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Spectral Results on Some Hamiltonian Properties of Graphs

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Abstract

Using Lotker's interlacing theorem on the Laplacian eigenvalues of a graph in [5] and Wang and Belardo's interlacing theorem on the signless Laplacian eigenvalues of a graph in [6], we in this note obtain spectral conditions for some Hamiltonian properties of graphs.

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We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. For a graph $G = (V, E)$, we use n and e to denote its order $|V|$ and size $|E|$, respectively. If u is a vertex of G, then $G - u$ is defined as the subgraph of G obtained from G by deleting u together with its incident edges. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path. A graph G is called Hamilton-connected if for each pair of vertices in G there is a Hamiltonian path between them. A cycle C in a graph G is said to be dominating if $V(G) - V(C)$ is independent. The connectivity and independence number of a graph G are denoted by $\kappa(G)$ and $\alpha(G)$, respectively. For a graph G of order n, We use $A(G)$ to denote the adjacency matrix of G and $D(G)$ to denote the diagonal matrix of the degree sequence of G. The Laplacian of G is defined as $L(G) = D(G) - A(G)$ and the signless Laplacian of G is defined as $Q(G) = D(G) + A(G)$. The eigenvalues $\mu_1(A) \leq \mu_2(A) \leq ... \leq \mu_n(A)$ of $A(G)$ are called the eigenvalues of G. The eigenvalues $\lambda_1(G) \geq \lambda_2(G) \geq ... \geq \lambda_n(G) = 0$ of $L(G)$ are called the Laplacian eigenvalues of G and the eigenvalues $q_1(G) \geq q_2(G) \geq ... \geq q_n(G)$ of $Q(G)$ are called the signless Laplacian eigenvalues of G.

The following interlacing theorem on the Laplacian eigenvalues of graphs was proved by Lotker in [5].

Theorem 1 ([5]) Let G be a graph of order n. If u is vertex in G, then

$$\lambda_{i+1}(G) - 1 \leq \lambda_i(G - u) \leq \lambda_i(G), where\ \ i = 1, 2, \cdots, (n - 1).$$

Using Theorem 1, Li in [4] obtained the following result.

Theorem 2 ([4]) Let G be a graph of order n. If G has an independent set I, then $|I| + \lambda_{n-|I|+1} \leq n$.

For the sake of completeness, we repeat the proofs in [4] below.

Proof of Theorem 2. Suppose $I = \{ v_1, v_2, \ldots v_r \}$ is an independent set in G. Set $N := V(G) - I = \{ u_1, u_2, \ldots u_s \}$. Then $s = n - r$. By Theorem 1, we have

$$\lambda_s(G - u_1) \geq \lambda_{s+1}(G) - 1,$$

$$\lambda_{s-1}(G - u_1 - u_2) \geq \lambda_s(G - u_1) - 1,$$

$$\lambda_{s-2}(G - u_1 - u_2 - u_3) \geq \lambda_{s-1}(G - u_1 - u_2) - 1,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\lambda_{s-(s-1)}(G - u_1 - u_2 - ... - u_s) \geq \lambda_{s-(s-2)}(G - u_1 - u_2 - ... - u_{s-1}) - 1.$$

Summing up the inequalities above, we have

$$\lambda_1(G - u_1 - u_2 - u_3 - ... - u_s) \geq \lambda_{s+1}(G) - s = \lambda_{s+1}(G) - (n - r).$$

Since there is no edge in the graph $(G - u_1 - u_2 - u_3 ... - u_s)$, $\lambda_1(G - u_1 - u_2 - u_3 - ... - u_k) = 0$. Thus $r + \lambda_{s+1} \leq n$. Namely, $|I| + \lambda_{n-|I|+1} \leq n$. □

The following interlacing theorem on the signless Laplacian eigenvalues of graphs was proved by Wang and Belardo in [6].

Theorem 3 ([6]) Let G be a graph of order n. If u is vertex in G, then

$$q_{i+1}(G) - 1 \leq q_i(G - u) \leq q_i(G), where \ \ i = 1, 2, \cdots, (n - 1).$$

Using Theorem 3, We can similarly prove the following theorem.

Theorem 4 Let G be a graph of order n. If G has an independent set I, then $|I| + q_{n-|I|+1} \leq n$.

In this note, we will use Theorem 2 and Theorem 4 to prove the following theorems on the Hamiltonian properties of graphs.

Theorem 5 Let G be a graph of order n with connectivity κ, Laplacian eigenvalues $\lambda_1(G) \geq \lambda_2(G) \geq ... \geq \lambda_n(G) = 0$, and signless Laplacian eigenvalues $q_1(G) \geq q_2(G) \geq ... \geq q_n(G)$.

(1) If $n \leq \kappa + \lambda_{n-\kappa}$, then G is Hamiltonian.

(2) If $n \leq \kappa + q_{n-\kappa}$, then G is Hamiltonian.

Remark 1 Let G be the non-Hamiltonian complete bipartite graph $K_{r,r+1}$ $(r \geq 2)$. Notice that $\kappa(G) = r$ and $\lambda_{n-\kappa} = q_{n-\kappa} = r$. Thus $n - 1 = 2r \leq \kappa + \lambda_{n-\kappa} = \kappa + q_{n-\kappa}$. Therefore (1) and (2) in Theorem 5 are best possible.

Theorem 6 Let G be a graph of order n with connectivity κ, Laplacian eigenvalues $\lambda_1(G) \geq \lambda_2(G) \geq ... \geq \lambda_n(G) = 0$, and signless Laplacian eigenvalues $q_1(G) \geq q_2(G) \geq ... \geq q_n(G)$.

(1) If $n \leq \kappa + \lambda_{n-\kappa-1} + 1$, then G is traceable.

(2) If $n \leq \kappa + q_{n-\kappa-1} + 1$, then G is traceable.

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Remark 2 Let G be the non-traceable complete bipartite graph $K_{r,r+2}$ $(r \geq 1)$. Notice that $\kappa(G) = r$ and $\lambda_{n-\kappa-1} = q_{n-\kappa-1} = r$. Thus $n - 1 = 2r + 1 \leq \kappa + \lambda_{n-\kappa-1} + 1 = \kappa + q_{n-\kappa-1} + 1$. Therefore (1) and (2) in Theorem 6 are best possible.

Theorem 7 Let G be a graph of order n with connectivity κ, Laplacian eigenvalues $\lambda_1(G) \geq \lambda_2(G) \geq ... \geq \lambda_n(G) = 0$, and signless Laplacian eigenvalues $q_1(G) \geq q_2(G) \geq ... \geq q_n(G)$.

(1) If $n \leq \kappa + \lambda_{n-\kappa+1} - 1$, then G is Hamilton-connected.

(2) If $n \leq \kappa + q_{n-\kappa+1} - 1$, then G is Hamilton-connected.

Remark 3 Let G be the non-Hamilton-connected complete bipartite graph $K_{r,r}$ graph $(r \geq 3)$. Notice that $\kappa(G) = r$ and $\lambda_{n-\kappa+1} = q_{n-\kappa+1} = r$. Thus $n - 1 = 2r - 1 \leq \kappa + \lambda_{n-\kappa+1} - 1 = \kappa + q_{n-\kappa+1} - 1$. Therefore (1) and (2) in Theorem 7 are best possible.

Theorem 8 Let G be a 2-connected triangle-free graph of order n with connectivity κ, Laplacian eigenvalues $\lambda_1(G) \geq \lambda_2(G) \geq ... \geq \lambda_n(G) = 0$, and signless Laplacian eigenvalues $q_1(G) \geq q_2(G) \geq ... \geq q_n(G)$.

(1) If $n \leq 2\kappa + \lambda_{n-2\kappa+2} - 2$, then every longest cycle in G is dominating.

(2) If $n \leq 2\kappa + q_{n-2\kappa+2} - 2$, then every longest cycle in G is dominating.

We need the following results to prove our theorems. Notice that Theorem 9 below can be proved by using slight modifications of the proofs in [2].

Theorem 9 ([2]) Let G be a graph of order n with connectivity κ and independence number α.

(1) If $\alpha \leq \kappa$, then G is Hamiltonian.

(2) If $\alpha \leq \kappa + 1$, then G is traceable.

(3) If $\alpha \leq \kappa - 1$, then G is Hamilton-connected.

Theorem 10 ([3]) Let G be a 2-connected triangle-free graph of order n with connectivity κ and independence number α. If $\alpha \leq 2\kappa - 2$, then every longest cycle in G is dominating.

Proof of Theorem 5. Let G be a graph satisfying the conditions in Theorem 5.

(1) If $\alpha \leq \kappa$, then, by (1) in Theorem 9, G is Hamiltonian. If $\alpha \geq \kappa + 1$, then there exists an independent set S in G such that $|S| = \kappa + 1$. Applying Theorem 2, we have that $\kappa+1+\lambda_{n-\kappa} = \kappa+1+\lambda_{n-(\kappa+1)+1} = |S|+\lambda_{n-|S|+1} \leq n$, which is a contradiction.

(2) If $\alpha \leq \kappa$, then, by (1) in Theorem 9, G is Hamiltonian. If $\alpha \geq \kappa + 1$, then there exists an independent set S in G such that $|S| = \kappa + 1$. Applying Theorem 4, we have that $\kappa+1+q_{n-\kappa} = \kappa+1+q_{n-(\kappa+1)+1} = |S|+q_{n-|S|+1} \leq n$, which is a contradiction.

Thus we complete the proof of Theorem 5. □

Proof of Theorem 6. Let G be a graph satisfying the conditions in Theorem 6.

(1) If $\alpha \leq \kappa + 1$, then, by (2) in Theorem 9, G is traceable. If $\alpha \geq \kappa + 2$, then there exists an independent set S in G such that $|S| = \kappa + 2$. Applying Theorem 2, we have that $\kappa+2+\lambda_{n-\kappa-1} = \kappa+2+\lambda_{n-(\kappa+2)+1} = |S|+\lambda_{n-|S|+1} \leq n$, which is a contradiction.

(2) If $\alpha \leq \kappa + 1$, then, by (2) in Theorem 9, G is traceable. If $\alpha \geq \kappa + 2$, then there exists an independent set S in G such that $|S| = \kappa + 2$. Applying Theorem 4, we have that $\kappa+2+q_{n-\kappa-1} = \kappa+2+q_{n-(\kappa+2)+1} = |S|+q_{n-|S|+1} \leq n$, which is a contradiction.

Thus we complete the proof of Theorem 6. □

Proof of Theorem 7. Let G be a graph satisfying the conditions in Theorem 7.

(1) If $\alpha \leq \kappa - 1$, then, by (3) in Theorem 9, G is Hamilton-connected. If $\alpha \geq \kappa$, then there exists an independent set S in G such that $|S| = \kappa$. Applying Theorem 2, we have that $\kappa + \lambda_{n-\kappa+1} = |S| + \lambda_{n-|S|+1} \leq n$, which is a contradiction.

(2) If $\alpha \leq \kappa - 1$, then, by (3) in Theorem 9, G is Hamilton-connected. If $\alpha \geq \kappa$, then there exists an independent set S in G such that $|S| = \kappa$. Applying Theorem 4, we have that $\kappa + q_{n-\kappa-1} = |S| + q_{n-|S|+1} \leq n$, which is a contradiction.

Thus we complete the proof of Theorem 7. □

Proof of Theorem 8. Let G be a graph satisfying the conditions in Theorem 8.

(1) If $\alpha \leq 2\kappa - 2$, then, by Theorem 10, we have that every longest

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cycle in G is dominating. If $\alpha \geq 2\kappa - 1$, then there exists an independent set S in G such that $|S| = 2\kappa - 1$. Applying Theorem 2, we have that $2\kappa - 1 + \lambda_{n-2\kappa+2} = 2\kappa - 1 + \lambda_{n-(2\kappa-1)+1} = |S| + \lambda_{n-|S|+1} \leq n$, which is a contradiction.

(2) If $\alpha \leq 2\kappa - 2$, then, by Theorem 10, we have that every longest cycle in G is dominating. If $\alpha \geq 2\kappa - 1$, then there exists an independent set S in G such that $|S| = 2\kappa - 1$. Applying Theorem 2, we have that $2\kappa - 1 + q_{n-2\kappa+2} = 2\kappa - 1 + q_{n-(2\kappa-1)+1} = |S| + \lambda_{n-|S|+1} \leq n$, which is a contradiction.

Thus we complete the proof of Theorem 8. □

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