in C_{r0}(p)$. Let $t \in \mathbb{R}$ such that $|t - t_0| < \delta^{\frac{M_1}{p_{ks}}}$ and $y = (y_{ks})$ defined by

$$y_{ks} = \begin{cases} t, & (k,s) = (n,m) \\ 0, & \text{otherwise} \end{cases}.$$

So $y = (y_{ks}) \in C_{r0}(p)$ and we have $\left\| y - t_0 e^{(nm)} \right\|_{C_{r0}(p)} = |t - t_0|^{\frac{p_{ks}}{M_1}} < \delta$. From (2.3), we find

$$|g(k,s,t) - g(k,s,t_0)| = \left\| P_g(y) - P_g\left(t_0 e^{(nm)}\right) \right\|_1 < \varepsilon.$$

Therefore, the function $g(k,s,.)$ is continuous on $\mathbb{R}$ for each $k, s \in \mathbb{N}$.

Conversely, assume that the function $g(k,s,.)$ is continuous on $\mathbb{R}$ for each $k, s \in \mathbb{N}$. We will show that P_g is continuous on $C_{r0}(p)$. Let $x = (x_{ks}) \in C_{r0}(p)$ and $\varepsilon > 0$. Since g satisfies $(2')$, then P_g acts from $C_{r0}(p)$ to $\mathcal{L}_1$ by Theorem 2.1. Hence, there exist $\alpha > 0$ and $(c_{ks}) \in \mathcal{L}_1$ such that

$$|g(k,s,t)| \le c_{ks} \text{ whenever } |t| \le \alpha \tag{2.4}$$

for all $k, s \in \mathbb{N}$. Since $(x_{ks}) \in C_{r0}(p) \subset M_u(p)$ and $(c_{ks}) \in \mathcal{L}_1$, there exists $N \in \mathbb{N}$ such that

$$|x_{ks}| \le \frac{\alpha}{2} \text{ for all } k, s \in \mathbb{N} \text{ with } max\{k,s\} \ge N$$

and

$$\sum_{\max\{k,s\} \ge N} c_{ks} < \frac{\varepsilon}{3}.$$

So, $|x_{ks}| \le \alpha$ for all $k, s \in \mathbb{N}$ with $max\{k,s\} \ge N$. From (2.4), we write $|g(k,s,x_{ks})| \le c_{ks}$ for all $k, s \in \mathbb{N}$ with $max\{k,s\} \ge N$. Hence, we have

$$\sum_{\max\{k,s\} \ge N} |g(k,s,x_{ks})| \le \sum_{\max\{k,s\} \ge N} c_{ks} < \frac{\varepsilon}{3}. \tag{2.5}$$

Since $g(k,s,.)$ is continuous at x_{ks} for all $k, s \in \{1, 2, \ldots, N-1\}$, there exists $\delta > 0$ with $\delta = \min\left\{1, \left(\frac{\alpha}{2}\right)^{\frac{p_{ks}}{M_1}}\right\}$ such that

$$|t - x_{ks}| < \delta^{\frac{M_1}{p_{ks}}} \text{ implies } |g(k,s,t) - g(k,s,x_{ks})| < \frac{\varepsilon}{3(N-1)} \tag{2.6}$$

for any $t \in \mathbb{R}$. Let $z = (z_{ks}) \in C_{r0}(p)$ be such that $\|z - x\|_{C_{r0}(p)} < \delta$. Thus,

$$|z_{ks} - x_{ks}|^{\frac{p_{ks}}{M_1}} \le \sup_{k,s \in \mathbb{N}} |z_{ks} - x_{ks}|^{\frac{p_{ks}}{M_1}} = \|z - x\|_{C_{r0}(p)} < \delta$$

for each $k, s \in \mathbb{N}$. By using (2.6), we find

$$|g(k,s,z_{ks}) - g(k,s,x_{ks})| < \frac{\varepsilon}{3(N-1)}$$

for all $k, s \in \{1, 2, \ldots, N-1\}$. Hence, we have

$$\sum_{k,s=1}^{N-1} |g(k,s,z_{ks}) - g(k,s,x_{ks})| < \frac{\varepsilon}{3}. \tag{2.7}$$

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Since $|z_{ks}| \leq |z_{ks} - x_{ks}| + |x_{ks}| < \delta^{\frac{M_1}{p_{ks}}} + \frac{\alpha}{2} \leq \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha$ for all $k, s \in \mathbb{N}$ with $max\{k,s\} \geq N$, we find that $|g(k,s,z_{ks})| \leq c_{ks}$ for all $k, s \in \mathbb{N}$ with $max\{k,s\} \geq N$ from (2.4). Hence, we have

$$\sum_{\max\{k,s\} \geq N} |g(k,s,z_{ks})| \leq \sum_{\max\{k,s\} \geq N} c_{ks} < \frac{\varepsilon}{3}.$$

So, we obtain

$$\begin{aligned}
\|P_g(z) - P_g(x)\| &= \sum_{k,s=1}^{\infty} |g(k,s,z_{ks}) - g(k,s,x_{ks})| \\
&\leq \sum_{k,s=1}^{N-1} |g(k,s,z_{ks}) - g(k,s,x_{ks})| + \sum_{\max\{k,s\} \geq N} |g(k,s,z_{ks})| + \\
&\quad + \sum_{\max\{k,s\} \geq N} |g(k,s,x_{ks})| \\
&< \varepsilon
\end{aligned}$$

by using (2.5) and (2.7). This completes the proof. □

Example 2.3. Let $g : \mathbb{N}^2 \times \mathbb{R} \to \mathbb{R}$ be defined by

$$g(k,s,t) = \frac{|t|^{\frac{p_{ks}}{M_1}}}{4^{k+s}}$$

for all $k, s \in \mathbb{N}$ and for all $t \in \mathbb{R}$. Since $g(k,s,.)$ is continuous on $\mathbb{R}$ for all $k, s \in \mathbb{N}$, then g satisfies $(2')$. Let $\alpha = 1$ and $|t| \leq 1$. Then for all $k, s \in \mathbb{N}$,

$$\begin{aligned}
|g(k,s,t)| &= \frac{|t|^{\frac{p_{ks}}{M_1}}}{4^{k+s}} \\
&\leq \frac{1}{4^{k+s}}.
\end{aligned}$$

Since $\sum_{k,s=1}^{\infty} \frac{1}{4^{k+s}} < \infty$, we put $c_{ks} = \frac{1}{4^{k+s}}$ for all $k, s \in \mathbb{N}$. By Theorem 2.1, we find that $P_g : C_{r0}(p) \to \mathcal{L}_1$. Since $g(k,s,.)$ is continuous on $\mathbb{R}$ for all $k, s \in \mathbb{N}$, then the superposition operator P_g is continuous on $C_{r0}(p)$ by Theorem 2.2.

3. SUPERPOSITION OPERATORS OF $C_{r0}(p)$ INTO $\mathcal{L}(q)$

In this section, by using the methods developed in [12] we extend our theorems proved in Section 2 to the superposition operator acting from the space $C_{r0}(p)$into $\mathcal{L}(q)$ where $p = (p_{ks})$ and $q = (q_{ks})$ are bounded double sequences of positive numbers . For characterization of the superposition operator $P_g : C_{r0}(p) \to \mathcal{L}(q)$, we will use the following proposition.

Proposition 3.1. *Let X be a double sequences space. If $\mathcal{L}_1 \subseteq X$ and $P_g : X \to M_u(q)$, then there exist $N \in \mathbb{N}$ and $\alpha > 0$ such that $(g(k,s,.))_{\max\{k,s\} \geq N}^{\infty}$ is uniformly bounded on $[-\alpha, \alpha]$ ([6]).*

Theorem 3.2. *Let $g : \mathbb{N}^2 \times \mathbb{R} \to \mathbb{R}$. Then $P_g : C_{r0}(p) \to \mathcal{L}(q)$ if and only if there exist $N \in \mathbb{N}$ and $\alpha > 0$ such that*

$$\sum_{\max\{k,s\} \geq N} \sup_{|t| \leq \alpha^{\frac{1}{p_{ks}}}} |g(k,s,t)|^{\frac{q_{ks}}{M_2}} < \infty. \tag{3.1}$$

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Proof. Suppose that P_g acts from $C_{r0}(p)$ to $\mathcal{L}(q)$. Since $\mathcal{L}_1 \subset C_{r0}(p)$ and $\mathcal{L}(q) \subset M_u(q)$, by Proposition 3.1 we see that there exist $N \in \mathbb{N}$ and $\alpha > 0$ such that $(g(k,s,.))_{\max\{k,s\}\geq N}^{\infty}$ is uniformly bounded on $\left[-\alpha^{\frac{1}{p_{ks}}}, \alpha^{\frac{1}{p_{ks}}}\right]$. Therefore, $\sup_{|t|\leq \alpha^{\frac{1}{p_{ks}}}} |g(k,s,t)|^{\frac{q_{ks}}{M_2}} < \infty$ for all $k,s \in \mathbb{N}$ with $\max\{k,s\} \geq N$. We define $B(k,s,\beta)$ by

$$B(k,s,\beta) = \sup_{|t|\leq\beta} |g(k,s,t)|^{\frac{q_{ks}}{M_2}} \tag{3.2}$$

for all $\beta \in \mathbb{R}$ with $0 < \beta \leq \alpha^{\frac{1}{p_{ks}}}$. We assert that $\sum_{\max\{k,s\}\geq N} B(k,s,\beta) < \infty$ for some $\beta \in \mathbb{R}$ with $0 < \beta \leq \alpha^{\frac{1}{p_{ks}}}$. To show that this is the case, we assume the contrary. Therefore, $\sum_{\max\{k,s\}\geq N} B\left(k,s,\alpha^{\frac{1}{p_{ks}}}\left(\frac{1}{i}+\frac{1}{j}\right)\right) = \infty$ for all $i,j \in \mathbb{N}$. Hence, there exist $n' > n$ and $m' > m$ such that

$$\sum_{k=n}^{n'}\sum_{s=1}^{m'-1} B\left(k,s,\alpha^{\frac{1}{p_{ks}}}\left(\frac{1}{i}+\frac{1}{j}\right)\right) + \sum_{k=1}^{n'-1}\sum_{s=m}^{m'} B\left(k,s,\alpha^{\frac{1}{p_{ks}}}\left(\frac{1}{i}+\frac{1}{j}\right)\right) + \sum_{k=n}^{n'}\sum_{s=m}^{m'} B\left(k,s,\alpha^{\frac{1}{p_{ks}}}\left(\frac{1}{i}+\frac{1}{j}\right)\right) > 1$$

for all $i,j \in \mathbb{N}$ and $n,m \geq N$. Then, there exist two subsequences $(n_k)_{k=1}^{\infty}$ of $(n)_{n=1}^{\infty}$ and $(m_k)_{k=1}^{\infty}$ of $(m)_{m=1}^{\infty}$ such that

$$\begin{aligned} &\sum_{k=n_i+1}^{n_{i+1}}\sum_{s=1}^{m_{j+1}-1} B\left(k,s,\alpha^{\frac{1}{p_{ks}}}\left(\frac{1}{i}+\frac{1}{j}\right)\right) + \sum_{k=1}^{n_{i+1}-1}\sum_{s=m_j+1}^{m_{j+1}} B\left(k,s,\alpha^{\frac{1}{p_{ks}}}\left(\frac{1}{i}+\frac{1}{j}\right)\right) + \\ &+ \sum_{k=n_i+1}^{n_{i+1}}\sum_{s=m_j+1}^{m_{j+1}} B\left(k,s,\alpha^{\frac{1}{p_{ks}}}\left(\frac{1}{i}+\frac{1}{j}\right)\right) \\ > &\ 1 \end{aligned}$$

for all $i,j \in \mathbb{N}$ and $n > n_1$, $m > m_1$. We put $\mathcal{F} = \{(k,s) : k \leq n_1 \text{ and } s \leq m_1\}$. If $(k,s) \in \mathcal{F}$, we take $x_{ks} = 0$. If $k > n_1$ and $s > m_1$, then there exist $i \in \mathbb{N}$ and $j \in \mathbb{N}$ such that $n_i < k \leq n_{i+1}$ and $m_j < s \leq m_{j+1}$. Hence, there exists $x_{ks} \in \left[-\alpha^{\frac{1}{p_{ks}}}\left(\frac{1}{i}+\frac{1}{j}\right), \alpha^{\frac{1}{p_{ks}}}\left(\frac{1}{i}+\frac{1}{j}\right)\right]$ such that

$$0 \leq B\left(k,s,\alpha^{\frac{1}{p_{ks}}}\left(\frac{1}{i}+\frac{1}{j}\right)\right) < |g(k,s,x_{ks})|^{\frac{q_{ks}}{M_2}} + 2^{-(k+s)} \tag{3.3}$$

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from (3.2). Therefore, it is obvious that $x_{ks}\in C_{r0}(p)$. By using (3.3), we write

$$
\begin{aligned}
r^2 &< \sum_{i=1}^{r}\sum_{j=1}^{r}\left(\sum_{k=n_i+1}^{n_{i+1}}\sum_{s=1}^{m_{j+1}-1} B\left(k,s,\alpha^{\frac{1}{p_{ks}}}\left(\frac{1}{i}+\frac{1}{j}\right)\right)+\sum_{k=1}^{n_{i+1}-1}\sum_{s=m_j+1}^{m_{j+1}} B\left(k,s,\alpha^{\frac{1}{p_{ks}}}\left(\frac{1}{i}+\frac{1}{j}\right)\right)+\right.\\
&\left.+\sum_{k=n_i+1}^{n_{i+1}}\sum_{s=m_j+1}^{m_{j+1}} B\left(k,s,\alpha^{\frac{1}{p_{ks}}}\left(\frac{1}{i}+\frac{1}{j}\right)\right)\right)\\
&= \sum_{k=n_1+1}^{n_{r+1}}\sum_{s=1}^{m_{r+1}-1} B\left(k,s,\alpha^{\frac{1}{p_{ks}}}\left(\frac{1}{i}+\frac{1}{j}\right)\right)+\sum_{k=1}^{n_{r+1}-1}\sum_{s=m_1+1}^{m_{r+1}} B\left(k,s,\alpha^{\frac{1}{p_{ks}}}\left(\frac{1}{i}+\frac{1}{j}\right)\right)\\
&+\sum_{k=n_1+1}^{n_{r+1}}\sum_{s=m_1+1}^{m_{r+1}} B\left(k,s,\alpha^{\frac{1}{p_{ks}}}\left(\frac{1}{i}+\frac{1}{j}\right)\right)\\
&< \sum_{k=n_1+1}^{n_{r+1}}\sum_{s=1}^{m_{r+1}-1}\left|g(k,s,x_{ks})\right|^{\frac{q_{ks}}{M_2}}+\sum_{k=1}^{n_{r+1}-1}\sum_{s=m_1+1}^{m_{r+1}}\left|g(k,s,x_{ks})\right|^{\frac{q_{ks}}{M_2}}+\sum_{k=n_1+1}^{n_{r+1}}\sum_{s=m_1+1}^{m_{r+1}}\left|g(k,s,x_{ks})\right|^{\frac{q_{ks}}{M_2}}+\\
&+\sum_{k=1}^{\infty}\sum_{s=1}^{\infty}2^{-(k+s)}.
\end{aligned}
$$

for all $r\in\mathbb{N}$. Hence, $(g(k,s,x_{ks}))_{k,s=1}^{\infty}\notin\mathcal{L}(q)$. This is a contradiction, because of $P_g: C_{r0}(p)\to\mathcal{L}(q)$.

Conversely, suppose that there exist $N\in\mathbb{N}$ and $\alpha>0$ such that

$$\sum_{\max\{k,s\}\geq N}\sup_{|t|\leq\alpha^{\frac{1}{p_{ks}}}}\left|g(k,s,t)\right|^{\frac{q_{ks}}{M_2}}<\infty.$$

To show that $P_g: C_{r0}(p)\to\mathcal{L}(q)$, let $x=(x_{ks})\in C_{r0}(p)$. Since $r-\lim|x_{ks}|^{p_{ks}}=0$, there exists $N'\geq N$ such that $|x_{ks}|\leq\alpha^{\frac{1}{p_{ks}}}$ for all $k,s\in\mathbb{N}$ with $\max\{k,s\}\geq N'$. Therefore, we find

$$\sum_{\max\{k,s\}\geq N'}\left|g(k,s,x_{ks})\right|^{\frac{q_{ks}}{M_2}}\leq\sum_{\max\{k,s\}\geq N'}\sup_{|t|\leq\alpha^{\frac{1}{p_{ks}}}}\left|g(k,s,t)\right|^{\frac{q_{ks}}{M_2}}<\infty.$$

Thus, we get $P_g(x)=g(k,s,x_{ks})\in\mathcal{L}(q)$. □

We need the following proposition to show the continuity of the superposition operator $P_g: C_{r0}(p)\to\mathcal{L}(q)$.

Proposition 3.3. *Let X be a double sequences space containing all finite double sequences, Y be a double sequences space such that $Y\subseteq M_u(q)$ and $\|.\|_X: X\to\mathbb{R}$, $\|.\|_Y: Y\to\mathbb{R}$ satisfy the conditions in (1.1). Suppose that*
(i) $P_g: X\to Y$,
(ii) there exist $\alpha>0$ such that $\|e^{mn}\|_X\leq\alpha$ for all $m,n\in\mathbb{N}$ and $a\in\mathbb{R}$ with $0<a\leq 1$ such that $\|\lambda x\|_X=|\lambda|^a\|x\|_X$ for all $\lambda\in\mathbb{R}$.
(iii) $\|.\|_{M_u(q)}\leq\beta\|.\|_Y$ on Y for some $\beta>0$.
If P_g is continuous at x, then for any $\varepsilon>0$ there exists $\delta>0$ such that

$$|t-x_{ks}|<\delta \text{ implies } |g(k,s,t)-g(k,s,x_{ks})|<\varepsilon$$

for all $k,s\in\mathbb{N}$ and $t\in\mathbb{R}$ ([6]).

Theorem 3.4. *If $P_g: C_{r0}(p)\to\mathcal{L}(q)$, then P_g is continuous on $C_{r0}(p)$ if and only if $g(k,s,.)$ is continuous on $\mathbb{R}$ for all $k,s\in\mathbb{N}$.*

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Proof. Since the conditions in Proposition 3.3 provided, it's not hard to see that the condition is necessary.

Conversely, let any $x=(x_{ks})\in C_{r0}(p)$ and assume that $g(k,s,.)$ is continuous at x_{ks} for all $k,s\in\mathbb{N}$. Hence, by Theorem 3.2 there exist $N_1\in\mathbb{N}$ and $\alpha>0$ such that

$$\sum_{\max\{k,s\}\geq N_1}\sup_{|t|\leq\alpha^{\frac{1}{p_{ks}}}}|g(k,s,t)|^{\frac{q_{ks}}{M_2}}<\infty. \tag{3.4}$$

Since $x=(x_{ks})\in C_{r0}(p)$, there exists $N_2\geq N_1$ such that $|x_{ks}|\leq\frac{\alpha^{\frac{1}{p_{ks}}}}{2}$ for all $k,s\in\mathbb{N}$ with $\max\{k,s\}\geq N_2$. Let $\varepsilon>0$. From (3.4), we see that

$$\sum_{k=1}^{N_1-1}\sum_{s=N_1}^{\infty}\sup_{|t|\leq\alpha^{\frac{1}{p_{ks}}}}|g(k,s,t)|^{\frac{q_{ks}}{M_2}}<\infty,\ \sum_{k=N_1}^{\infty}\sum_{s=1}^{N_1-1}\sup_{|t|\leq\alpha^{\frac{1}{p_{ks}}}}|g(k,s,t)|^{\frac{q_{ks}}{M_2}}<\infty,\ \sum_{k=N_1}^{\infty}\sum_{s=N_1}^{\infty}\sup_{|t|\leq\alpha^{\frac{1}{p_{ks}}}}|g(k,s,t)|^{\frac{q_{ks}}{M_2}}<\infty.$$

Therefore, there exists $N\in\mathbb{N}$ with $N\geq N_2$ such that

$$\sum_{k=1}^{N_1-1}\sum_{s=N}^{\infty}\sup_{|t|\leq\alpha^{\frac{1}{p_{ks}}}}|g(k,s,t)|^{\frac{q_{ks}}{M_2}}<\frac{\varepsilon}{3.2^{\frac{q_{ks}}{M_2}+1}}$$

$$\sum_{k=N}^{\infty}\sum_{s=1}^{N_1-1}\sup_{|t|\leq\alpha^{\frac{1}{p_{ks}}}}|g(k,s,t)|^{\frac{q_{ks}}{M_2}}<\frac{\varepsilon}{3.2^{\frac{q_{ks}}{M_2}+1}}$$

$$\sum_{k=N_1}^{N-1}\sum_{s=N}^{\infty}\sup_{|t|\leq\alpha^{\frac{1}{p_{ks}}}}|g(k,s,t)|^{\frac{q_{ks}}{M_2}}+\sum_{k=N}^{\infty}\sum_{s=N_1}^{N-1}\sup_{|t|\leq\alpha^{\frac{1}{p_{ks}}}}|g(k,s,t)|^{\frac{q_{ks}}{M_2}}+\sum_{k=N}^{\infty}\sum_{s=N}^{\infty}\sup_{|t|\leq\alpha^{\frac{1}{p_{ks}}}}|g(k,s,t)|^{\frac{q_{ks}}{M_2}}<\frac{\varepsilon}{3.2^{\frac{q_{ks}}{M_2}+1}}.$$

Consequently, we obtain that there exists $N\in\mathbb{N}$ with $N\geq N_2$ such that

$$\sum_{\max\{k,s\}\geq N}\sup_{|t|\leq\alpha^{\frac{1}{p_{ks}}}}|g(k,s,t)|^{\frac{q_{ks}}{M_2}}<\frac{\varepsilon}{2^{\frac{q_{ks}}{M_2}+1}}. \tag{3.5}$$

Since $g(k,s,.)$ is continuous at x_{ks} for all $k,s\in\{1,2,...,N-1\}$, there is $\delta\in\mathbb{R}$ with $0<\delta\leq\left(\frac{\alpha}{2^{p_{ks}}}\right)^{\frac{1}{M_1}}$ such that

$$|g(k,s,t)-g(k,s,x_{ks})|<\left[\frac{\varepsilon}{2(N-1)}\right]^{\frac{q_{ks}}{M_2}}\quad\text{whenever}\quad|t-x_{ks}|<\delta^{\frac{M_1}{p_{ks}}}. \tag{3.6}$$

Let $z=(z_{ks})\in C_{r0}(p)$ satisfying $\|z-x\|_{C_{r0}(p)}<\delta$. Thus, $|z_{ks}-x_{ks}|^{\frac{p_{ks}}{M_1}}\leq\|z-x\|_{C_{r0}(p)}<\delta$. From (3.6), we find $|g(k,s,z_{ks})-g(k,s,x_{ks})|^{\frac{q_{ks}}{M_2}}<\frac{\varepsilon}{2(N-1)}$ for all $k,s\in\{1,2,...,N-1\}$. We write $|z_{ks}|\leq|z_{ks}-x_{ks}|+|x_{ks}|<\delta^{\frac{M_1}{p_{ks}}}+\frac{\alpha^{\frac{1}{p_{ks}}}}{2}\leq\frac{\alpha^{\frac{1}{p_{ks}}}}{2}+\frac{\alpha^{\frac{1}{p_{ks}}}}{2}=\alpha^{\frac{1}{p_{ks}}}$ for all $k,s\in\mathbb{N}$ with $\max\{k,s\}\geq N$. We have

$$\begin{aligned}|g(k,s,z_{ks})-g(k,s,x_{ks})|^{\frac{q_{ks}}{M_2}}&\leq 2^{\frac{q_{ks}}{M_2}}\max\left\{|g(k,s,z_{ks})|^{\frac{q_{ks}}{M_2}},|g(k,s,x_{ks})|^{\frac{q_{ks}}{M_2}}\right\}\\&\leq 2^{\frac{q_{ks}}{M_2}}\sup_{|t|\leq\alpha^{\frac{1}{p_{ks}}}}|g(k,s,t)|^{\frac{q_{ks}}{M_2}}\end{aligned}$$

for all $k,s\in\mathbb{N}$ with $\max\{k,s\}\geq N$. By using (3.5), we obtain

$$\sum_{\max\{k,s\}\geq N}|g(k,s,z_{ks})-g(k,s,x_{ks})|^{\frac{q_{ks}}{M_2}}\leq 2^{\frac{q_{ks}}{M_2}}\sum_{\max\{k,s\}\geq N}\sup_{|t|\leq\alpha^{\frac{1}{p_{ks}}}}|g(k,s,t)|^{\frac{q_{ks}}{M_2}}<\frac{\varepsilon}{2}.$$

Therefore,

$$\begin{aligned}\sum_{k,s=1}^{\infty}\left|g\left(k,s,z_{ks}\right)-g\left(k,s,x_{ks}\right)\right|^{\frac{q_{ks}}{M_2}} &= \sum_{k,s=1}^{N-1}\left|g\left(k,s,z_{ks}\right)-g\left(k,s,x_{ks}\right)\right|^{\frac{q_{ks}}{M_2}}+\\ &+\sum_{\max\{k,s\}\geq N}\left|g\left(k,s,z_{ks}\right)-g\left(k,s,x_{ks}\right)\right|^{\frac{q_{ks}}{M_2}}\\ &< (N-1)\frac{\varepsilon}{2(N-1)}+\frac{\varepsilon}{2}<\varepsilon.\end{aligned}$$

Hence, we get $\left\|P_g(z)-P_g(x)\right\|_{\mathcal{L}(q)}=\sum_{k,s=1}^{\infty}\left|g\left(k,s,z_{ks}\right)-g\left(k,s,x_{ks}\right)\right|^{\frac{q_{ks}}{M_2}}<\varepsilon$. This completes the proof.

□

Example 3.5. Let $g:\mathbb{N}^2\times\mathbb{R}\to\mathbb{R}$ defined by

$$g(k,s,t)=\left(\frac{|t|^{p_{ks}}}{2^{k+s}}\right)^{\frac{M_2}{q_{ks}}}$$

for all $k,s\in\mathbb{N}$ and for all $t\in\mathbb{R}$. Let $\alpha=2$ and $|t|\leq 2^{\frac{1}{p_{ks}}}$. Then for all $k,s\in\mathbb{N}$,

$$\sum_{\max\{k,s\}\geq N}\sup_{|t|\leq 2^{\frac{1}{p_{ks}}}}|g(k,s,t)|^{\frac{q_{ks}}{M_2}}=\sum_{\max\{k,s\}\geq N}\sup_{|t|\leq 2^{\frac{1}{p_{ks}}}}\frac{|t|^{p_{ks}}}{2^{k+s}}\leq\sum_{\max\{k,s\}\geq N}\frac{2}{2^{k+s}}\leq\sum_{k,s=1}^{\infty}\frac{2}{2^{k+s}}<\infty.$$

By Theorem 3.2, we find that $P_g:C_{r0}(p)\to\mathcal{L}(q)$. Since $g(k,s,.)$ is continuous and bounded on $\mathbb{R}$ for all $k,s\in\mathbb{N}$, then the superposition operator P_g is continuous on $C_{r0}(p)$ by Theorem 3.4.

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