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COINCIDENCE RESULTS WITH COMPACTNESS ASSUMPTIONS FOR FAMILIES OF CORRESPONDENCES CONTAINING UPPER SEMI-CONTINUOUS MULTIMAPS, AND THEIR APPLICATIONS

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Abstract

In this paper we apply some *fixed-point results* to deduce new *coincidence* and *quasi-coincidence* theorems for two families of multimaps. The notion of *quasi-coincidence point* will be introduced in this work. Also, we apply some of these results, to obtain some *constrained equilibrium theorems* for an abstract economy with two companies, having or not preference relations. We consider the families of multimaps in the *completely metrizable locally convex spaces* setting, by using some *compactness* hypothesis and assuming that some multimaps are *upper semi-continuous*.

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1. Introduction

a) Some definitions

Let X and Y be nonempty sets. A *multimap* or a *correspondence* $T: X \to 2^Y$ is a function from X to the power set 2^Y of Y (the class of all subsets of Y).

A multimap T is *nonempty-valued*, if $T(x)$ is a nonempty set, for all $x \in X$.

If X and Y are two vector spaces, a multimap $T: X \to 2^Y$ is *convex-valued*, if for each $x \in X$, the set $T(x)$ is a convex set.

If $A \subseteq X$, then $T(A) = \{T(x) \subseteq Y \mid x \in A\}$.

If $B \subseteq X$, then $T^{-1}(B) = \{x \in X \mid T(x) \subset B\}$; in particular, for $y \in Y$, we have: $x \in T^{-1}(y)$ if and only if $y \in T(x)$.

Now let X and Y be two topological spaces. For a set $A \subseteq X$ we will denote by $\operatorname{int} A$, its *interior* A.

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Consider a multimap $T : X \to 2^Y$. We say that:

1) T is *compact-valued*, if $T(x)$ is a compact set, for all $x \in X$;
2) T is *compact*, if there exists a compact subset $K \subseteq Y$, such that $T(X) \subseteq K$. In particular, T is compact if Y is compact.
3) T is *upper semi-continuous*, if for every $x \in X$ and every open set $D \subseteq Y$, with $T(x) \subseteq D$, there exists a neighborhood V of x, such that $T(x') \subseteq D$, for all $x' \in V$.

If X is a nonempty set and $T : X \to 2^X$ is a nonempty-valued multimap, an element $x \in X$ is called a *fixed-point* for T, if $x \in T(x)$.

If X and Y are nonempty sets and $T : X \to 2^Y$ and $S : Y \to 2^X$ are two nonempty-valued multimaps, an element $(x, y) \in X \times Y$ is called a *coincidence point* for T and S, if $y \in T(x)$ and $x \in S(y)$.

Let now I be an arbitrary index set, $(E_i)_{i \in I}$ a family of topological vector spaces, $(X_i)_{i \in I}$ a family of nonempty convex sets such that $X_i \subseteq E_i$, for each $i \in I$, $X = \prod_{i \in I} X_i$, and $(T_i)_{i \in I}$, with $T_i : X \to 2^{X_i}$, $i \in I$, a family of nonempty-valued multimaps. An element $x = (x_i)_{i \in I} \in X$ is called a *collectively fixed-point* (in short a *fixed-point*) for the family $(T_i)_{i \in I}$, if $x_i \in T_i(x)$, for each $i \in I$.

If I and J are two index sets, $(X_i)_{i \in I}$ and $(Y_j)_{j \in J}$ are two families of nonempty sets, we denote by X and Y the following sets: $X = \prod_{i \in I} X_i$ and $Y = \prod_{j \in J} Y_j$. Let $(S_j)_{j \in J}$ and $(T_i)_{i \in I}$ be two families of nonempty-valued multimaps, with $S_j : X \to 2^{Y_j}$ and $T_i : Y \to 2^{X_i}$. An element $(x, y) \in X \times Y$, with $x = (x_i)_{i \in I}$, $y = (y_j)_{j \in J}$, is called a *collectively coincidence point* (in short a *coincidence point*) for the families $(T_i)_{i \in I}$ and $(S_j)_{j \in J}$, if $y_j \in S_j(x)$ and $x_i \in T_i(y)$, for all $j \in J$ and $i \in I$.

Now we will introduce a *new definition*. Firstly we will specify the *framework*.
Consider I an index set, and $(X_i)_{i \in I}$ and $(Y_i)_{i \in I}$ two families of nonempty sets,

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$X=\prod_{i\in I} X_i$ and $Y=\prod_{i\in I} Y_i$. For any fixed $i\in I$, denote $X^i=\prod_{l\in I, l\neq i} X_l$ and $Y^i=\prod_{k\in I, k\neq i} Y_k$. Let $(P_i)_{i\in I}$ and $(Q^i)_{i\in I}$ be two families of nonempty multimaps, where $P_i : Y^i \to 2^{X_i}$ and $Q^i : X_i \to 2^{Y^i}$, for all $i\in I$; similarly, let $(Q_i)_{i\in I}$ and $(P^i)_{i\in I}$ be two families of nonempty multimaps, with $P^i : Y_i \to 2^{X^i}$, and $Q_i : X^i \to 2^{Y_i}$.

We call an element $(x,y)\in X\times Y$, with $x=(x_i)_{i\in I}$, and $y=(y_i)_{i\in I}$, a *quasi-coincidence point* for:

a) $(P_i)_{i\in I}$ and $(Q^i)_{i\in I}$, if $y^i\in Q^i(x_i)$ and $x_i\in P_i(y^i)$, for all $i\in I$, where y^i is the projection of y on Y^i;

b) $(Q_i)_{i\in I}$ and $(P^i)_{i\in I}$, if $x^i\in P^i(y_i)$ and $y_i\in Q_i(x^i)$, for all $i\in I$, where x^i is the projection of x on X^i.

b) A brief history of the coincidence results

The need *to model economic phenomena*, and also *certain situations* that arise between different economical agents determined to apply the results of pure mathematics for its various disciplines such as mathematical analysis, algebra, optimization, differential equations, etc.

In this context the *fixed-point theory* reformulated some classical results in mathematics. Also were been introduced new notions as: *equilibrium point*, *maximal point*, *coincidence point*. As reference works on this subject, we can consider the book of K. C. Border, *Fixed-point theorems with applications to economics and game theory*, Cambridge Univ. Press, Cambridge, UK, 1995, and also, the book of N. C. Yannelis, *Lecture notes in general equilibrium theory*, Department of Economics Univ. of Illinois, Urbana-Champaign, August, 2003. But in this domain worked many other famous mathematicians: S. Kakutani (1941), K. Fan (1961), F. E. Browder (1968), C. J. Himmelberg (1972), E. Tarafdar (1977), G. Mehta (1987), S. Park (1989, 2002), P. Deguire (1995).

Other remarkable papers which *apply the fixed-point theorems for multimaps* in the *general equilibrium theory* belong to: G. Tian (1992), D. J. Rim and W. K. Kim (1992), K. K. Tan and G. X. Z. Yuan (1994), G. X. Z. Yuan and E. Tarafdar (1996), X. Wu (1997), G. Mehta, K.-K. Tan, G. X. Z. Yuan(1997), L.-J. Lin, S. Park, Z. T. Yu (1999), S. Chelbi and M. Florenzano (1999), G. X. Z. Yuan(1999), S. P. Singh, E. Tarafdar and B. Watson (2000), W. K. Kim and K. K. Tan (2001), L.-J. Lin and H. I.

Chen (2001), L.-J. Lin (2001), L.-J. Lin, S. F. Cheng, X. Y. Liu and Q. H. Ansari (2001), L.-J. Lin, Z. T. Yu, Q. H. Ansari and L.-P. Lai (2003). Also we mention the papers of R.-M. Dăneţ (2008), R.-M. Dăneţ and M.-V. Popescu (2008), R.-M. Dăneţ, I.-M. Popovici and F. Voicu (2006, 2009). An important step was made once with the demonstration of the fact that the *fixed-point theorem* of Browder (see F. E. Browder, *The fixed point theory of multi-valued mappings in topological vector spaces*, Math. Ann. **177** (1968), 283-301) it is equivalent with a theorem of *maximal point* (see N. C. Yannelis and N. D. Prabhakar, *Existence of maximal elements and equilibria in linear topological spaces*, J. Math. Econom. **12** (1983), 233-245). This leads to find some *new mathematical results starting from economical contexts*.

Over the last years, many more *generalized forms* of the Browder's fixed-point theorem were found. This led to setting some theorems of *maximal element* for families of multimaps. The utility of finding some maximal element results consists in finding the existence of a solution for the equilibrium in abstract economies and generalized games and also for systems of variational inequalities.

In 1984, F. E. Browder, combining the Kakutani-Fan fixed-point theorem and Fan-Browder theorem, obtained a *coincidence theorem* for multimaps. Also, different authors have demonstrated coincidence theorems for multimaps: H. Komiya (1986, using the Browder's fixed-point theorem), C. Horvat (1990, using a generalization of the classical KKM theorem), S. Park (1994), X. P. Ding (1997, giving a coincidence theorem for two multimaps, both of them without convex values and having the property of open inverse values), Y.-C. Cherng (2005), Z. D. Mitrovic (2006, demonstrating some coincidence results in generalized convex spaces).

In 2006, R.-M. Dăneţ, I.-M. Popovici and F. Voicu (see [9]) proved a *coincidence theorem* for two families of multimaps, which were defined on a product space, this being a consequence of some fixed-points results which were obtained by these authors. In 2007, M.-V. Popescu and R.-M. Dăneţ obtained two *coincidence results* for two families of multimaps (see [13]), having the index sets not necessarily equal. In the same year, M. Balaj and L. J. Lin, by using the fixed-point theorem of Fan-Browder, gave new *coincidence* and *maximal element theorems*. In 2008, R.-M. Dăneţ and M.-V. Popescu obtained some fixed-point results (see [5]), and then they applied these results in economics, giving two general equilibrium theorems.

Our work begins with some *fixed-points results* for multimaps, which lead us to obtain some *coincidence points* results for two families of multimaps in the **completely metrizable locally convex spaces setting**, some multimaps being *compact-valued* and *upper semi-continuous*. Then, we will use these coincidence theorems for obtaining some *equilibrium points* in a *generalized abstract economy with two companies* (or an *abstract generalized game with two families of players*), where the companies can have a different number of factories. In the last section, we will give some *quasi-coincidence results*, and then we will apply them to *abstract economies with two companies*.

c) Some results

In this section we will give *two fixed-point results* for a family of *upper semi-continuous* multimaps (or correspondences) in the *completely metrizable locally convex spaces* setting. It is well known (Theorem 5.35 in [1]) that in any completely metrizable locally convex space, the closed convex hull of a compact set is compact (see also [4]).

Note that the completely metrizable locally convex spaces setting includes the *Banach spaces* (with their norm topology) setting. Note also that the proofs of the results in this paper did not require that entire space, for example, to be completely metrizable. The same argument works provided that $\overline{\text{co}}K$ is compact for some compact set $K \subseteq E$, if $\overline{\text{co}}K$ lies in a subset of E, that is, completely metrizable.

Lemma 1.1 (see Lemma 2.2 in [11], or [2]) *Let X and Y be two topological spaces and $T : X \to 2^Y$ be a multimap. If X is compact and T is upper semi-continuous, nonempty-valued and compact-valued, then $T(X)$ is compact (and hence T is compact).*

The following two results appeared, for example, in [8], having a starting point a result of [3], generalized in a certain sense in [12].

Lemma 1.2 *Let I be an arbitrary index set. Let $(E_i)_{i\in I}$ be a family of topological vector spaces and $(X_i)_{i\in I}$ another family of nonempty convex sets, such that $X_i \subseteq E_i$, for each $i \in I$, and $X = \prod_{i\in I} X_i$. Let C be a nonempty compact subset of X, and for each $i \in I$, let $T_i : X \to 2^{X_i}$ be a nonempty-valued and convex-valued multimap. Suppose that for each $i \in I$, the following conditions hold:*

1) $X = \bigcup_{x_i \in X_i} \text{int}T_i^{-1}(x_i)$;

2) If X is not compact, assume that there exists a nonempty compact convex set C_i in X_i, such that $X \setminus C \subseteq \bigcup_{y_i \in C_i} \text{int}T_i^{-1}(y_i)$.

Then, there exists $\tilde{x} = (\tilde{x}_i)_{i\in I} \in X$ *, such that* $\tilde{x}_i \in T_i(\tilde{x})$*, for all* $i \in I$*, i.e.* $\tilde{x}$ *is a fixed-point for the family* $(T_i)_{i\in I}$.

The following result belongs to Q. H. Ansari and C. D. Yao (see [3]).

Lemma 1.3 *Let* I, $(E_i)_{i\in I}$, $(X_i)_{i\in I}$, X *and* C *be like in the previous lemma. Let also* S_i, $T_i : X \to 2^{X_i}$ *(*$i \in I$*) be two families of multimaps,* S_i *being nonempty-valued. Suppose that for each* $i \in I$*, the following conditions hold:*

1) *for each* $x \in X$, $\mathrm{co}S_i(x) \subseteq T_i(x)$;
2) $X = \bigcup_{x_i \in X_i} \mathrm{int}S_i^{-1}(x_i)$;
3) *If* X *is not compact, assume that there exists a nonempty compact convex set* $C_i \subseteq X_i$*, such that* $X \setminus C = \bigcup_{y_i \in C_i} \mathrm{int}S_i^{-1}(y_i)$.

Then, there exists $\tilde{x} = (\tilde{x}_i)_{i\in I} \in X$ *, such that* $\tilde{x}_i \in T_i(\tilde{x})$*, for all* $i \in I$.

Remark that in the above lemma appeared two families of multimaps $(T_i)_{i\in I}$ and $(S_i)_{i\in I}$ defined on X, such that, for all $i \in I$ and $x \in X$, $\mathrm{co}S_i(x) \subseteq T_i(x)$. Then we obtain the existence of a fixed-point for the family $(T_i)_{i\in I}$, if we impose some conditions to the family $(S_i)_{i\in I}$.

In the following theorem, formulated in the completely metrizable locally convex spaces setting, the multimaps are compact (see [6]).

Lemma 1.4 *Let* I *be an arbitrary index set, and for each* $i \in I$*, let* $X_i \subseteq E_i$ *be a nonempty convex set in a completely metrizable locally convex space* E_i*. Let also* $X = \prod_{i\in I} X_i$ *and let* $T_i : X \to 2^{X_i}$ *(*$i \in I$*) be a nonempty-valued and convex-valued multimap. Suppose that for each* $i \in I$*, the following conditions hold:*

1) $X = \bigcup_{x_i \in X_i} \mathrm{int}T_i^{-1}(x_i)$;
2) T_i *is a compact multimap (that is, there exists a nonempty compact subset* $K_i \subseteq X_i$ *such that* $T_i(X) \subseteq K_i$*).*

Then, there exists $\tilde{x} = (\tilde{x}_i)_{i\in I} \in X$ *, such that* $\tilde{x}_i \in T_i(\tilde{x})$*, for each* $i \in I$ *(hence* $\tilde{x}$ *is a fixed-point for the family* $(T_i)_{i\in I}$*).*

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Proof. We apply Lemma 1.2, for the sets $(C_i)_{i\in I}$, such that $C_0=\prod_{i\in I}C_i=\overline{\text{co}}K$, where $K=\prod_{i\in I}K_i$. Because E_i is a completely metrizable locally convex space, according, for example to [1, Corollary 5.33], it follows that C_i is compact and therefore $T_i:X\to 2^{C_i}$. □

Lemma 1.5 *Let* I, $(E_i)_{i\in I}$, $(X_i)_{i\in I}$, *and* X *be like in Lemma 1.4. Let also* S_i, $T_i:X\to 2^{X_i}$ ($i\in I$) *be two families of multimaps,* S_i *being nonempty-valued. Suppose that for each* $i\in I$, *the following conditions hold:*

1) for each $x\in X$, $\text{co}S_i(x)\subseteq T_i(x)$;

2) $X=\bigcup_{x_i\in X_i}\text{int}S_i^{-1}(x_i)$;

3) S_i *is a compact multimap (i.e., there exists a nonempty compact subset* $K_i\subseteq X_i$, *such that* $S_i(X)\subseteq K_i$).

Then, there exists $\tilde{x}=(\tilde{x}_i)_{i\in I}\in X$, *such that* $\tilde{x}_i\in T_i(\tilde{x})$, *for each* $i\in I$ *(that is,* $\tilde{x}$ *is a fixed-point for the family* $(T_i)_{i\in I}$).

The proof of Lemma 1.5 is similar with the proof of Lemma 1.4, but it uses Lemma 1.3 instead of Lemma 1.2.

2. Coincidence results and their applications in abstract economies with two companies

In this section we will give some *coincidence results* and then, we will apply them to abstract economies with two companies, finding a *constrained equilibrium point*. We will work in the completely metrizable locally convex spaces setting, and we will consider compactness assumptions on the families of multimaps, some of these correspondences being upper semi-continuous.

In the following result appear two families of multimaps.

Theorem 2.1 *Let* I *and* J *be two arbitrary index sets. Let* $(E_i)_{i\in I}$ *and* $(F_j)_{j\in J}$ *be two families of completely metrizable locally convex spaces and* $(X_i)_{i\in I}$, $(Y_j)_{j\in J}$ *two families of nonempty convex sets such that,* $X_i\subseteq E_i$ *and* $Y_j\subseteq F_j$, *for all* $i\in I$ *and*

$j \in J$. Let also $X = \prod_{i \in I} X_i$, $Y = \prod_{j \in J} Y_j$ and let $S_j : X \to 2^{Y_j}$ ($j \in J$) and $T_i : Y \to 2^{X_i}$ ($i \in I$) be two families of nonempty-valued and convex-valued multimaps. Suppose that, for all $i \in I$ and $j \in J$, the following conditions hold:

i) $X = \bigcup_{y_j \in Y_j} \operatorname{int} S_j^{-1}(y_j)$;

ii) $Y = \bigcup_{x_i \in X_i} \operatorname{int} T_i^{-1}(x_i)$;

iii) S_j is upper semi-continuous and compact-valued multimap;

iv) X is compact.

Then, there exists a coincidence point $(\tilde{x}, \tilde{y}) \in X \times Y$ for the families $(S_j)_{j \in J}$ and $(T_i)_{i \in I}$ (that is, $\tilde{x} = (\tilde{x}_i)_{i \in I}$, $\tilde{y} = (\tilde{y}_j)_{j \in J}$ are such that $\tilde{x}_i \in T_i(\tilde{y})$ and $\tilde{y}_j \in S_j(\tilde{x})$, for all $i \in I$ and $j \in J$).

Proof. We will apply Lemma 1.4. Let us define for all $i \in I$ and $j \in J$, the multimap $U_{ji} : X \times Y \to 2^{Y_j \times X_i}$, by $U_{ji}(x, y) = S_j(x) \times T_i(y)$, where $x \in X$ and $y \in Y$. From the hypothesis "*i)*" , it follows that, for each $i \in I$ and $j \in J$, the following relations hold:

$$X \times Y = \left(\bigcup_{y_j \in Y_j} \operatorname{int} S_j^{-1}(y_j) \right) \times \left(\bigcup_{x_i \in X_i} \operatorname{int} T_i^{-1}(x_i) \right) \subseteq$$

$$\subseteq \bigcup_{(y_j, x_i) \in Y_j \times X_i} \left(\operatorname{int} S_j^{-1}(y_j) \times \operatorname{int} T_i^{-1}(x_i) \right) \subseteq$$

$$\subseteq \bigcup_{(y_j, x_i) \in Y_j \times X_i} \operatorname{int} \left(S_j^{-1}(y_j) \times T_i^{-1}(x_i) \right) \subseteq X \times Y.$$

But, obviously, $U_{ji}^{-1}(y, x) = S_j^{-1}(y_j) \times T_i^{-1}(x_i)$. Hence the hypothesis "*i)*" implies that

$$X \times Y = \bigcup_{(y_j, x_i) \in Y_j \times X_i} \operatorname{int} U_{ji}^{-1}(y_j, x_i).$$

From the hypothesis "*iii)*" and "*iv)*", by applying Lemma 1.1, it follows that $S_j(X)$ is compact, that is, S_j is compact (for each $j \in J$). Also the conditions "*iv)*" implies that T_i is compact (for each $i \in I$) and so we obtain that the multimap $U_{ji} : X \times Y \to 2^{Y_j \times X_i}$ is compact. Therefore, there exist $K_i \subseteq X_i$ and $L_j \subseteq Y_j$, two nonempty compact subsets, such that $S_j(x) \subseteq L_j$ and $T_i(Y) \subseteq K_i$. Denoting by

$C_{ji} = L_j \times K_i$ the Cartesian product of these compact sets, we obtain a nonempty compact subset in $Y_j \times X_i$, such that, for each $x \in X$ and $y \in Y$, we have

$$U_{ji}\left(x,y\right) = S_j\left(x\right) \times T_i\left(y\right) \in S_j\left(X\right) \times T_i\left(Y\right) \subseteq L_j \times K_i = C_{ji},$$

that is, $U_{ji}\left(X \times Y\right) \subseteq C_{ji}$. By using Lemma 1.4, we obtain a fixed-point $\left(\tilde{x},\tilde{y}\right) \in X \times Y$ for the family $\left(U_{ji}\right)_{i\in I, j\in J}$. Hence, if $\tilde{x} = \left(\tilde{x}_i\right)_{i\in I}$, $\tilde{y} = \left(\tilde{y}_j\right)_{j\in J}$, then $\tilde{y}_j \in S_j\left(\tilde{x}\right)$ and $\tilde{x}_i \in T_i\left(\tilde{y}\right)$. □

Now we give an **economic interpretation** of the Theorem 2.1, for I and J two finite index sets.

Let I and J be finite index sets and, for all $i \in I$ and $j \in J$, let E_i, F_j, X_i, Y_j, X, Y, A_j and B_i be like in Theorem 2.1. Let $\Gamma = \left(X_i, Y_j, S_j, T_i\right)_{i\in I, j\in J}$ be an abstract economy with two companies and without preferences relations. In the first company, the set X_i denotes the *strategy set* of the i th factory $\left(i \in I\right)$. In the second company, the set Y_j denotes the *strategy set* of the j th factory $\left(j \in J\right)$. The set $X = \prod_{i\in I} X_i$ denotes the *strategies set* of the first company and the set $Y = \prod_{j\in J} Y_j$ represents the *strategies set* of the second company. The nonempty-valued and convex-valued multimaps $S_j : X \to 2^{Y_j}$ ($j \in J$) and $T_i : Y \to 2^{X_i}$ $\left(i \in I\right)$ are the *constraints* of the economy Γ. For example, for each fixed j in J, the constraint S_j restricts the strategy set of the j th factory in the second company to its subset $S_j\left(X\right)$ when the first company has chosen its strategy $x = \left(x_i\right)_{i\in I}$ in X. In others words, the strategy of the j th factory in the second company depends on all the strategies in the first company.

Then, under the hypothesis of the Theorem 2.1, including compactness assumptions and the upper semi-continuity of all constraints in one of the two companies, there exists an *equilibrium point* for Γ, that is, a coincidence point $\left(\tilde{x},\tilde{y}\right) \in X \times Y$, for the constraint multimaps $\left(S_j\right)_{j\in J}$ and $\left(T_i\right)_{i\in I}$. Hence $\tilde{x} = \left(\tilde{x}_i\right)_{i\in I}$, $\tilde{y} = \left(\tilde{y}_j\right)_{j\in J}$ are such that $\tilde{x}_i \in T_i\left(\tilde{y}\right)$ and $\tilde{y}_j \in S_j\left(\tilde{x}\right)$, for all $i \in I$ and $j \in J$.

In the following theorem, we will consider four families of multimaps and, imposing some conditions on the two of these families, we will obtain the existence of a *coincidence point* for the other two families.

Theorem 2.2 *Let* I, J, $(E_i)_{i\in I}$, $(F_j)_{j\in J}$, $(X_i)_{i\in I}$, $(Y_j)_{j\in J}$, X *and* Y *be like in the previous theorem. Let also* $S_j, R_j : X \to 2^{Y_j}$ $(j\in J)$ *and* V_i, $T_i : Y \to 2^{X_i}$ $(i\in I)$ *be four families of multimaps, such that* S_j *and* V_i *are nonempty-valued. Assume that for all* $i\in I$ *and* $j\in J$, *the following conditions hold:*

i) *for all* $x\in X$, $\mathrm{co}S_j(x)\subseteq R_j(x)$;

ii) *for all* $y\in Y$, $\mathrm{coV}_i(y)\subseteq T_i(y)$;

iii) $Y=\bigcup\limits_{x_i\in X_i}\mathrm{int}V_i^{-1}(x_i)$;

iv) $X=\bigcup\limits_{y_j\in Y_j}\mathrm{int}S_j^{-1}(y_j)$;

v) S_j *is upper semi-continuous and compact-valued;*

vi) X *is compact.*

Then, there exists a coincidence point $(\tilde{x},\tilde{y})\in X\times Y$ *for the families* $(R_j)_{j\in J}$ *and* $(T_i)_{i\in I}$, *that is,* $\tilde{x}=(\tilde{x}_i)_{i\in I}\in X$ *and* $\tilde{y}=(\tilde{y}_j)_{j\in J}\in Y$ *are such that* $\tilde{x}_i\in T_i(\tilde{y})$, *and* $\tilde{y}_j\in R_j(\tilde{x})$, *for all* $i\in I$ *and* $j\in J$.

Proof. For each $i\in I$ and $j\in J$, define the multimaps:

$$U_{ji}:X\times Y\to 2^{Y_j\times X_i}\text{, by } U_{ji}(x,y)=S_j(x)\times V_i(y),$$

for $x\in X$ and $y\in Y$, and

$$W_{ji}:X\times Y\to 2^{Y_j\times X_i}\text{, by } W_{ji}(x,y)=R_j(x)\times T_i(y),$$

for $x\in X$ and $y\in Y$. Then, from the hypothesis "*i)*" and "*ii)*" it follows that:

$$0)\ \mathrm{co}U_{ji}(x,y)\subseteq W_{ji}(x,y),$$

for all $i\in I$ and $j\in J$, and for all $x\in X$ and $y\in Y$. Indeed, let

$z\in \mathrm{co}U_{ji}(x,y)\Rightarrow z=\sum\limits_{k=1}^{p}\lambda_k z_k$, where $z_k=(a_k,b_k)\in S_j(x)\times V_i(y)$, $\lambda_k\geq 0$, for all $k=1,...,p$ and $\sum\limits_{k=1}^{p}\lambda_k=1$. Then:

$$z=\sum_{k=1}^{p}\lambda_k\left(a_k,b_k\right)=\left(\sum_{k=1}^{p}\lambda_k a_k,\sum_{k=1}^{p}\lambda_k b_k\right)\overset{i),ii)}{\Rightarrow} z\in R_j\left(x\right)\times T_i\left(y\right)=W_{ji}\left(x,y\right),$$

because $\sum_{k=1}^{p}\lambda_k a_k\in \mathrm{co}S_j\left(x\right)\overset{i)}{\subseteq}R_j\left(x\right)$ and $\sum_{k=1}^{p}\lambda_k b_k\in \mathrm{co}V_i\left(y\right)\overset{ii)}{\subseteq}T_i\left(y\right)$.

Now, from the hypothesis "*iii)*" and "*iv)*", we have

$$X\times Y=\bigcup_{\left(y_j,x_i\right)\in Y_j\times X_i}\mathrm{int}\left(S_j^{-1}\left(y_j\right)\times V_i^{-1}\left(x_i\right)\right),$$

for all $j\in J$ and $i\in I$. But $S_j^{-1}\left(y_j\right)\times V_i^{-1}\left(x_i\right)=U_{ji}^{-1}\left(y_j,x_i\right)$; therefore it follows that:

$$1)\ X\times Y=\bigcup_{\left(y_j,x_i\right)\in Y_j\times X_i}\mathrm{int}U_{ji}^{-1}\left(y_j,x_i\right),$$

for all $j\in J$ and $i\in I$. From the hypothesis "*v)*" and "*vi)*", with Lemma 1.1., it follows that S_j is compact. Also, from "*vi)*", it follows that X_i is compact, for all $i\in I$, that is, V_i is compact. Because

$$U_{ji}\left(x,y\right)=S_j\left(x\right)\times V_i\left(y\right)$$

(for all $x\in X$ and $y\in Y$), it follows that U_{ji} is a compact multimap. By using Lemma 1.5, we obtain a *fixed-point* $\left(\tilde{y},\tilde{x}\right)\in Y\times X$ for the family $\left(W_{ji}\right)_{i\in I,\,j\in J}$, that is, if $\tilde{x}=\left(\tilde{x}_i\right)_{i\in I}$, $\tilde{y}=\left(\tilde{y}_j\right)_{j\in J}$, we have $\tilde{y}_j\in R_j\left(\tilde{x}\right)$ and $\tilde{x}_i\in T_i\left(\tilde{y}\right)$, for all $j\in J$ and $i\in I$. Therefore $\left(\tilde{y},\tilde{x}\right)$ is a *coincidence point* for the families $\left(R_j\right)_{j\in J}$ and $\left(T_i\right)_{i\in I}$. □

In what follows, we find *equilibrium points* for abstract economies having two companies, some upper semi-continuous constraint multimaps, and preference relations.

Theorem 2.3 *Let I and J be two arbitrary index sets, and $\left(E_i\right)_{i\in I}$ and $\left(F_j\right)_{j\in J}$, two families of completely metrizable locally convex spaces. For all $i\in I$ and $j\in J$, let also $X_i\subseteq E_i$ and $Y_j\subseteq F_j$ be nonempty convex sets, and let $X=\prod_{i\in I}X_i$, and $Y=\prod_{j\in J}Y_j$. Consider $\Gamma=\left(X_i, A_i, B_i, P_i, Y_j, C_j, D_j, Q_j\right)_{\substack{i\in I\\ j\in J}}$ a generalized abstract economy with two companies* (*or a generalized abstract game with two families of players*) *with the constraint multimaps A_i, B_i, C_j, D_j and the preference multimaps P_i, Q_j, where*

$$A_i,\,B_i,\,P_i:Y\to 2^{X_i}\ \textit{and}\ C_j,\,D_j,\,Q_j:X\to 2^{Y_j},$$

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for each $i \in I$ *and* $j \in J$. *Suppose that, for all* $j \in J$ *and* $i \in I$, *the following conditions hold:*

1) for all $y \in Y$, $\mathrm{co}A_i(y) \subseteq B_i(y)$, *and* $A_i(y)$ *is a nonempty set;*

2) for all $x \in X$, $\mathrm{co}C_j(x) \subseteq D_j(x)$, *and* $C_j(x)$ *is a nonempty set;*

3) the multimaps C_j *and* Q_j *are upper semi-continuous and compact-valued;*

4) X *is compact;*

5) $X = \bigcup_{y_j \in Y_j} \mathrm{int}\left(C_j^{-1}(y_j) \cap \left(Q_j^{-1}(y_j) \cup H_j\right)\right)$, *where*

$$H_j = \left\{x \in X \mid C_j(x) \cap Q_j(x) = \varnothing\right\};$$

6) $Y = \bigcup_{x_i \in X_i} \mathrm{int}\left(A_i^{-1}(x_i) \cap \left(P_i^{-1}(x_i) \cup G_i\right)\right)$, *where*

$$G_i = \left\{y \in Y \mid A_i(y) \cap P_i(y) = \varnothing\right\};$$

7) for all $\tilde{x} = (\tilde{x}_i)_{i \in I} \in X$ *and* $\tilde{y} = (\tilde{y}_j)_{j \in J} \in Y$, $x_i \notin \mathrm{co}P_i(y)$ *and* $y_j \notin \mathrm{co}Q_j(x)$.

Then, there exists $(\tilde{x}, \tilde{y})$, *an equilibrium point for* Γ, *that is,* $\tilde{x} = (\tilde{x}_i)_{i \in I} \in X$ *and* $\tilde{y} = (\tilde{y}_j)_{j \in J} \in Y$ *are such that, for all* $i \in I$ *and* $j \in J$,

$$\tilde{x}_i \in B_i(\tilde{y}),\ A_i(\tilde{y}) \cap P_i(\tilde{y}) = \varnothing \ \text{(i.e. } \tilde{y} \in G_i\text{)}$$
$$\tilde{y}_j \in D_j(\tilde{x}),\ C_j(\tilde{x}) \cap Q_j(\tilde{x}) = \varnothing \ \text{(i.e. } \tilde{x} \in H_j\text{)}.$$

Proof. For all $i \in I$ and $j \in J$, define:

$$V_i, T_i : Y \to 2^{X_i},\ S_j, R_j : X \to 2^{Y_j} \text{ by}$$

$$V_i(y) = \begin{cases} A_i(y) & , \text{if } y \in G_i \\ A_i(y) \cap \mathrm{co}P_i(y), & \text{if } y \notin G_i \end{cases}; \quad T_i(y) = \begin{cases} B_i(y) & , \text{if } y \in G_i \\ B_i(y) \cap \mathrm{co}P_i(y), & \text{if } y \notin G_i \end{cases}$$

$$S_j(x) = \begin{cases} C_j(x) & , \text{if } x \in H_j \\ C_j(x) \cap \mathrm{co}Q_j(x), & \text{if } x \notin H_j \end{cases}; \quad R_j(x) = \begin{cases} D_j(x) & , \text{if } x \in H_j \\ D_j(x) \cap \mathrm{co}Q_j(x), & \text{if } x \notin H_j \end{cases}$$

We will prove that all the hypothesis of Theorem 2.2 are fulfilled. So:

i) for each $j \in J$ and $x \in X$, $\mathrm{co}S_j \subseteq R_j(x)$.

Indeed, let $z=\sum_{p=1}^{n}\lambda_p z_p \in \operatorname{co}S_j(x)$, where $\sum_{p=1}^{n}\lambda_p = 1$, $\lambda_p \geq 0$ and $z_p \in S_j(x)$, for all $p=1,\dots,n$. Hence

$$z_p \in \begin{cases} C_j(x) & , \textit{if } x \in H_j \\ C_j(x) \cap \operatorname{co}Q_j(x), & \textit{if } x \notin H_j \end{cases}.$$

If $x \notin H_j$ ($\Leftrightarrow C_j(x) \cap Q_j(x) \neq \varnothing$) and for all $p=1,\dots,n$, $z_p \in C_j(x) \cap \operatorname{co}Q_j(x)$ $\Rightarrow z \in \operatorname{co}C_j(x) \cap \operatorname{co}Q_j(x) \overset{2)}{\subseteq} D_j(x) \cap \operatorname{co}Q_j(x) = R_j(x)$, hence $z \in R_j(x)$. If $x \in H_j$ ($\Leftrightarrow C_j(x) \cap \operatorname{co}Q_j(x) = \varnothing$) and for all $p=1,\dots,n$, $z_p \in C_j(x) \Rightarrow z \in \operatorname{co}C_j(x) \overset{2)}{\subseteq}$ $D_j(x) = R_j(x)$, hence $z \in R_j(x)$. Hence, $\operatorname{co}S_j(x) \subseteq R_j(x)$. Similarly, we have:

ii) for all $i \in I$ and $y \in Y$, $\operatorname{co}V_i(y) \subseteq T_i(y)$.

Also, the following is true:

iii) for each $i \in I$, $Y = \bigcup_{x_i \in X_i} \operatorname{int}V_i^{-1}(x_i)$.

Indeed, we have:

$$V_i^{-1}(x_i) = \left(A_i^{-1}(x_i) \cap (\operatorname{co}P_i)^{-1}(x_i) \cap C_Y G_i\right) \cup \left(A_i^{-1}(x_i) \cap G_i\right) \qquad \text{(I)}$$

(If $y \in V_i^{-1}(x_i)$ then $x_i \in V_i(y)$, that is,

$$x_i \in A_i(y) \cap \operatorname{co}P_i(y), \text{ if } y \notin G_i, \text{ or}$$
$$x_i \in A_i(y), \text{ if } y \in G_i.$$

Therefore, $y \in A_i^{-1}(x_i)$, $y \in (\operatorname{co}P_i)^{-1}(x_i)$ and $y \notin G_i$, or $y \in A_i^{-1}(x_i)$ and $y \in G_i$.)

From (I), it follows that:

$$V_i^{-1}(x_i) \supseteq \left(A_i^{-1}(x_i) \cap P_i^{-1}(x_i) \cap C_Y G_i\right) \cup \left(A_i^{-1}(x_i) \cap G_i\right) \Rightarrow$$
$$V_i^{-1}(x_i) \supseteq A_i^{-1}(x_i) \cap \left(P_i^{-1}(x_i) \cup G_i\right). \qquad \text{(II)}$$

Now we will prove *"iii)"*. If $y \in Y$, then, from the hypothesis *"6)"*, it follows that for each $i \in I$, there exists $x_i \in X_i$, such that

$$y \in \operatorname{int}\left(A_i^{-1}(x_i) \cap \left(P_i^{-1}(x_i) \cup G_i\right)\right) \overset{\text{(II)}}{\subseteq} \operatorname{int}V_i^{-1}(x_i).$$

Hence, $y \in \bigcup_{x_i \in X_i} \operatorname{int}V_i^{-1}(x_i)$. Similarly, it follows that:

iv) for each $j \in J$, $X = \bigcup_{y_j \in Y_j} \operatorname{int} S_j^{-1}(y_j)$.

Finally, we remark that the conditions *"v)"* and *"vi)"* of the Theorem 2.2 are fulfilled, that is:

v) for all $j \in J$, S_j is upper semi-continuous and compact-valued;

vi) Y is compact.

Indeed, "*vi)*" is exactly the hypothesis "*4)*" of Theorem 2.3. For the proof of "*v)*", we remark that, obviously, S_j is compact-valued (because C_j and A_i are compact-valued). Now, let us prove that S_j is upper semi-continuous, that is, for each $x \in X$ and an open set $D_j \subseteq Y_j$ with $S_j(x) \subseteq D_j$, there exists a neighborhood U of x such that for all $x' \in U$, $S_j(x') \subseteq D_j$. If $x \in H_j$, then $S_j(x) = C_j(x)$ and the above statements follows by using that C_j is upper semi-continuous. If $x \notin H_j$, then $S_j(x) = C_j(x) \cap Q_j(x)$. Because C_j and Q_j are upper semi-continuous, there exist V and W two neighborhoods of x, such that for all $x' \in V$ and $x'' \in W$, $C_j(x') \subseteq D_j$ and $Q_j(x'') \subseteq D_j$. Consider the neighborhood U of x, given by $U = V \cap W$. Obviously, for each $x' \in U$, $S_j(x') = C_j(x') \cap Q_j(x') \subseteq D_j$. Then, we can apply Theorem 2.2, finding a *coincidence point*, for the families $(T_j)_{j \in J}$ and $(R_i)_{i \in I}$, that is, there exists $(\tilde{x}, \tilde{y}) \in X \times Y$, $\tilde{x} = (\tilde{x}_i)_{i \in I} \in X$ and $\tilde{y} = (\tilde{y}_j)_{j \in J} \in Y$, such that, for all $i \in I$ and $j \in J$,

$$\tilde{x}_i \in T_i(\tilde{y}) \text{ and } \tilde{y}_j \in Q_j(\tilde{x}).$$

According to the hypothesis *"7"*, we have that

$$\tilde{x}_i \notin \mathrm{coP}_i(\tilde{y}) \text{ and } \tilde{y}_j \notin \mathrm{co}Q_j(\tilde{x}).$$

By using *"5)"*, and "*6)*" it follows that, for each $i \in I$ and $j \in J$,

$$\tilde{x}_i \in B_i(\tilde{y}),\ A_i(\tilde{y}) \cap P_i(\tilde{y}) = \varnothing \text{ and}$$
$$\tilde{y}_j \in D_j(\tilde{x}),\ C_j(\tilde{x}) \cap Q_j(\tilde{x}) = \varnothing .$$

Therefore $(\tilde{x}, \tilde{y})$ is an *equilibrium point* for Γ. □

3. Quasi-coincidence results and their applications to abstract economies with two companies

In this section we will give some *quasi-coincidence results*, and then we will apply them to abstract economies with two companies. We will work in the completely metrizable locally convex spaces setting and will use only compactness assumptions.

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Note that, the following result is in the line of Theorem 3.11 in [10], but the hypothesis and the proof are different. Firstly, we will specify the *framework*.

Let I be a finite index set, let $(E_i)_{i\in I}$ be a family of completely metrizable locally convex spaces and let $(X_i)_{i\in I}$ be a family of nonempty convex sets, such that $X_i \subseteq E_i$, for all $i \in I$. Let $X = \prod_{i\in I} X_i$ and for each $i \in I$, denote $X^i = \prod_{j\in I, j\neq i} X_j$, write $X = X^i \underline{\times} X_i$ and identify with $X_i \underline{\times} X^i$. For each $x \in X$, $x_i \in X_i$ denotes its i th coordinate and $x^i \in X^i$ the projection of x onto X^i. We also write (like in [11], p. 529), $x = (x^i, x_i) \in X^i \underline{\times} X_i$. Now we will give sufficient conditions to obtain a *quasi-coincidence point* for two families of multimaps $(H_i)_{i\in I}$ and $(H^i)_{i\in I}$, which are not defined on the whole product space X, but only on X^i and on X_i, respectively, where $i \in I$.

Theorem 3.1 *Let I be a finite index set, let $(E_i)_{i\in I}$ be a family of completely metrizable locally convex spaces and let $(X_i)_{i\in I}$ be a family of nonempty convex sets such that $X_i \subseteq E_i$, for all $i \in I$. For all $i \in I$, let $H_i : X^i \to 2^{X_i}$ and $H^i : X_i \to 2^{X^i}$ be nonempty-valued and convex-valued multimaps. Suppose that, for all $i \in I$, the following conditions hold:*

1) $X^i = \bigcup_{x_i \in X_i} \operatorname{int} H_i^{-1}(x_i)$ and $X_i = \bigcup_{x^i \in X^i} \operatorname{int}(H^i)^{-1}(x^i)$;

2) X_i is compact.

Then, there exists a quasi-coincidence point $(\tilde{x}, \tilde{y}) \in X \times X$ for H_i and H^i $(i \in I)$, that is, there exist $\tilde{x} = (\tilde{x}_i)_{i\in I} \in X$ and $\tilde{y} = (\tilde{y}_i)_{i\in I} \in X$, such that $\tilde{x}_i \in H_i(\tilde{y}^i)$ and $\tilde{y}^i \in H^i(\tilde{x}_i)$, for all $i \in I$.

Proof. Because we wrote $X = X^i \underline{\times} X_i$, we can define

$$U_i = H^i \underline{\times} H_i : X \to 2^{X_i \underline{\times} X^i}, \text{ by}$$
$$U_i(x^i, x_i) = H^i(x_i) \underline{\times} H_i(x^i),$$

for all $x = (x_i)_{i\in I} \in X$, understanding by $H^i(x_i) \underline{\times} H_i(x^i)$ the set of all $y = (y_i)_{i\in I} \in X$ such that $y^i \in H^i(x_i)$ and $y_i \in H_i(x^i)$. Also, for all $i \in I$, denote by P_i and respectively P^i the canonical projections $P_i : 2^X \to 2^{X_i}$ and $P^i : 2^X \to 2^{X^i}$ (that is, for each $A \subseteq X$, $P_i(A) = \bigcup_{x \in A} P_i(x) = \bigcup_{x \in A} x_i \subseteq X_i$ and $P^i(A) = \bigcup_{x \in A} P^i(x) = \bigcup_{x \in A} x^i \subseteq X^i$).

Denote also $M_i : X \to 2^{X_i}$ and $M^i : X \to 2^{X_i}$ the multimaps $M_i = P_i \circ U_i$ and $M^i = P^i \circ U_i$. ($M_i : X \xrightarrow{U_i} 2^X \xrightarrow{P_i} 2^{X_i}$ and $M^i : X \xrightarrow{U_i} 2^X \xrightarrow{P^i} 2^{X^i}$)

Hence, for each $x = (x_i)_{i \in I} \in X$,

$$M_i(x) = P_i(U_i(x)) = P_i\left(H^i(x_i) \underline{\times} H_i(x^i)\right) = H_i(x^i)$$

and

$$M^i(x) = P^i(U_i(x)) = P^i\left(H^i(x^i) \underline{\times} H_i(x_i)\right) = H^i(x_i)$$

(because $H_i(x^i) \subseteq X_i$ and $H^i(x_i) \subseteq X^i$). Obviously, because H^i and H_i are nonempty-valued and convex-valued multimaps, it follows that M_i and M^i are nonempty-valued and convex-valued multimaps, too. Indeed, let us prove, for example, that M_i is *nonempty-valued*: for all $x = (x_i)_{i \in I} \in X$,

$$M_i(x) = P_i(U_i(x)) = P_i\left(H^i(x_i) \underline{\times} H_i(x^i)\right) = H_i(x^i) \neq \varnothing,$$

because H_i is nonempty-valued. Now, let us prove that M_i is *convex-valued*. Let $x = (x_i)_{i \in I} \in X$; we will prove that the set $M_i(x)$ is convex. For this aim, let $y_i, z_i \in M_i(x)$, and $\alpha, \beta \geq 0$, such that $\alpha + \beta = 1$. Because $M_i(x) = P_i\left(H^i(x_i) \underline{\times} H_i(x^i)\right) = H_i(x^i)$ and H_i is convex-valued it follows that $H_i(x^i)$ is convex, and hence, $\alpha y_i + \beta z_i \in M_i(x) = H^i(x_i)$. Also, because X_i is compact for all $i \in I$, it follows that X^i is compact, too. Let us prove that, for all $i \in I$, $X = \bigcup_{y_i \in X_i} \operatorname{int} M_i^{-1}(y_i)$. Let $x = (x_i)_{i \in I} \in X$, where $x_i \in X_i$, for all $i \in I$. But the hypothesis "*1)*" assumes that

$$X^i = \bigcup_{y_i \in X_i} \operatorname{int} H_i^{-1}(y_i) \text{ and } X_i = \bigcup_{z^i \in X^i} \operatorname{int}\left(H^i\right)^{-1}(z^i).$$

Hence we have:

a) $x^i \in X^i \overset{1)}{\Rightarrow} \exists\, y_i \in X_i$, such that $x^i \in \operatorname{int} H_i^{-1}(y_i) \Rightarrow \exists\, y_i \in X_i$ and $\exists\, D^i \subset X^i$, an open subset, such that $x^i \in D^i \subseteq H_i^{-1}(y_i) \Leftrightarrow x^i \in D^i$ and $\forall\, z^i \in D^i$, $z^i \in H_i^{-1}(y_i)$, that is, $y_i \in H_i(z^i)$.

b) $x_i \in X_i \overset{1)}{\Rightarrow} \exists\, v^i \in X^i$, such that $x_i \in \operatorname{int} H_i^{-1}(v_i) \Rightarrow \exists\, v^i \in X^i$, and $\exists\, E_i \subset X_i$, an open subset, such that $x_i \in E_i \subseteq H_i^{-1}(v^i) \Leftrightarrow x_i \in E_i$ and $\forall\, t_i \in E_i$,

$t_i \in \left(H^i\right)^{-1}\left(v^i\right)$, that is, $v^i \in H^i\left(x_i\right)$.

Denote $F_i = D^i \underline{\times} E_i \subset X^i \underline{\times} X_i = X$, understanding by $D^i \underline{\times} E_i$, the set of all $y = \left(y_i\right)_{i\in I} \in X$, such that $y^i \in D^i$ and $y_i \in E_i$.

I) We remark that F_i is an open subset of X. Indeed, if $a = \left(a_i\right)_{i\in I} \in F_i$, then $a^i \in D^i$ and $a_i \in E_i$ (for all $i \in I$). But D^i and E_i are open sets and hence they are neighborhood of $a^i \in X^i$ and $a_i \in X_i$, respectively. Therefore F_i is a neighborhood of a.

II) Let $x = \left(x_i\right)_{i\in I} \in X$. We have $x^i \in X^i \overset{a)}{\Rightarrow} \exists\, y_i \in X_i$ and $\exists\, D^i \subset X^i$, an open subset, such that $x^i \in D^i$ and $\forall\, z^i \in D^i$, $y_i \in H_i\left(z^i\right)$. Also we have $x_i \in X_i \overset{b)}{\Rightarrow} \exists\, v^i \in X^i$ and $\exists\, E_i \subseteq X_i$, an open subset, such that $x_i \in E_i$ and $\forall\, t_i \in E_i$, $v^i \in H^i\left(t_i\right)$.

Denote $v^i = y^i$ and $t_i = z_i$, therefore $y = \left(y^i, y_i\right)$ and $z = \left(z^i, z_i\right) \in X^i \underline{\times} X_i$. Consider $F_i = D^i \underline{\times} E_i$, an *open* set (see "I)"). We have $x = \left(x^i, x_i\right) \in F_i$ and, for all $z = \left(z^i, z_i\right) \in F_i$,

$$y \in H^i\left(z_i\right) \underline{\times} H_i\left(z^i\right) \Rightarrow y_i \in H_i\left(z_i\right) \subset P_i\left(y\right) = M_i\left(z\right)$$

$\left(\Leftrightarrow z \in M_i^{-1}\left(y_i\right)\right)$. Hence $\exists\, y_i \in X_i$ and $\exists\, F_i \subseteq X$, an open subset such that $x \in F_i$ and $\forall\, z \in F_i,\ z \in M_i^{-1}\left(y_i\right) \Rightarrow x \in \bigcup_{y_i \in X_i} \operatorname{int} M_i^{-1}\left(y_i\right)$. Hence $X = \bigcup_{y_i \in X_i} \operatorname{int} M_i^{-1}\left(y_i\right)$. Similarly, $X = \bigcup_{y^i \in X^i} \operatorname{int}\left(M^i\right)^{-1}\left(y^i\right)$. We remark that the multimaps $M_i : X \to 2^{X_i}$ and $M^i : X \to 2^{X^i}$ satisfy the hypothesis of Corollary 2.2 in [7], for $J = I$ and $Y = X$. So, there exists a quasi-coincidence point $\left(\tilde{x}, \tilde{y}\right) \in X \times X$, that is, there exists $\tilde{x} = \left(\tilde{x}_i\right)_{i\in I} \in X$ and $\tilde{y} = \left(\tilde{y}_i\right)_{i\in I} \in X$ such that, for all $i \in I$, $\tilde{x}_i \in M_i\left(\tilde{y}\right)$ and $\tilde{y}^i \in M^i\left(\tilde{x}\right)$, that is,

$$\tilde{x}_i \in P_i\left(U_i\left(\tilde{y}\right)\right) = P_i\left(H^i\left(y_i\right) \underline{\times} H_i\left(y^i\right)\right) = H_i\left(\tilde{y}^i\right) \text{ and}$$

$$\tilde{y}^i \in P^i\left(U_i\left(\tilde{x}\right)\right) = P^i\left(H^i\left(\tilde{x}_i\right) \underline{\times} H_i\left(\tilde{x}^i\right)\right) = H^i\left(\tilde{x}^i\right).$$

Hence $\tilde{x}_i \in H_i\left(\tilde{y}^i\right)$ and $\tilde{y}^i \in H^i\left(\tilde{x}^i\right)$, for all $i \in I$. $\square$

In the following theorem appear two families $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ of nonempty convex subsets of some completely metrizable locally convex spaces, and four families of multimaps, P_i, Q_i, P^i, Q^i $(i\in I)$, defined on X^i, Y^i, X_i, and Y_i, respectively, with values in the power sets of Y_i, X_i, Y^i, and X^i, respectively. We will give sufficient conditions to obtain two *quasi-coincidence points* for $(Q_i)_{i\in I}, (P^i)_{i\in I}$ and $(P_i)_{i\in I}, (Q^i)_{i\in I}$, respectively.

Theorem 3.2 *Let I be a finite index set, let $(E_i)_{i\in I}$, $(F_i)_{i\in I}$ be two families of completely metrizable locally convex spaces. For all $i\in I$, let also $X_i \subseteq E_i$ and $Y_i \subseteq F_i$ be nonempty convex sets, and let $X=\prod_{i\in I} X_i$, and $Y=\prod_{i\in I} Y_i$.*

For all $i\in I$, let $P_i : X^i \to 2^{Y_i}$, $Q_i : Y^i \to 2^{X_i}$, $P^i : X_i \to 2^{Y^i}$ and $Q^i : Y_i \to 2^{X^i}$ be nonempty-valued and convex-valued multimaps. Suppose that, for all $i\in I$, the following conditions hold:

1) $X^i=\bigcup_{y_i\in Y_i} \operatorname{int} P_i^{-1}(y_i)$ *and* $X_i=\bigcup_{y^i\in Y^i} \operatorname{int}(P^i)^{-1}(y^i)$; $Y^i=\bigcup_{x_i\in X_i} \operatorname{int} Q_i^{-1}(x_i)$ *and* $Y_i=\bigcup_{x^i\in X^i} \operatorname{int}(Q_i)^{-1}(x^i)$;

2) P_i, Q_i, P^i *and* Q^i *are compact.*

Then, there exist two quasi-coincidence points, $(\tilde{x},\tilde{t})$ and $(\tilde{y},\tilde{s})$ in $X\times Y$, for the families $(Q_i)_{i\in I}, (P^i)_{i\in I}$ and $(P_i)_{i\in I}, (Q^i)_{i\in I}$, respectively, that is, if $\tilde{x}=(\tilde{x}_i)_{i\in I}\in X$, $\tilde{t}=(\tilde{t}_i)_{i\in I}\in Y$, and $\tilde{y}=(\tilde{y}_i)_{i\in I}\in Y$, $\tilde{s}=(\tilde{s}_i)_{i\in I}\in X$, it follows that $\tilde{x}_i\in Q_i(\tilde{t}^i)$, $\tilde{t}^i\in P^i(\tilde{x}_i)$, $\tilde{y}_i\in P_i(\tilde{s}^i)$ and $\tilde{s}^i\in Q^i(\tilde{y}_i)$, for all $i\in I$.

Proof. Consider $Z_i=X_i\times Y_i$, $i\in I$ and $Z=\prod_{i\in I} Z_i$. Define:

$$U_i : Z^i \to 2^{Z_i} \text{ and } U^i : Z_i \to 2^{Z^i}$$

by the following formulas:

$$U_i(z^i)=Q_i(y^i)\times P_i(x^i) \text{ and } U^i(z_i)=Q^i(y_i)\times P^i(x_i),$$

where $z_i=(x_i,y_i)\in Z_i$, for $x_i\in X_i$ and $y_i\in Y_i$, for all $i\in I$. From "*1)*" it follows that, for all $i\in I$, we have:

$$Z^i=\bigcup_{z_i\in Z_i} \operatorname{int} U_i^{-1}(z_i) \text{ and}$$

$$Z_i = \bigcup_{z^i \in Z^i} \mathrm{int}\left(U^i\right)^{-1}\left(z^i\right).$$

Indeed, for example, we have:

$$Z^i = X^i \times Y^i \overset{1)}{=} \left(\bigcup_{y_i \in Y_i} \mathrm{int}\, P_i^{-1}\left(y_i\right) \right) \times \left(\bigcup_{x_i \in X_i} \mathrm{int}\, Q_i^{-1}\left(x_i\right) \right) =$$

$$= \bigcup_{z_i = (x_i, y_i) \in X_i \times Y_i} \mathrm{int}\left(P_i^{-1}\left(y_i\right) \times Q_i^{-1}\left(x_i\right)\right) = \bigcup_{z_i \in Z_i} \mathrm{int}\, U_i^{-1}\left(z_i\right).$$

The last two equalities can be proved as in our paper [6], Theorem 3.2. Also, from "*2)*", it follows that U_i and U^i are compact multimaps. Indeed, for example, because P_i and Q_i are compact, there exist the compact sets $L_i \subseteq Y_i$ and $K_i \subseteq X_i$ such that $P_i\left(X^i\right) \subseteq L_i$ and $Q_i\left(Y^i\right) \subseteq X_i$. Then, by using Theorem 3.1, we find $\tilde{z} = \left(\tilde{z}_i\right)_{i \in I} \in Z$ and $\tilde{u} = \left(\tilde{u}_i\right)_{i \in I} \in Z$, such that

$$\tilde{z}_i \in U_i\left(\tilde{u}^i\right) \text{ and } \tilde{u}^i \in U_i\left(\tilde{z}_i\right), \text{ for all } i \in I.$$

Putting $\tilde{z}_i = \left(\tilde{x}_i, \tilde{y}_i\right)$ and $\tilde{u}_i = \left(\tilde{s}_i, \tilde{t}_i\right)$, we obtain

$$\left(\tilde{x}_i, \tilde{y}_i\right) \in Q_i\left(\tilde{t}^i\right) \times P_i\left(\tilde{s}^i\right) \text{ and}$$

$$\left(\tilde{s}^i, \tilde{t}^i\right) \in Q^i\left(\tilde{y}_i\right) \times P^i\left(\tilde{x}_i\right),$$

that is, $\tilde{x}_i \in Q_i\left(\tilde{t}^i\right)$, $\tilde{y}_i \in P_i\left(\tilde{s}^i\right)$, $\tilde{s}^i \in Q^i\left(\tilde{y}_i\right)$ and $\tilde{t}^i \in P^i\left(\tilde{x}_i\right)$, for all $i \in I$. □

Now we will give an **economic interpretation** for the Theorem 3.2, with I a finite set.

Let I, E_i, F_i, X_i, Y_i, X, Y, P_i, Q_i, P^i, Q^i be like in Theorem 3.2. Consider I a finite index set. Let $\Gamma = \left(X_i, Y, P_i, Q_i, P^i, Q^i\right)_{i \in I}$ be an abstract economy with two companies having the same number of factories and without preference relations.

We suppose that:

1) The products between the factories in the same company are different;
2) The financial systems and management systems are independent between the factories in the same company, while;
3) Some collections of products are the same and some collections of products are different between different factories in the two companies.

In the first company, the set X_i denotes the *strategy set* of the i th factory ($i \in I$). In the second company, the set Y_i denotes the *strategy set* for the i th factory ($i \in I$). The set $X = \left(X_i\right)_{i \in I}$ represents the *strategies set* of the first company and the set

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$Y=\left(Y_i\right)_{i\in I}$ is considered as the *strategies set* of the second company. Also, for all $i\in I$, $X^i=\prod_{l\in I, l\neq i} X_l$ is the *strategies set* for all other factories different from the i th factory, in the first company, and $Y^i=\prod_{k\in I, k\neq i} Y_k$ is the *strategies set* for all other factories different from the i th factory, in the second company. The nonempty-valued and convex-valued multimaps $P_i : X^i \to 2^{Y_i}$, $Q_i : Y^i \to 2^{X_i}$, $P^i : X_i \to 2^{Y^i}$ and $Q^i : Y_i \to 2^{X^i}$, $i\in I$ are the *constraints* of the economy Γ. For example, for each fixed i in I:

a) $P^i : X_i \to 2^{Y^i}$ is the *constraint* which restricts the strategies of all factories, different from the i th factory in the second company to its subset $P^i\left(X_i\right)$ (when the factory has chosen its strategy x_i in X_i); that is, their strategies depend on the strategy of the i th factory in the first company.

b) $Q_i : Y^i \to 2^{X_i}$ is the *constraint* which restricts the strategies of the i th factory in the first company to its subset $Q^i\left(Y^i\right)$ (when all the factories of the second company, different from the i th factory, have chosen their strategies y^i in Y^i); that is, its strategy depends on the all strategies of the factories of the second company different from the i th factory.

Then, under the hypothesis of the Theorem 3.2, including compactness assumptions on all constraint multimaps in Γ, there exists a *strategies combination* $\left(\tilde{x},\tilde{t}\right)$ and $\left(\tilde{y},\tilde{s}\right)$ in $X\times Y$, such that, if $\tilde{x}=\left(\tilde{x}_i\right)_{i\in I}\in X$, $\tilde{t}=\left(\tilde{t}_i\right)_{i\in I}\in Y$ and $\tilde{y}=\left(\tilde{y}_i\right)_{i\in I}\in Y$, $\tilde{s}=\left(\tilde{s}_i\right)_{i\in I}\in X$, the following are true:

$$\tilde{x}_i\in Q_i\left(\tilde{t}^i\right),\ \tilde{t}^i\in P^i\left(\tilde{x}_i\right),\ \tilde{y}_i\in P_i\left(\tilde{s}^i\right) \text{ and } \tilde{s}^i\in Q^i\left(\tilde{y}_i\right), \text{ for all } i\in I.$$

With this strategy combination, each factory in this economy can chose a collection of products which are suitable for its welfare.

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