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The Exponential Change of Finsler Metric and Relation between Imbedding Class Numbers of their Tangent Riemannian Spaces

By

B. N. PRASAD*, **H. S. Shukla and O. P. Pandey**

*C-10, Surajkund Avas Vikas Colony, Gorakhpur, India
Department of Mathematics & Statistics
DDU Gorakhpur University,Gorakhpur, India
e-mail : baijnath_prasad2003@yahoo.co.in,
profhsshuklagkp@rediffmail.com, oppandey1988@gmail.com

Abstract

In the present paper the relation between imbedding class numbers of tangent Riemannian spaces of (M^n, L) and (M^n, L^*) have been obtained, where the Finsler metric L^* is obtained from L by $L^* = Le^{\beta/L}$ and M^n is the differentiable manifold.

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1. Introduction

Let (M^n, L) be an $n-$dimensional Finsler space on a differentiable manifold M^n, equipped with the fundamental function $L(x, y)$. In 1971, Matsumoto [2] introduced the transformation of Finsler metric:

$$L^*(x, y) = L(x, y) + \beta(x, y) \tag{1.1}$$

where $\beta(x, y) = b_i(x)y^i$ is a differentiable one-form on M^n. In 1984 Shibata [5] has studied the properties of Finsler space (M^n, L^*) whose metric function $L^*(x, y)$ is obtained from $L(x, y)$ by the relation $L^*(x, y) = f(L, \beta)$ where f is positively homogeneous of degree one in L and β. This change of metric function is called a $\beta-$change. The change (1.1) is a particular case of $\beta-$change called Randers change. The following theorem has importance under Randers change.

Theorem 1.1. [2] Let (M^n, L^*) be a locally Minkowskian $n-$space obtained from a locally Minkowskian $n-$space (M^n, L) by the change (1.1). If the tangent Riemannian $n-$space (M^n_X, g_X) to (M^n, L) is of imbedding class r, then the tangent Riemannian $n-$space (M^n_X, g^*_X) to (M^n, L^*) is of imbedding class at most $r+2$.

Another particular $\beta-$change of Finsler metric function is a Kropina change of metric function given by

$$L^*(x,y) = \frac{L^2(x,y)}{\beta(x,y)}. \tag{1.2}$$

If $L(x,y)$ reduces to the metric function of Riemannian space then $L^*(x,y)$ reduces to the metric function of Kropina space [4]. Due to this reason the transformation (1.2) has been called the Kropina change of Finsler metric.

In 2003, Singh, Prasad and Kumari [6] introduced the Kropina change of Finsler metric given by (1.2) and proved that Theorem 1.1 is valid for this transformation also.

In 1990, Prasad, Shukla and Singh [3] introduced the same transformation (1.1) under the condition that $\beta = b_i(x,y)\,y^i$ where $b_i(x,y)$ are components of the $h-$vector field. They proved that the above theorem is valid for this transformation also.

In the present paper we consider an exponential change of Finsler metric given by

$$L^* = Le^{\beta/L}$$

and we have proved that Theorem 1.1 is valid for this transformation also.

2. The Finsler Space (M^n, L^*)

Let (M^n, L) be a given Finsler space and let $b_i(x)dx^i$ be a one-form on M^n. We shall define on M^n a function $L^*(x,y)\,(>0)$ by the equation

$$L^* = Le^{\beta/L}, \tag{2.1}$$

where we put $\beta(x,y) = b_i(x)y^i$. To find the metric tensor g^*_{ij}, the angular metric tensor h^*_{ij}, the Cartan tensor C^*_{ijk} and the $v-$curvature tensor of (M^n, L^*) we use the following results:

$$\dot\partial_i \beta = b_i \qquad \dot\partial_i L = l_i, \qquad \dot\partial_j l_i = \frac{1}{L}h_{ij}, \tag{2.2}$$

where $\dot\partial_i$ stands for $\frac{\partial}{\partial y^i}$ and h_{ij} are components of angular metric tensor of (M^n, L) given by $h_{ij} = g_{ij} - l_i l_j = L\dot\partial_i\dot\partial_j L$.

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The successive differentiation of (2.1) with respect to y^i and y^j give

$$l_i^* = \left(1 - \frac{\beta}{L}\right) e^{\beta/L} l_i + e^{\beta/L} b_i, \tag{2.3}$$

$$h_{ij}^* = e^{2\beta/L} \left\{ \left(1 - \frac{\beta}{L}\right) h_{ij} + \frac{\beta^2}{L^2} l_i l_j - \frac{\beta}{L}(l_i b_j + l_j b_i) + b_i b_j \right\}. \tag{2.4}$$

From (2.3) and (2.4) we get the following relation between metric tensors of (M^n, L) and (M^n, L^*).

$$\begin{aligned} g_{ij}^* = e^{2\beta/L} \Big\{ & \left(1 - \frac{\beta}{L}\right) g_{ij} - \frac{\beta}{L} \left(1 - \frac{2\beta}{L}\right) l_i l_j \\ & + \left(1 - \frac{2\beta}{L}\right)(l_i b_j + l_j b_i) + 2 b_i b_j \Big\}. \end{aligned} \tag{2.5}$$

The contravariant components of the metric tensor of (M^n, L^*) derived from (2.5), are given by

$$\begin{aligned} g^{*ij} = e^{-2\beta/L} \Big[& \frac{L}{L-\beta} g^{ij} + \frac{(L-2\beta)(\beta - L\triangle)}{(L-\beta)\{(1-\triangle)L - \beta\}} l^i l^j \\ & - \frac{L(L-2\beta)}{(L-\beta)\{(1-\triangle)L-\beta\}}(l^i b^j + l^j b^i) \\ & - \frac{L^2}{(L-\beta)\{(1-\triangle)L-\beta\}} b^i b^j \Big], \end{aligned} \tag{2.6}$$

where we put $b^2 = g^{ij} b_i b_j$, $b^i = g^{ij} b_j$, $l^i = g^{ij} l_j$ and $\triangle = \frac{\beta^2}{L^2} - b^2$.

Differentiating (2.5) with respect to y^k and using (2.2) we get the following relation between the Cartan tensors of (M^n, L) and (M^n, L^*):

$$\begin{aligned} C_{ijk}^* = \frac{1}{2}\dot{\partial}_k g_{ij}^* = e^{2\beta/L} \Big[& \left(1 - \frac{\beta}{L}\right) C_{ijk} + \frac{1}{2L}\left(1 - \frac{2\beta}{L}\right) \\ & \times (h_{ij} m_k + h_{jk} m_i + h_{ik} m_j) + \frac{2}{L} m_i m_j m_k \Big], \end{aligned} \tag{2.7}$$

where $m_i = b_i - \frac{\beta}{L} l_i$. It is to be noted that

$$m_i l^i = 0, \quad m_i b^i = b^2 - \frac{\beta^2}{L^2} = -\triangle, \quad h_{ij} l^j = 0, \quad h_{ij} m^i = h_{ij} b^i = m_j, \tag{2.8}$$

where $m^i = g^{ij} m_j = b^i - \frac{\beta}{L} l^i$.

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The quantities corresponding to (M^n, L^*) will be denoted by putting $*$ on those quantities. To find $C^{*i}_{jk} = g^{*ih}C^*_{jhk}$ we use (2.6), (2.7) and (2.8). We get

$$\begin{aligned}
C^{*i}_{jk} = C^i_{jk} &- \frac{1}{(1-\triangle)L-\beta}C_{.jk}n^i + \frac{L-2\beta}{2L(L-\beta)} \\
&\times(h^i_j m_k + h_{jk}m^i + h^i_k m_j) + \frac{2}{L-\beta}m_j m_k m^i \\
&- \frac{(1-2\triangle)L-2\beta}{L(L-\beta)\{(1-\triangle)L-\beta\}}m_j m_k n^i \\
&+ \frac{\triangle(L-2\beta)}{2L\{(1-\triangle)L-\beta\}(L-\beta)}h_{jk}n^i,
\end{aligned} \tag{2.9}$$

where $n^i = (L-2\beta)l^i + Lb^i$ and $C_{.jk} = C_{hjk}b^h$.

Throughout this paper we use the symbol . to denote the contraction with b^i. To find the $v-$curvature tensor of (M^n, L^*) we use the following:

$$\begin{aligned}
&C_{rij}n^r = LC_{.ij}, \quad C_{rij}m^r = C_{.ij}, \quad C_{rij}h^r_h = C_{ijh}, \quad m_r n^r = -\triangle L, \\
&h_{jr}n^r = Lm_j, \quad h^r_h m_r = m_h, \quad h_{jr}h^r_h = h_{jh}, \quad m_r m^r = -\triangle.
\end{aligned} \tag{2.10}$$

The $v-$curvature tensor S^*_{hijk} of (M^n, L^*) is defined as

$$\begin{aligned}
S^*_{hijk} &= C^{*r}_{hk}C^*_{rij} - C^{*r}_{hj}C^*_{rik} \\
&= g^*_{ri}[\dot\partial_k C^{*r}_{hj} - \dot\partial_j C^{*m}_{hk} + C^{*m}_{hj}C^{*r}_{mk} - C^{*m}_{hk}C^{*r}_{mj}].
\end{aligned} \tag{2.11}$$

From (2.7), (2.8), (2.9), (2.10) and (2.11) we get the following relation between $v-$curvature tensors of (M^n, L) and (M^n, L^*):

$$S^*_{hijk} = e^{2\beta/L}\frac{L-\beta}{L}S_{hijk} + d_{ik}d_{hj} - d_{ij}d_{hk} + E_{ij}E_{hk} - E_{ik}E_{hj}, \tag{2.12}$$

where

$$d_{ij} = \frac{e^{\beta/L}\sqrt{L-\beta}}{\sqrt{(1-\triangle)L-\beta}}\left\{C_{.ij} - \frac{L-2\beta}{2L^2}h_{ij} - \frac{1}{L-\beta}m_i m_j\right\}, \tag{2.13}$$

$$E_{ij} = \frac{e^{\beta/L}}{2L}\left\{\frac{L-2\beta}{L}h_{ij} + \frac{3L-2\beta}{L-\beta}m_i m_j\right\}. \tag{2.14}$$

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By direct calculation we get the following results which will be used in the latter section of the paper.

$$\text{(a)}\quad \dot{\partial}_k\left(e^{\beta/L}\sqrt{1-\frac{\beta}{L}}\right) = \frac{(L-2\beta)e^{\beta/L}}{2L\sqrt{L}\,\sqrt{L-\beta}}m_k,$$

$$\text{(b)}\quad \dot{\partial}_k\left(\frac{e^{\beta/L}\sqrt{L-\beta}}{\sqrt{(1-\triangle)L-\beta}}\right) = \frac{e^{\beta/L}[2(L-\beta)-\triangle(L-2\beta)]}{2\sqrt{L-\beta}[(1-\triangle)L-\beta]^{3/2}}m_k + \frac{e^{\beta/L}.L\sqrt{L-\beta}}{[(1-\triangle)L-\beta]^{3/2}}C_{..k},$$

$$(2.15)\quad \text{(c)}\quad \dot{\partial}_k\left(\frac{L-2\beta}{2L^2}\right) = -\frac{1}{L^2}m_k - \frac{L-2\beta}{2L^3}l_k,$$

$$\text{(d)}\quad \dot{\partial}_k\left(\frac{1}{L-\beta}\right) = -\frac{1}{(L-\beta)^2}(l_k-b_k),$$

$$\text{(e)}\quad \dot{\partial}_k\left(\frac{e^{\beta/L}}{2L}\right) = \frac{e^{\beta/L}}{2L^2}(m_k-l_k),$$

$$\text{(f)}\quad \dot{\partial}_k\left(\frac{L-2\beta}{L}\right) = -\frac{2}{L}m_k,$$

$$\text{(g)}\quad \dot{\partial}_k\left(\frac{3L-2\beta}{L-\beta}\right) = \frac{L}{(L-\beta)^2}m_k, \quad \text{(h)}\quad \dot{\partial}_k\triangle = 2C_{..k} + \frac{2\beta}{L^2}m_k.$$

3. Imbedding Class Numbers

The tangent vector space M^n_X to M^n at every point x is considered as the Riemannian $n-$space (M^n_X, g_X) with the Riemannian metric $g_X = g_{ij}(x,y)dy^i dy^j$. Then the components of the Cartan tensor are the Christoffel symbols associated with g_X:

$$C^i_{jk} = \frac{1}{2}g^{ih}(\dot{\partial}_k g_{jh} + \dot{\partial}_j g_{hk} - \dot{\partial}_h g_{jk}).$$

Thus C^i_{jk} defines the components of the Riemannian connection on M^n_X and $v-$covariant derivative say

$$(3.0)\qquad X_i|_j = \dot{\partial}_j X_i - X_h C^h_{ij}$$

is the covariant derivative of covariant vector X_i with respect to Riemannian connection C^i_{jk} on M^n_X. It is observed that the $v-$curvature tensor S_{hijk} of (M^n, L) is the Riemannian Christoffel curvature tensor of the Riemannian space (M^n_X, g_X) at a point X. The space (M^n_X, g_X) equipped with such a Riemannian connection is called the tangent Riemannian $n-$space [2].

It is well known [1] that any Riemannian $n-$space V^n can be imbedded isometrically in a Euclidean space of dimension $\frac{n(n+1)}{2}$ in the analytics case. If $n+r$ is the lowest dimension of the Euclidean space in which V^n is imbedded isometrically, then the integer r is called the imbedding class number of V^n. The fundamental theorem of isometric imbedding ([1] page 190) is that the tangent Riemannian $n-$space (M^n_X, g_X) is locally imbedded isometrically in a Euclidean $(n+r)-$space if and only if there exist $r-$number $\epsilon_P = \pm 1$, $r-$symmetric tensors $H_{(P)ij}$ and $\frac{r(r-1)}{2}$ covariant vector fields $H_{(P,Q)i} = -H_{(Q,P)i}$; $P, Q = 1, 2, \ldots, r$, satisfying the Gauss equations

$$S_{hijk} = \sum_P \epsilon_P \{H_{(P)hj} H_{(P)ik} - H_{(P)ij} H_{(P)hk}\}, \tag{3.1}$$

the Codazzi equations

$$H_{(P)ij}|_k - H_{(P)ik}|_j = \sum_Q \epsilon_Q \{H_{(Q)ij} H_{(Q,P)k} - H_{(Q)ik} H_{(Q,P)j}\}, \tag{3.2}$$

and the Ricci-Kühne equations

$$\begin{aligned} H_{(P,Q)i}|_j - H_{(P,Q)j}|_i + \sum_R \epsilon_R \{H_{(R,P)i} H_{(R,Q)j} - H_{(R,P)j} H_{(R,Q)i}\} \\ + g^{hk} \{H_{(P)hi} H_{(Q)kj} - H_{(P)hj} H_{(Q)ki}\} = 0. \end{aligned} \tag{3.3}$$

The numbers $\epsilon_P = \pm 1$ are the indicators of unit normal vector N_P to M^n and $H_{(P)ij}$ are the second fundamental tensors of M^n with respect to the normals N_P.

The following imbedding theorem is main result of the present paper.

Theorem 3.1. *Let (M^n, L^*) be a Finsler space obtained from a Finsler space (M^n, L) by the exponential change (2.1). If the tangent Riemannian $n-$space (M^n_X, g_X) to (M^n, L) is of imbedding class r, then the tangent Riemannian $n-$space (M^n_X, g^*_X) to (M^n, L^*) is of imbedding class at most $r+2$.*

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Proof. In order to prove the theorem, we put

$$
\begin{aligned}
&H^*_{(P)ij} = e^{\beta/L}\sqrt{1-\frac{\beta}{L}}\,H_{(P)ij}, \qquad \epsilon^*_P = \epsilon_P,\ \ P=1,2,\dots,r\\
(3.4)\qquad &H^*_{(r+1)ij} = d_{ij}, \qquad \epsilon^*_{r+1} = 1\\
&H^*_{(r+2)ij} = E_{ij}, \qquad \epsilon^*_{r+2} = -1.
\end{aligned}
$$

Then it follows from (2.12) and (3.1) that

$$
S^*_{hijk} = \sum_{\mu=1}^{r+2} \epsilon^*_\mu \{H^*_{(\mu)hj}H^*_{(\mu)ik} - H^*_{(\mu)hk}H^*_{(\mu)ij}\},
$$

which is the Gauss equation of (M^n_X, g^*_X).

Moreover to verify Codazzi and Ricci-Kühne equations of (M^n_X, g^*_X) we put

$$
\begin{aligned}
&H^*_{(P,Q)i} = -H^*_{(Q,P)i} = H_{(P,Q)i}, \quad P,Q = 1,2,,\dots,r\\
&H^*_{(P,r+1)i} = -H^*_{(r+1,P)i} = \frac{\sqrt{L}}{\sqrt{(1-\triangle)L-\beta}}H_{(P).i}, \quad P=1,2,\dots,r\\
(3.5)\quad &H^*_{(P,r+2)i} = -H^*_{(r+2,P)i} = 0, \quad P=1,2,\dots,r.\\
&H^*_{(r+1,r+2)i} = -H^*_{(r+2,r+1)i} = \frac{1}{\sqrt{L-\beta}\sqrt{(1-\triangle)L-\beta}}m_i.
\end{aligned}
$$

The Codazzi equations of (M^n_X, g^*_X) consists of the following three equations:

$$
\begin{aligned}
(3.6)\ (a)\quad H^*_{(P)ij}\overset{*}{|}_k - H^*_{(P)ik}\overset{*}{|}_j = &\sum_Q \epsilon^*_Q\{H^*_{(Q)ij}H^*_{(Q,P)k} - H^*_{(Q)ik}H^*_{(Q,P)j}\}\\
&+\epsilon^*_{r+1}\{H^*_{(r+1)ij}H^*_{(r+1,P)k} - H^*_{(r+1)ik}H^*_{(r+1,P)j}\}\\
&+\epsilon^*_{r+2}\{H^*_{(r+2)ij}H^*_{(r+2,P)k} - H^*_{(r+2)ik}H^*_{(r+2,P)j}\}
\end{aligned}
$$

$$
\begin{aligned}
(b)\quad H^*_{(r+1)ij}\overset{*}{|}_k - H^*_{(r+1)ik}\overset{*}{|}_j = &\sum_Q \epsilon^*_Q\{H^*_{(Q)ij}H^*_{(Q,r+1)k} - H^*_{(Q)ik}H^*_{(Q,r+1)j}\}\\
&+\epsilon^*_{r+2}\{H^*_{(r+2)ij}H^*_{(r+2,r+1)k} - H^*_{(r+2)ik}H^*_{(r+2,r+1)j}\}\\
(c)\quad H^*_{(r+2)ij}\overset{*}{|}_k - H^*_{(r+2)ik}\overset{*}{|}_j = &\sum_Q \epsilon^*_Q\{H^*_{(Q)ij}H^*_{(Q,r+2)k} - H^*_{(Q)ik}H^*_{(Q,r+2)j}\}\\
&+\epsilon^*_{r+1}\{H^*_{(r+1)ij}H^*_{(r+1,r+2)k} - H^*_{(r+1)ik}H^*_{(r+1,r+2)j}\}.
\end{aligned}
$$

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To prove these equations we note that for any symmetric tensor X_{ij} satisfying $X_{ij}l^i = X_{ij}l^j = 0$, we have from (2.9) and (3.0),

$$(3.7)\quad X_{ij}\overset{*}{|}_k - X_{ik}\overset{*}{|}_j = X_{ij}|_k - X_{ik}|_j + \frac{L}{(1-\triangle)L-\beta}\{C_{.ik}X_{.j} - C_{.ij}X_{.k}\} + \frac{L-2\beta}{2L(L-\beta)}(X_{ik}m_j - X_{ij}m_k) + \frac{L}{(L-\beta)\{(1-\triangle)L-\beta\}} \times (X_{.k}m_j - X_{.j}m_k)m_i + \frac{L-2\beta}{2L\{(1-\triangle)L-\beta\}}(h_{ij}X_{.k} - h_{ik}X_{.j}).$$

In view of (3.4) and (3.5), equation (3.6)a is equivalent to

$$(3.8)\quad \left(e^{\beta/L}\sqrt{1-\frac{\beta}{L}}.H_{(P)ij}\right)\overset{*}{\Big|}_k - \left(e^{\beta/L}\sqrt{1-\frac{\beta}{L}}.H_{(P)ik}\right)\overset{*}{\Big|}_j = \left(e^{\beta/L}\sqrt{1-\frac{\beta}{L}}\right).\sum_Q \epsilon_Q\{H_{(Q)ij}H_{(Q,P)k} - H_{(Q)ik}H_{(Q,P)j}\} - \frac{\sqrt{L}}{\sqrt{(1-\triangle)L-\beta}}\{H_{(P).k}d_{ij} - H_{(P).j}d_{ik}\}.$$

Applying formula (3.7) for $H_{(P)ij}$ and using equation (2.13) and (2.15)a, we get

$$\left(e^{\beta/L}\sqrt{1-\frac{\beta}{L}}.H_{(P)ij}\right)\overset{*}{|}_k - \left(e^{\beta/L}\sqrt{1-\frac{\beta}{L}}.H_{(P)ik}\right)\overset{*}{|}_j = \left(e^{\beta/L}\sqrt{1-\frac{\beta}{L}}\right) \times\{H_{(P)ij}|_k - H_{(P)ik}|_j\} - \frac{\sqrt{L}}{\sqrt{(1-\triangle)L-\beta}}\{H_{(P).k}d_{ij} - H_{(P).j}d_{ik}\},$$

which after using equation (3.2), gives equation (3.8).

In view of (3.4) and (3.5), equation (3.6)b is equivalent to

$$(3.9)\quad d_{ij}\overset{*}{|}_k - d_{ik}\overset{*}{|}_j = \sum_Q \epsilon_Q\{H_{(Q)ij}H_{(Q).k} - H_{(Q)ik}H_{(Q).j}\}\left(\frac{e^{\beta/L}\sqrt{L-\beta}}{\sqrt{(1-\triangle)L-\beta}}\right) + \frac{1}{\sqrt{L-\beta}\sqrt{(1-\triangle)L-\beta}}\{E_{ij}m_k - E_{ik}m_j\}.$$

To verify (3.9) we note that

$$(3.10)\quad C_{.ij}|_k - C_{.ik}|_j = b_h S^h_{ijk}$$

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$$(3.11)\qquad b|_k = -\frac{1}{b}C_{..k}, \qquad h_{ij}|_k - h_{ik}|_j = \frac{1}{L}(h_{ij}l_k - h_{ik}l_j)$$

$$(3.12)\qquad m_i|_k = -C_{.ik} - \frac{\beta}{L^2}h_{ik} - \frac{1}{L}l_i m_k.$$

The $v-$covariant differentiation of (2.13) and use of (2.15)b, c, d will give the value of $d_{ij}|_k$. Then taking skew-symmetric part of $d_{ij}|_k$ in j and k using (3.10), (3.11), (3.12), we get

$$(3.13)\quad d_{ij}|_k - d_{ik}|_j = \frac{e^{\beta/L}\sqrt{L-\beta}}{[(1-\triangle)L-\beta]^{3/2}}\{L[C_{.ij}C_{..k} - C_{.ik}C_{..j}]$$
$$-\frac{L-2\beta}{2L}(h_{ij}C_{..k} - h_{ik}C_{..j}) + [(1-\triangle)L-\beta]b_h S^h_{ijk}$$
$$-\frac{L}{L-\beta}(C_{..k}m_j - C_{..j}m_k)m_i + \frac{\triangle(L+2\beta)}{2(L-\beta)}(C_{.ij}m_k - C_{.ik}m_j)$$
$$\left.+\frac{(L-2\beta)[2(L-\beta)-\triangle(3L+2\beta)]}{4L^2(L-\beta)}(h_{ij}m_k - h_{ik}m_j)\right\}.$$

Applying formula (3.7) for d_{ij} and using (3.13), we get

$$(3.14)\quad d_{ij}\overset{*}{|}_k - d_{ik}\overset{*}{|}_j = \frac{e^{\beta/L}\sqrt{L-\beta}}{\sqrt{(1-\triangle)L-\beta}}b_h S^h_{ijk}$$
$$+\frac{e^{\beta/L}L-2\beta}{2L^2\sqrt{L-\beta}\sqrt{(1-\triangle)L-\beta}}(h_{ij}m_k - h_{ik}m_j).$$

Substituting (3.1) and (2.14) in the right hand side of (3.14), we get equation (3.9).

In view of (3.4) and (3.5), equation (3.6)c is equivalent to

$$(3.15)\quad E_{ij}\overset{*}{|}_k - E_{ik}\overset{*}{|}_j = \frac{1}{\sqrt{L-\beta}\sqrt{(1-\triangle)L-\beta}}(d_{ij}m_k - d_{ik}m_j).$$

The $v-$covariant differentiation of (2.14) and use of equations (2.15)e, f, g will give the value of $E_{ij}|_k$. Then taking skew-symmetric part of $E_{ij}|_k$ in j and k and using (3.11), (3.12), we get

$$(3.16)\qquad E_{ij}|_k - E_{ik}|_j = \frac{e^{\beta/L}(3L-2\beta)}{2L(L-\beta)}(C_{.ij}m_k - C_{.ik}m_j)$$
$$-\frac{e^{\beta/L}(L-2\beta)}{2L^2(L-\beta)}(h_{ij}m_k - h_{ik}m_j).$$

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Applying formula (3.7) for E_{ij} and using (3.16), we get (3.15). This completes the proof of Codazzi equations of (M^n_X, g^*_X).

The Ricci Kühne equations of (M^n_X, g^*_X) consist of the following four equations:

$$\text{(3.17)(a)} \quad H^*_{(P,Q)i}\overset{*}{\big|}_j - H^*_{(P,Q)j}\overset{*}{\big|}_i + \sum_Q \epsilon^*_Q\{H^*_{(R,P)i}H^*_{(R,Q)j} - H^*_{(R,P)j}H^*_{(R,Q)i}\} + \epsilon^*_{r+1}\{H^*_{(r+1,P)i}H^*_{(r+1,Q)j} - H^*_{(r+1,P)j}H^*_{(r+1,Q)i}\} + \epsilon^*_{r+2}\{H^*_{(r+2,P)i}H^*_{(r+2,Q)j} - H^*_{(r+2,P)j}H^*_{(r+2,Q)i}\} + g^{*hk}\{H^*_{(P)hi}H^*_{(Q)kj} - H^*_{(P)hj}H^*_{(Q)ki}\} = 0, \; P, Q = 1, 2, \ldots, r$$

$$\text{(b)} \quad H^*_{(P,r+1)i}\overset{*}{\big|}_j - H^*_{(P,r+1)j}\overset{*}{\big|}_i + \sum_R \epsilon^*_R\{H^*_{(R,P)i}H^*_{(R,r+1)j} - H^*_{(R,P)j}H^*_{(R,r+1)i}\} + \epsilon^*_{r+2}\{H^*_{(r+2,P)i}H^*_{(r+2,r+1)j} - H^*_{(r+2,P)j}H^*_{(r+2,r+1)i}\} + g^{*hk}\{H^*_{(P)hi}H^*_{(r+1)kj} - H^*_{(P)hj}H^*_{(r+1)ki}\} = 0, \; P = 1, 2, \ldots, r$$

$$\text{(c)} \quad H^*_{(P,r+2)i}\overset{*}{\big|}_j - H^*_{(P,r+2)j}\overset{*}{\big|}_i + \sum_R \epsilon^*_R\{H^*_{(R,P)i}H^*_{(R,r+2)j} - H^*_{(R,P)j}H^*_{(R,r+2)i}\} + \epsilon^*_{r+1}\{H^*_{(r+1,P)i}H^*_{(r+1,r+2)j} - H^*_{(r+1,P)j}H^*_{(r+1,r+2)i}\} + g^{*hk}\{H^*_{(P)hi}H^*_{(r+2)kj} - H^*_{(P)hj}H^*_{(r+2)ki}\} = 0, \; P = 1, 2, \ldots, r$$

$$\text{(d)} \quad H^*_{(r+1,r+2)i}\overset{*}{\big|}_j - H^*_{(r+1,r+2)j}\overset{*}{\big|}_i + \sum_R \epsilon^*_R\{H^*_{(R,r+1)i}H^*_{(R,r+2)j} - H^*_{(R,r+1)j} \times H^*_{(R,r+2)i}\} + g^{*hk}\{H^*_{(r+1)hi}H^*_{(r+2)kj} - H^*_{(r+1)hj}H^*_{(r+2)ki}\} = 0.$$

In view of (3.4) and (3.5), equation (3.17)a is equivalent to

$$\text{(3.18)} \quad H_{(P,Q)i}\overset{*}{\big|}_j - H_{(P,Q)j}\overset{*}{\big|}_i + \sum_R \epsilon_R\{H_{(R,P)i}H_{(R,Q)j} - H_{(R,P)j} \times H_{(R,Q)i}\} + \frac{L}{(1-\triangle)L-\beta}\{H_{(P).i}H_{(Q).j} - H_{(P).j}H_{(Q).i}\} + g^{*hk}\{H_{(P)hi}H_{(Q)kj} - H_{(P)hj}H_{(Q)ki}\}e^{2\beta/L}\left(1-\frac{\beta}{L}\right) = 0.$$

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Since $H_{(P,Q)i}\overset{*}{|}_j - H_{(P,Q)j}\overset{*}{|}_i = H_{(P,Q)i}|_j - H_{(P,Q)j}|_i$, equation (3.18) follows from (3.3), (2.6) and the facts that $H_{(P)ij}l^i = 0 = H_{(P,Q)i}l^i$.

By virtue of (3.4) and (3.5), equation (3.17)b may be written as

$$\left(\frac{\sqrt{L}}{\sqrt{(1-\triangle)L-\beta}}H_{(P).i}\right)\overset{*}{\Big|}_j - \left(\frac{\sqrt{L}}{\sqrt{(1-\triangle)L-\beta}}H_{(P).j}\right)\overset{*}{\Big|}_i + \sum_R \epsilon_R\{H_{(R,P)i}H_{(R).j} - H_{(R,P)j}H_{(R).i}\}\frac{\sqrt{L}}{\sqrt{(1-\triangle)L-\beta}} + g^{*hk}\{H_{(P)hi}d_{kj} - H_{(P)hj}d_{ki}\}\left(e^{\beta/L}\sqrt{1-\frac{\beta}{L}}\right) = 0. \tag{3.19}$$

Now

$$\left(\frac{\sqrt{L}}{\sqrt{(1-\triangle)L-\beta}}H_{(P).i}\right)\overset{*}{\Big|}_j = \frac{\sqrt{L}}{\sqrt{(1-\triangle)L-\beta}}H_{(P).i}\overset{*}{|}_j + \dot{\partial}_j\left(\frac{\sqrt{L}}{\sqrt{(1-\triangle)L-\beta}}\right)H_{(P).i}$$

Since $\dot{\partial}_j\left(\frac{\sqrt{L}}{\sqrt{(1-\triangle)L-\beta}}\right) = \frac{\sqrt{L}}{2[(1-\triangle)L-\beta]^{3/2}}\left\{\left(1+\frac{2\beta}{L}\right)m_j + 2LC_{..j}\right\}$ and $H_{(P).i}\overset{*}{|}_j - H_{(P).j}\overset{*}{\Big|}_i = H_{(P).i}|_j - H_{(P).j}|_i$, we have

$$\left(\frac{\sqrt{L}}{\sqrt{(1-\triangle)L-\beta}}H_{(P).i}\right)\overset{*}{\Big|}_j - \left(\frac{\sqrt{L}}{\sqrt{(1-\triangle)L-\beta}}H_{(P).j}\right)\overset{*}{\Big|}_i = \frac{\sqrt{L}}{\sqrt{(1-\triangle)L-\beta}}[H_{(P).i}|_j - H_{(P).j}|_i] + \frac{\sqrt{L}}{2[(1-\triangle)L-\beta]^{3/2}}\left\{\left(1+\frac{2\beta}{L}\right)(H_{(P).i}m_j - H_{(P).j}m_i) + 2L(H_{(P).i}C_{..j} - H_{(P).j}C_{..i})\right\}. \tag{3.20}$$

Since $H_{(P).i}|_j - H_{(P).j}|_i = (H_{(P)hi}|_j - H_{(P)hj}|_i)b^h - (H_{(P)hi}C^h_{.j} - H_{(P)hj}C^h_{.i})$, the equation (3.20) may be written as

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$$\left(\frac{\sqrt{L}}{\sqrt{(1-\triangle)L-\beta}}H_{(P).i}\right)\Big|^{*}_{j}-\left(\frac{\sqrt{L}}{\sqrt{(1-\triangle)L-\beta}}H_{(P).j}\right)\Big|^{*}_{i}$$

$$=\frac{\sqrt{L}}{\sqrt{(1-\triangle)L-\beta}}[(H_{(P)hi}|_j-H_{(P)hj}|_i)b^h+(H_{(P)hj}C^h_{.i}-H_{(P)hi}C^h_{.j})]$$

$$+\frac{\sqrt{L}}{2[(1-\triangle)L-\beta]^{3/2}}\left\{\left(1+\frac{2\beta}{L}\right)(H_{(P).i}m_j-H_{(P).j}m_i)\right.$$
$$\left.+2L(H_{(P).i}C_{..j}-H_{(P).j}C_{..i})\right\}.$$

Substituting these values in (3.19) and using (2.6), (2.13) and Codazzi equation (3.2) for (M^n, L), we obtain that equation (3.19) is identically satisfied.

In view of (3.4) and (3.5), equation (3.17)c may be written as

$$\frac{\sqrt{L}}{\sqrt{L-\beta}\{(1-\triangle)L-\beta\}}(H_{(P).j}m_i-H_{(P).i}m_j)$$
$$+g^{*hk}\{H_{(P)hi}E_{kj}-H_{(P)hj}E_{ki}\}\left(e^{\beta/L}\sqrt{1-\frac{\beta}{L}}\right)=0, \tag{3.21}$$

which is identically satisfied by virtue of equations (2.6), (2.14) and the facts that $H_{(P)hi}l^i=0$, $E_{ij}l^i=0$.

In view of (3.4) and (3.5), equation (3.17)d may be written as

$$\left(\frac{1}{\sqrt{L-\beta}\sqrt{(1-\triangle)L-\beta}}m_i\right)\Big|^{*}_{j}-\left(\frac{1}{\sqrt{L-\beta}\sqrt{(1-\triangle)L-\beta}}m_j\right)\Big|^{*}_{i}$$
$$+g^{*hk}(d_{hi}E_{kj}-d_{hj}E_{ki})=0. \tag{3.22}$$

Since

$$\dot{\partial}_j\left(\frac{1}{\sqrt{L-\beta}\sqrt{(1-\triangle)L-\beta}}\right)=-\left[\frac{2(L-\beta)-\triangle(2L-\beta)}{2(L-\beta)^{3/2}\{(1-\triangle)L-\beta\}^{3/2}}\right]l_j$$
$$+\left[\frac{2(L-\beta)-\triangle L}{2(L-\beta)^{3/2}\{(1-\triangle)L-\beta\}^{3/2}}\right]b_j+\frac{L}{\sqrt{L-\beta}\{(1-\triangle)L-\beta\}^{3/2}}C_{..j}$$
$$+\frac{2\beta}{L\{(1-\triangle)L-\beta\}}m_j$$

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and $m_i\overset{*}{|}_j - m_j\overset{*}{|}_i = m_i|_j - m_j|_i$, we have

$$\left(\frac{1}{\sqrt{L-\beta}\sqrt{(1-\triangle)L-\beta}}m_i\right)\overset{*}{\Big|}_j - \left(\frac{1}{\sqrt{L-\beta}\sqrt{(1-\triangle)L-\beta}}m_j\right)\overset{*}{\Big|}_i$$

$$\begin{aligned} &= \frac{1}{\sqrt{L-\beta}\sqrt{(1-\triangle)L-\beta}}(m_i|_j - m_j|_i) \\ &- \left(\frac{2(L-\beta)-\triangle(2L-\beta)}{2(L-\beta)^{3/2}\{(1-\triangle)L-\beta\}^{3/2}}\right)(l_j m_i - l_i m_j) \\ &+ \left(\frac{2(L-\beta)-\triangle L}{2(L-\beta)^{3/2}\{(1-\triangle)L-\beta\}^{3/2}}\right)(b_j m_i - b_i m_j) \\ &+ \frac{L}{\sqrt{L-\beta}\{(1-\triangle)L-\beta\}^{3/2}}(m_i C_{..j} - m_j C_{..i}). \end{aligned} \tag{3.23}$$

Using equations (2.6), (2.13), (2.14), (3.11), (3.12) and (3.23) one can show that (3.22) is identically satisfied. Thus the Ricci-Kühne equations are satisfied for (M^n_X, g^*_X). This completes the proof of Theorem 3.1. □

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