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Controllability Results For First Order Impulsive Stochastic Functional Differential Systems with State-Dependent Delay

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Abstract

In this paper, we study the controllability results of first order impulsive stochastic differential and neutral differential systems with state-dependent delay by using semigroup theory. The controllability results are derived by the means of Leray-Schauder Alternative fixed point theorem. An example is provided to illustrate the theory.

Keywords and Phrases: Controllability, Impulsive neutral stochastic differential equations, State-dependent delay, Fixed point, Semigroup theory.
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1 Introduction

Stochastic differential equations have been considered extensively through discussion in the finite and infinite dimensional spaces. As a matter of fact, there exist broad literature on the related to the topic and it has played an important role in many ways such as option pricing, forecast of the growth of population, etc., and as an applications which cover the generalizations of stochastic differential equations arising in the fields such as electromagnetic theory, population dynamics, and heat conduction in material with memory and stochastic differential equations are obtained by including random fluctuations in ordinary differential equations which have been deduced from phenomological or physical laws. Random differential and integral equations play an important role in characterizing numerous social, physical, biological and engineering problems. For more details reader may refer [11], [17], [19], [33], [38], [41] and reference therein.

Impulsive systems arise naturally in various fields, such as mechanical systems, economics, engineering, biological systems and population dynamics, undergo abrupt changes in their state at certain moments between intervals of continuous evolution. Since many evolution

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process, optimal control models in economics, stimulated neural networks, frequency- modulated systems and some motions of missiles or aircrafts are characterized by the impulsive dynamical behavior. Nowadays, there has been increasing interest in the analysis and synthesis of impulsive systems due to their significance both in theory and applications. Thus the theory of impulsive differential equations has seen considerable development. For more details, see the monographs of Lakshmikantham et al. [34], Bainov and Simeonov [3] and Samoilenko and Perestuk [46].

Controllability play an important role in the analysis and design of control systems. Any control system is said to be controllable if every state corresponding to this process can be affected or controlled in respective time by some control signals. If the system cannot be controlled completely then different types of controllability can be defined such as approximate, null, local null and local approximate null controllability. For more details reader may refer the papers [2], [4], [5], [7], [16], [15], [31], [35], [36], [40], [39], [47] and reference therein. Functional differential equations with state-dependent delay appear frequently in applications as model equations and for this reason the study of such equations gave received much attention in last few years, see for an instance [1], [12], [21], [22], [32] and reference therein. The partial differential with differential equations with state dependent delay have been examine recently [23], [25], [26], [24], [27], [28], [14]. For more details reader may refer the papers of [6], [9], [10], [29], [37], [44], [45] and reference therein.

In [27], the authors E. Hernandez et al. have proved existence for an impulsive abstract partial differential equation with state-dependent delay by using Leray-Schauder nonlonear alternative fixed point theorem, whereas P. Balasubramaniam et al. [8] have establish controllability of neutral stochastic functional differential inclusions with infinite delay in abstract space by using Nonlinear alternative for Kakutani maps, and Yong Ren et al.[43] have proved Controllability of impulsive neutral stochastic functional differential inclusions with infinite delay by using Dhage fixed point theorem. More recently, Z. Yan et al.[48] have examine Existence of solutions for impulsive partial stochastic neutral differential equations with state-dependent delay by using Krasnoselskii-Schaefer fixed point theorem.

Inspired by the above mentioned works [8], [27], [43], [48], the main purpose of this paper is to establish the controllability results for the following first order impulsive stochastic differential equations with state-dependent delay of the form

$$d[x(t)] = \Big[Ax(t) + Bu(t)\Big]dt + F(t, x_{\rho(t,x_t)})dw(t), \quad t \in J := [0, b], \tag{1.1}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{1.2}$$

$$\Delta x(t_k) = I_k(x_{t_k}), \qquad k = 1, 2, \dots, m, \tag{1.3}$$

where, the state variable $x(\cdot)$ takes the values in a real separable Hilbert space H with inner product $(\cdot, \cdot)$ and the norm $\|\cdot\|$ and the control function $u(\cdot)$ takes values in $L^2(J, U)$, a Banach space of admissible control functions for a separable Hilbert space U. Also, A is the infinitesimal generator of an analytic semigroup of bounded linear operator $\{T(t)\}_{t\geq 0}$ in the Hilbert space H and B is a bounded linear operator from U into H. The history

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$x_t : (-\infty, 0] \to H, x_t(s) = x(t+s), s \le 0$, belong to an abstract phase space $\mathcal{B}$, which will be described axiomatically in Preliminaries. Let K be the another separable Hilbert space with inner product $(\cdot,\cdot)_K$ and the norm $\|\cdot\|_K$. Suppose, $\{w(t) : t \ge 0\}$ is a given K- valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q \ge 0$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a normal filtration $\{\mathcal{F}_t\}_{t\ge0}$, which generated by the Wiener process w. We now employing the same notation $\|\cdot\|$ for the norm $L(K;H)$, where $L(K;H)$ denotes the space of all bounded linear operator from K into H. Assume that $F : J \times \mathcal{B} \to L_Q(K,H)$, $\rho : J \times \mathcal{B} \to (-\infty, b]$, are measurable mapping in H-norm and $L_Q(K,H)$ norm respectively, where $L_Q(K,H)$ denotes the space of all Q-Hilbert-Schmidt operators from K into H which will be defined in Section 2. $I_k : \mathcal{B} \to H, k = 1, 2, \ldots, m$ are bounded functions. Furthermore, the fixed times t_k satisfies $0 = t_0 < t_1 < t_2 < \ldots < t_m < b$, $x(t_k^+)$ and $x(t_k^-)$ denote the right and left limits of $x(t)$ at $t = t_k$. And $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ represents the jump in the state x at time t_k, where I_k determines the size of the jump.

The rest of this paper is organized as follows. In Section 2, we introduce some basic notations and necessary preliminaries. In Section 3, we establish the controllability of impulsive stochastic differential systems. In Section 4, we derive the Controllability of neutral impulsive stochastic differential systems. Finally, in Section 5, paper concludes with an example is to illustrate the obtained results.

2 Preliminaries

Let $(K, \|\cdot\|_K)$ and $(H, \|\cdot\|_H)$ be the two separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_K$ and $\langle\cdot,\cdot\rangle_H$, respectively. We denote $\mathcal{L}(K,H)$ be the set of all linear bounded operator from K into H, equipped with the usual operator norm $\|\cdot\|$. In this article, we use the symbol $\|\cdot\|$ to denote norms of operator regardless of the space involved when no confusion possibly arises.

Let $(\Omega, \mathcal{F}, P, H)$ be the complete probability space furnished with a complete family of right continuous increasing σ- algebra $\{\mathcal{F}_t, t \in J\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$. An H- valued random variable is an $\mathcal{F}$- measurable function $x(t) : \Omega \to H$ and a collection of random variables $S = \{x(t,\omega) : \Omega \to H \setminus t \in J\}$ is called stochastic process. Usually we write $x(t)$ instead of $x(t,\omega)$ and $x(t) : J \to H$ in the space of S. Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of K. Suppose that $\{w(t) : t \ge 0\}$ is a cylindrical K-valued wiener process with a finite trace nuclear covariance operator $Q \ge 0$, denote $\mathrm{Tr}(Q) = \sum_{i=1}^{\infty} \lambda_i = \lambda < \infty$, which satisfies that $Qe_i = \lambda_i e_i$. So, actually, $\omega(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i}\omega_i(t)e_i$, where $\{\omega_i(t)\}_{i=1}^{\infty}$ are mutually independent one-dimensional standard Wiener processes. We assume that $\mathcal{F}_t = \sigma\{\omega(s) : 0 \le s \le t\}$ is the σ-algebra generated by ω and $\mathcal{F}_t = \mathcal{F}$. Let $\Psi \in \mathcal{L}(K,H)$ and define

$$\|\Psi\|_Q^2 = \mathrm{Tr}(\Psi Q \Psi^*) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\Psi e_n\|^2.$$

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If $\|\Psi\|_Q < \infty$, then Ψ is called a Q-Hilbert-Schmidt operator. Let $\mathcal{L}_Q(K,H)$ denote the space of all Q-Hilbert-Schmidt operators $\Psi : K \to H$. The completion $\mathcal{L}_Q(K,H)$ of $\mathcal{L}(K,H)$ with respect to the topology induced by the norm $\|\cdot\|_Q$ where $\|\Psi\|_Q^2 = \langle \Psi, \Psi\rangle$ is a Hilbert space with the above norm topology. For more details reader may refer the reference [17].

The collection of all strongly measurable, square integrable, H-valued random variables, denoted by $L_2(\omega, H)$ is a Banach space equipped with norm $\|x(\cdot)\|_{L_2} = (E\|x(\cdot,w)\|^2)^{\frac{1}{2}}$, where the expectation, E is defined by $Ex = \int_\omega x(w)dP$. Let $C(J, L_2(\omega, H))$ be the Banach space of all continuous maps from J into $L_2(\omega, H)$ satisfying the conditions $\sup_{0\le t\le b} E\|x(t)\|^2 < \infty$. Let $L_2^0(\omega, H)$ denote the family of all $\mathcal{F}_0$-measurable, H-valued random variable $x(0)$.

Throughout this paper, we assume that $A : D(A) \subset H \to H$ is a the infinitesimal generator of a compact semigroup of linear operators $(T(t))_{t\ge 0}$ defined on a Hilbert space H and M_1 is a constant such that $\|T(t)\|^2 \le M_1$ for every $t \in J = [0,b]$. For more details about semigroup theory. For more details we refer [42] and reference therein.

To consider the impulsive condition (1.3), it is convenient to introduce some additional concepts and notations. We say that a function $u : [\sigma,\tau] \to H_\alpha$ is a normalized piecewise continuous function on $[\sigma,\tau]$ if u is piecewise continuous and left continuous on $(\sigma,\tau]$. We denote by $\mathcal{PC}([\sigma,\tau]; H_\alpha)$ the space formed by the normalized piecewise continuous, $\mathcal{F}_t$-adapted measurable process from $[\sigma,\tau]$ into H_α. In particular, we introduce the space $\mathcal{PC}$ formed by all $\mathcal{F}_t$-adapted measurable, H_α-valued stochastic process $u : [0,a] \to H_\alpha$ such that u is continuous at $t \ne t_k, u(t_k^-) = u(t_k)$ and $u(t_k^+)$ exists, for all $k = 1,\dots,m$. In this paper we always assume that $\mathcal{PC}$ is endowed with the norm $\|u\|_{\mathcal{PC}} = (\sup_{s\in J} E\|u(s)\|_\alpha^2)^{\frac{1}{2}}$. It is clear that $(\mathcal{PC}, \|\cdot\|_{\mathcal{PC}})$ is a Banach space.

To simplify the notations, we put $t_0 = 0$, $t_{m+1} = a$ and for $u \in \mathcal{PC}$ we denote by $\tilde{u}_k \in C([t_k, t_{k+1}]; X)$, $k = 0,1,\dots,m$, the function given by

$$\widetilde{u}_k(t) = \begin{cases} u(t), & \text{for } t \in (t_k, t_{k+1}], \\ u(t_k^+), & \text{for } t = t_k. \end{cases}$$

Moreover, for $N \subseteq \mathcal{PC}$ we denote by $\widetilde{N}_k$, $k = 0,1,\dots,m$, the set $\widetilde{N}_k = \{\tilde{u}_k : u \in N\}$. The notation $B_r[x,H]$ stands for the closed ball with center at x and radius $r > 0$ in H.

Lemma 2.1 *A set $N \subseteq \mathcal{PC}$ is relatively compact in $\mathcal{PC}$ if, and only if, the set $\widetilde{N}_k$ is relatively compact in $C([t_k, t_{k+1}]; L_2(\Omega, H_\alpha))$, for every $k = 0,1,\dots,m$.*

Lemma 2.2 *Let $x : (-\infty, b] \to H_\alpha$ be an $\mathcal{F}_t$- adapted measurable process such that the $\mathcal{F}_0$-adapted process $x_0 = \varphi(t) \in L_2^0(\Omega, \mathcal{B})$ and $x|_J \in \mathcal{PC}(J, H_\alpha)$, then*

$$\|x_s\|_{\mathcal{B}} \le M_b E\|\varphi\|_{\mathcal{B}} + K_b \sup_{0\le s\le b} E\|x(s)\|_\alpha,$$

where $K_b = \sup\{K(t) : 0 \le t \le b\}$, $M_b = \sup\{M(t) : 0 \le t \le b\}$.

In this work we will employ an axiomatic definition for the phase space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a seminormed linear space of $\mathcal{F}_0$-measurable functions maping $(-\infty, 0]$ into H_α and satisfies the following conditions[20, 30]:

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(A) If $x : (-\infty, \sigma + b] \to H_\alpha$, $b > 0$, is such that $x|_{[\sigma,\sigma+b]} \in \mathcal{PC}([\sigma, \sigma + b] : H_\alpha)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + b]$ the following conditions hold:

(i) x_t is in $\mathcal{B}$,

(ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$,

(iii) $\|x_t\|_{\mathcal{B}} \leq K(t-\sigma) \sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$,

where $H \geq 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, K is continuous, M is locally bounded, and H, K, M are independent of $x(\cdot)$.

(B) The space $\mathcal{B}$ is complete.

Lemma 2.3 *(Leray-Schauder Alternative[18]) Let D be a closed convex subset of a Banach space Z and assume that $0 \in D$. Let $\Psi : D \to D$ be a completely continuous map. Then, either the set $\{z \in D : z = \Psi(z), 0 < \lambda < 1\}$ is unbounded or the map Ψ has a fixed point in D.*

3 Controllability Results For First Order Impulsive Stochastic Systems

In this section, we prove the controllability of impulsive stochastic differential systems with state-dependent delay. Let $J_1 = (-\infty, b]$, here we present by defining the mild solution for the impulsive stochastic differential systems (1.1)-(1.3).

Definition 3.1 *An $\mathcal{F}_t$-adapted stochastic process $x : (-\infty, b] \to H$ is called mild solution of the system (1.1)-(1.3) if $x_0 = \varphi \in \mathcal{B}$ on J_0 satisfying $\|\varphi\|^2_{\mathcal{B}} < \infty$; the restrictions of $x(\cdot)$ to the interval $[0, b)$ is continuous stochastic process, for each $s \in [0, t)$ the function $T(t-s)F(t, x_{\rho(s,x_s)})$ is integrable and $\Delta x(t_k) = I_k(x(t_k)), k = 1, 2, \dots, m$ such that*

$$\begin{aligned} x(t) = T(t)\varphi(0) &+ \int_0^t T(t-s)F(t, x_{\rho(s,x_s)})dw(s) \\ &+ \int_0^t T(t-s)Bu(s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(x_{t_k}), \quad t \in J. \end{aligned}$$

Definition 3.2 *The nonlinear stochastic differential equations (1.1)-(1.3) is said to be controllable on the interval J_1, if for every continuous initial stochastic process $\varphi \in \mathcal{B}$ defined on J_0, there exists a stochastic control $u \in L_2(J, U)$ which is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that the solution $x(\cdot)$ of (1.1)-(1.3) satisfies $x(b) = x_1$ where x_1 and b are preassigned terminal state and time, respectively.*

In order to prove the main theorem, we always assume that $\rho : J \times \mathcal{B} \to (-\infty, b]$ is continuous and that $\varphi \in \mathcal{B}$. we assume the following hypotheses:

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- (H_φ) The function $t \to \varphi_t$ is continuous from $\mathcal{R}(\rho^-) = \{\rho(s,\psi) \le 0, (s,\psi) \in J \times \mathcal{B}\}$ into $\mathcal{B}$ and there exists a continuous and bounded function $J^\varphi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that $\|\varphi_t\| \le J^\varphi(t)\|\varphi\|_{\mathcal{B}}$ for each $t \in \mathcal{R}(\rho^-)$.

- (H1) A is the infinitesimal generator of an analytic semigroup of bounded linear operator $T(t), t > 0$ and there exists a constant M_1 such that

$$\|T(t)\|^2 \le M_1 \quad \text{for all } t \ge 0,$$

- (H2) The linear operator $W : L^2(J,U) \to L^2(\omega; H)$, defined by

$$Wu = \int_0^b T(b-s)Bu(s)ds,$$

has an induced inverse W^{-1} which takes values in $L^2(J,U)/KerW$[13, 35] and there exist two positive constants M_2 and M_3 such that

$$\|B\|^2 \le M_2 \qquad \text{and} \qquad \|W^{-1}\|^2 \le M_3.$$

- (H3) The maps I_k are completely continuous and there are positive constants $c_k^j, j = 1, 2,$ such that $\|I_k(x)\|^2 \le c_k^1\|x\|_{\mathcal{B}}^2 + c_k^2, k = 1, 2, \ldots, m,$ for every $x \in \mathcal{B}$.

- (H4) The function $I_k : \mathcal{B} \to H_\alpha$ are continuous and there are positive constants $M_{I_k}, k = 1, 2, \ldots, m$ such that

$$E\|I_k(x) - I_k(y)\|^2 \le M_{I_k}\|x - y\|^2, \quad x, y \in \mathcal{B}, k = 1, 2, \ldots, m.$$

- (H5) The function $F : J \times \mathcal{B} \to L_Q(K,H)$ satisfies the following conditions:

  - (i) The function $F(\cdot,\psi) : J \to L_Q(K,H)$ is strongly measurable.
  - (ii) The function $F(t,\cdot) : \mathcal{B} \to L_Q(K,H)$ is continuous for each $t \in J$.
  - (iii) There exists integrable function $p(t) : J \to [0,\infty)$ such that

$$E\|F(t,\varphi)\|^2 \le p(t)\Omega(\|\varphi\|_{\mathcal{B}}^2), \quad (t,\varphi) \in J \times \mathcal{B},$$

  where $\Omega : [0,\infty) \times (0,\infty)$ is a continuous nondecreasing function.

  - (iv) For every positive constant r, there exists an $h_r \in L^1(J)$ such that

$$\sup_{\|\varphi\|^2 \le r} \|F(t,\varphi)\|^2 \le h_r(t).$$

  - (v) $F : J \times \mathcal{B} \to L(K,H)$ is completely continuous. Then the operator

$$\Psi x(t) = \int_0^t T(t-s)F(s,x(s))dw(s) + \int_0^t T(t-s)(Bu_x)(s)ds, \quad t \in [0,b],$$

  is completely continuous.

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(H6) Let $S(a) = \{x : (-\infty, b] \to H : x_0 = 0; x|_J \in \mathcal{PC}\}$ endowed with norm of uniform convergence on J_1 and $y : (-\infty, b] \to H$ be the function defined by $y_0 = \varphi$ on $(-\infty, 0]$ and $y(t) = T(t)\varphi(0)$ on J_1.

Lemma 3.4 *If $x : (-\infty, b] \to H$ is a function such that $x_0 = \varphi$ and $x|_I \in \mathcal{PC}(I : H)$, then*

$$\|x_s\|_{\mathcal{B}} \le (M_b + J^{\varphi})\|\varphi\|_{\mathcal{B}} + K_b \sup\{\|x(\theta)\|;\ \theta \in [0, \max\{0, s\}]\}, \quad s \in \mathcal{R}(\rho^-) \cup J,$$

where $J^{\varphi} = \sup_{t\in\mathcal{R}(\rho^-)} J^{\varphi}(t)$, $M_b = \sup_{t\in J} M(t)$ and $K_b = \sup_{t\in J} K(t)$.

Theorem 3.1 *Assume that the assumptions (H_φ), (H1)-(H5) hold. Then the system (1.1)-(1.3) is controllable on J_1 provided that*

$$(4 + 16^2 b^2 M_1 M_2 M_3)\left[K_b\left(M_1 Tr(Q) \liminf_{\xi\to\infty} \frac{\Omega(\xi)}{\xi}\int_0^b p(s)ds + M_1\sum_{k=1}^{m} M_{I_k}\right)\right] < 1.$$

Proof: Consider the space $Y = \{x \in \mathcal{PC} : u(0) = \varphi(0)\}$ endowed with the uniform convergence topology. Using the assumption (H2), for an arbitrary function $x(\cdot)$, we define the control

$$u(t) = W^{-1}\left\{x_1 - T(b)\varphi(0) - \int_0^b T(b-s)F(s, x_{\rho(s,x_s)})dw(s) - \sum_{k=1}^{m} T(b-t_k)I_k(x_{t_k})\right\}(t).$$

Using this control, we shall show that the operator $\Psi : Y \to Y$ defined by

$$\begin{aligned}\Psi x(t) &= T(t)\varphi(0) + \int_0^t T(t-s)F(s, \bar{x}_{\rho(s,\bar{x}_s)})dw(s) + \sum_{0<t_k<t} T(t-t_k)I_k(\bar{x}_{t_k})\\ &\quad + \int_0^t T(t-\eta)BW^{-1}\left\{x_1 - T(b)\varphi(0) - \int_0^b T(b-s)F(s, \bar{x}_{\rho(s,\bar{x}_s)})dw(s)\right.\\ &\quad \left. - \sum_{k=1}^{m} T(b-t_k)I_k(\bar{x}_{t_k})\right\}(\eta)d\eta, t \in J,\end{aligned}$$

has a fixed point $x(\cdot)$. This fixed point $x(\cdot)$ is then a mild solution of the system (1.1)-(1.3). Clearly, $(\Psi x)(b) = x_1$, which means that the control u steers the systems from the initial state φ to x_1 in time b, provided we can obtain a fixed point of the operator Ψ which implies that the systems is controllable. Here $\bar{x} : (-\infty, b] \to H$ is such that $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on J. From the axiom (A) and our assumption on φ, it is easy to see that $\Psi x \in \mathcal{PC}$.

Next we claim that there exists $r > 0$ such that $\Psi(B_r(y_{|_J}, Y)) \subseteq (B_r(y_{|_J}, Y))$. If we assume this property is false, then for every $r > \|\varphi\|$ there exist $x^r \in (B_r(y_{|_J}, Y))$ and $t^r \in J$ such that $r < E\|\Psi x^r(t^r)\|^2$. Then by using Lemma 3.4 we get

$$\begin{aligned} r &< E\|\Psi x^r(t^r)\|^2\\ &\le E\|T(t)\varphi(0) + \int_0^{t^r} T(t-s)F(s, \bar{x}_{\rho(s,\bar{x}_s)})dw(s)\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{t^r} T(t-\eta)BW^{-1}\Big\{x_1 - T(b)\varphi(0) - \int_0^b T(b-s)F(s,\bar{x}_{\rho(s,\bar{x}_s)})dw(s)\\
&-\sum_{k=1}^{n} T(b-t_k)I_k(\bar{x}_{t_k})\Big\}(\eta)d\eta + \sum_{0<t_k<t} T(t-t_k)I_k(\bar{x}_{t_k})\|^2\\
\leq\ & 16M_1H\|\varphi\|_{\mathcal{B}}^2 + 16M_1Tr(Q)\int_0^{t_r} p(s)\Omega(\|\overline{x^r}_{\rho(s,\bar{x}_s^r)})\|_{\mathcal{B}}^2)ds + 16^2M_1M_2M_3\int_0^{t_r}\Big\{\|x_1\|^2\\
&+M_1H\|\varphi\|_{\mathcal{B}}^2 + M_1Tr(Q)\int_0^b p(s)\Omega(\|\overline{x^r}_{\rho(s,\bar{x}_s^r)})\|_{\mathcal{B}}^2)ds\\
&+M_1\sum_{k=1}^{m}(M_{I_k}\|(\bar{x}_{t_k})\|_{\mathcal{B}}^2 + \|I_k(0))\|^2\Big\}d\eta\\
&+16M_1\sum_{k=1}^{m}(M_{I_k}\|(\bar{x}_{t_k})\|_{\mathcal{B}}^2 + \|I_k(0)\|^2)\\
\leq\ & 16M_1H\|\varphi\|_{\mathcal{B}}^2 + 16M_1Tr(Q)\Omega((M_b+J_0^{\varphi})\|\varphi\|_{\mathcal{B}}^2 + K_br)\int_0^b p(s)ds\\
&+16^2b^2M_1M_2M_3\Big\{\|x_1\|^2 + M_1H\|\varphi\|_{\mathcal{B}}^2 + M_1Tr(Q)\Omega((M_b+J_0^{\varphi})\|\varphi\|_{\mathcal{B}}^2 + K_br)\int_0^b p(s)ds\\
&+M_1\sum_{k=1}^{m}(M_{I_k}((M_b+J_0^{\varphi})\|\varphi\|_{\mathcal{B}}^2 + K_br) + \|I_k(0)\|^2)\Big\}\\
&+16M_1\sum_{k=1}^{m}(M_{I_k}((M_b+J_0^{\varphi})\|\varphi\|_{\mathcal{B}}^2 + K_br) + \|I_k(0)\|^2),
\end{aligned}
$$

and hence

$$
1 \leq \left(16 + 16^2b^2M_1M_2M_3\right)\left[K_b\left(M_1Tr(Q)\liminf_{\xi\to\infty}\frac{\Omega(\xi)}{\xi}\int_0^b p(s)ds + M_1\sum_{k=1}^{m}M_{I_k}\right)\right],
$$

which is the contrary to the our assumption.

Let $r > 0$ be such that $\Psi(B_r(y_{|_J}, Y)) \subset (B_r(y_{|_J}, Y))$. In order to prove that Ψ is a condensing map on $\Psi(B_r(y_{|_J}, Y))$ into $(B_r(y_{|_J}, Y))$. We decompose Ψ as Ψ_1 and Ψ_2 (i.e) $\Psi = \Psi_1 + \Psi_2$ where

$$
\begin{aligned}
\Psi_1 x(t) &= \sum_{0<t_k<t} T(t-t_k)I_k(\bar{x}_{t_k}), && t\in J,\\
\Psi_2 x(t) &= T(t)\varphi(0) + \int_0^t T(t-s)F(t,\bar{x}_{\rho(s,\bar{x}_s)})dw(s) + \int_0^t T(t-s)Bu(s)ds, && t\in J.
\end{aligned}
$$

Now

$$
\begin{aligned}
E\|Bu(s)\|^2 \leq 16M_2M_3\Big[&\|x_1\|^2 + M_1H\|\varphi(0)\|^2 + Tr(Q)M_1\int_0^b h_r ds\\
&+M_1\sum_{k=1}^{m}c_k^1 r + M_1\sum_{k=1}^{m}c_k^2\Big] = G_0.
\end{aligned}
$$

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Step 1. The set $\Psi_2(B_r(y_{|_J}, Y))(t) = \{\Psi_2 x(t) : x \in (B_r(y_{|_J}, Y))\}$ is relatively compact in X for every $t \in J$. The case $t = 0$ is obvious. Let $0 < \epsilon < t \leq b$. If $x \in (B_r(y_{|_J}, Y))$, from Lemma 3.4 it follows that

$$\|\bar{x}_{\rho(s,\bar{x}_s)}\|_{\mathcal{B}}^2 \leq r^* = (M_b + \tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}}^2 + K_b r,$$

and so

$$\| \int_0^{\tau} T(\tau - s)F(t, \bar{x}_{\rho(s,\bar{x}_s)})dw(s)\|^2 \leq r^{**} = Tr(Q)\Omega(r^*)M_1 \int_0^b p(s)ds, \quad t \in J,$$

and

$$\| \int_0^{\tau} T(\tau - s)Bu(s)ds\|^2 \leq g^* = M_1 \int_0^b G_0 ds, \quad \tau \in J.$$

Consequently, for $x \in (B_r(y_{|_J}, Y))$, we define that

$$\begin{aligned} E\|\Psi_2 x(t)\|^2 &= E\|T(t)\varphi(0) + T(\epsilon)\int_0^{t-\epsilon} T(t-\epsilon-s)f(t, \bar{x}_{\rho(s,\bar{x}_s)})dw(s) + \\ &\quad + \int_{t-\epsilon}^{t} T(t-s)f(t, \bar{x}_{\rho(s,\bar{x}_s)})dw(s) + T(\epsilon)\int_0^{t-\epsilon} T(t-\epsilon-s)Bu(s)ds \\ &\quad + \int_{t-\epsilon}^{t} T(t-s)Bu(s)ds\|^2 \\ &\in 9\{T(t)\phi(0)\} + 9T(\epsilon)B_{r^{**}}(0, H) + 9C_\epsilon + 9T(\epsilon)B_{g^*}(0, H) + 9G_\epsilon, \end{aligned}$$

where $\mathrm{diam}(C_\epsilon) \leq 2M_1 Tr(Q)\Omega(r^*)\int_{t-\epsilon}^{t} p(s)ds$ and $\mathrm{diam}(G_\epsilon) \leq M_1 \int_{t-\epsilon}^{t} G_0 ds$ which proves that $\Psi_2(B_r(y_{|_J}, Y))(t)$ is relatively compact in H.

Step 2. The function $\Psi_2(B_r(y_{|_J}, Y))$ is equicontinuous on J. Let $0 < t < b$ and $\epsilon > 0$. Since the semigroup $(T(t))_{t\geq 0}$ is strongly continuous and $\Psi_2(B_r(y_{|_J}, Y))$ is relatively compact in H, there exists $0 < \delta \leq b - t$ such that

$$E\|T(h)x - x\|^2 < \epsilon, \quad x \in \Psi_2(B_r(y_{|_J}, Y)), \quad 0 < h < \delta.$$

Under these conditions, for $x \in \Psi_2(B_r(y_{|_J}, Y))$ and $0 < h < \delta$, we get

$$\begin{aligned} E\|\Psi_2 x(t+h) - \Psi_2 x(t)\|^2 &\leq E\|T(t+h)\varphi(0) - T(t)\varphi(0)\|^2 + E\|T(h)x - x\|^2 \\ &\quad + E\|\int_t^{t+h} T(t-s)F(t, \bar{x}_{\rho(s,\bar{x}_s)})dw(s)\|^2 \\ &\quad + E\|\int_t^{t+h} T(t-s)Bu(s)ds\|^2 \\ &\leq 9M_1\|(T(t+h) - I)\varphi(0)\|^2 + 9\epsilon + 9M_1 Tr(Q)\Omega(r^*)\int_t^{t+h} p(s)ds \\ &\quad + 9M_1 \int_t^{t+h} G_0 ds, \end{aligned}$$

which proves that the set function $\Psi_2(B_r(y_{|_J}, Y))$ is right equicontinuous at $t \in (0, b)$. Similarly, we can prove the right equicontinuity at zero and left equicontinuity at $t \in (0, b]$. Thus $\Psi_2(B_r(y_{|_J}, Y))$ is equicontinuous on J.

Step 3. The map $\Psi_2(\cdot)$ is continuous on $(B_r(y_{|_J}, Y))$. Let $(x^n)_{n\in N}$ be a sequence in $(B_r(y_{|_J}, Y))$ and $x \in (B_r(y_{|_J}, Y))$ such that $x^n \to x$ in $\mathcal{PC}$. From the Axiom **A**, it is easy to see that $(\overline{x^n})_s \to \bar{x}_s$ as $n \to \infty$ uniformly for $s \in (-\infty, b]$ as $n \to \infty$. By assumption, we have

$$F(t, \overline{x^n}_{\rho(s,\bar{x}_s)}) \to F(t, \bar{x}_{\rho(s,\bar{x}_s)}) \quad \text{as} \quad n \to \infty,$$

for each $s \in [0, t]$, and since

$$\|F(t, \overline{x^n}_{\rho(s,\bar{x}^n_s)}) - F(t, \bar{x}_{\rho(s,\bar{x}_s)})\|^2 \leq 2p(s)\Omega(r^*) \quad \text{as} \quad n \to \infty.$$

Now, a standard application of Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
E\|\Psi_2 x^n - \Psi_2 x\|^2_{\mathcal{B}} \leq E\| & \int_0^t T(t-s)[F(t, \overline{x^n}_{\rho(s,\bar{x}^n_s)}) - F(t, \bar{x}_{\rho(s,\bar{x}_s)})]dw(s) \\
& + \int_0^t T(t-\eta)B\Big[W^{-1}\Big\{x_1 - T(b)[\varphi(0)] \\
& - \int_0^b T(b-s)F(s, \overline{x^n}_{\rho(s,\bar{x}^n_s)})dw(s) \\
& - \sum_{k=1}^m T(b-t_k)I_k(\overline{x^n}_{t_k})\Big\} - W^{-1}\Big\{x_1 - T(b)[\varphi(0)] \\
& - \int_0^b T(b-s)F(s, \bar{x}_{\rho(s,\bar{x}_s)})dw(s) - \sum_{k=1}^m T(b-t_k)I_k(\bar{x}_{t_k})\Big\}\Big](\eta)d\eta\|^2 \\
\leq & \; 4Tr(Q)M_1\int_0^t E\|F(t, \overline{x^n}_{\rho(s,\bar{x}_s)}) - F(t, \bar{x}_{\rho(s,\bar{x}_s)})\|^2 ds \\
& + 4M_1M_2M_3\int_0^b \Big[M_1\int_0^b \|F(t, \overline{x^n}_{\rho(s,\bar{x}_s)}) - F(t, \bar{x}_{\rho(s,\bar{x}_s)})\|^2 ds \\
& + M_1\sum_{k=1}^m \|I_k(\overline{x^n}_{t_k}) - I_k(\bar{x}_{t_k})\|^2\Big]d\eta \\
& \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Thus, $\Psi_2(\cdot)$ is continuous.

Step 4. The map $\Psi_1(\cdot)$ is a contraction on $(B_r(y_{|_J}, Y))$

$$\|\Psi_1 x - \Psi_1 y\|^2 \leq K_b M_1 \sum_{k=1}^m M_{I_k}\|x - y\|^2.$$

It follows that Ψ_1 is a contraction on $(B_r(y_{|_J}, Y))$ which implies that Ψ is a condensing operator on $(B_r(y_{|_J}, Y))$ into $(B_r(y_{|_J}, Y))$.

Finally, from Lemma 2.3, Ψ has a fixed point in Y which implies that any fixed point $\Psi(\cdot)$ is a mild solution of the problem (1.1)-(1.3). This completes the proof.

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Theorem 3.2 *Assume that the hypotheses* (H_φ)*(H1)-(H6) satisfied. Further assume that* $\rho(t,\psi) \le t$ *for every* $(t,\psi) \in J \times \mathcal{B}$*, the maps* I_k *are completely continuous and there are constants* $c_k^j, k = 1,2,\dots,m, j = 1,2,$ *such that* $\|I_k(\psi)\| \le c_k^1\|\psi\|^2 + c_k^2$*, for every* $\psi \in \mathcal{B}$*. If* $\gamma = \left[1 - 18(1+16b^2M_1M_2M_3)K_bM_1\sum_{k=1}^m c_k^1\right] > 0$ *and*

$$\frac{N_2}{\gamma}\int_0^b p(s)ds < \int_C^\infty \frac{ds}{\Omega(s)},$$

where

$$\begin{aligned}
\overline{M} &= 144b^2M_1M_2M_3\left[\|x_1\|^2 + M_1\|\varphi(0)\|^2 + M_1\sum_{k=1}^m c_k^2\right] + 9M_1\sum_{k=1}^m c_k^2,\\
C &= v_\lambda(0) = \frac{N_1}{\gamma},\\
N_1 &= 2(M_b + J_0^\varphi + K_bM_1H)\|\varphi\|_{\mathcal{B}}^2 + 2K_b\overline{M},\\
N_2 &= 18(1+16b^2M_1M_2M_3)K_bM_1Tr(Q).
\end{aligned}$$

Then there exists a mild solutions of (1.1)-(1.3) and the systems (1.1)-(1.3) is controllable on J_1*.*

Proof: On the space $\mathcal{BPC} = \{u : (-\infty, b] \to H, u_0 = 0, u|_J \in \mathcal{PC}\}$ endowed with the norm $\|\cdot\|_{\mathcal{PC}}$, we define the operator $\Psi : \mathcal{BPC} \to \mathcal{BPC}$ by $(\Psi u)_0 = 0$ and

$$\begin{aligned}
\Psi x(t) = &\int_0^t T(t-s)f(s,\bar{x}_{\rho(s,\bar{x}_s)})dw(s) + \int_0^t T(t-\eta)BW^{-1}\Big[x_1 - T(b)\varphi(0)\\
&- \int_0^b T(b-s)f(s,\bar{x}_{\rho(s,\bar{x}_s)})dw(s) - \sum_{k=1}^m T(b-t_k)I_k(\bar{x}_{t_k})\Big](\eta)d\eta\\
&+ \sum_{k=1}^m T(t-t_k)I_k(\bar{x}_{t_k}), \quad t \in J,
\end{aligned}$$

where $\bar{x} = x + y$ on $(0,b]$ and $y(\cdot)$ is the function introduced in (H6). In order to use Lemma 2.3, we establish a priori estimates for the solutions of the integral equation $z = \lambda\Psi z, \lambda \in (0,1)$. By using Lemma 3.4, the notation $\alpha^\lambda(s) = \sup_{\theta\in[0,s]} E\|x^\lambda(\theta)\|^2$ and the fact that $\rho(s,(\bar{x})_s) \le s$, for each $s \in J$, we have that

$$\begin{aligned}
x^\lambda(t) = &\int_0^t T(t-s)F(s,\overline{x^\lambda}_{\rho(s,\bar{x}_s)})dw(s) + \int_0^t T(t-s)BW^{-1}\Big[x_1 - T(b)\varphi(0)\\
&- \int_0^b T(b-s)F(s,\overline{x^\lambda}_{\rho(s,\bar{x}_s)})dw(s) - \sum_{k=1}^m T(b-t_k)I_k(\overline{x^\lambda})_{t_k}\Big](\eta)d\eta\\
&+ \lambda\sum_{k=1}^m T(t-t_k)I_k(\overline{x^\lambda})_{t_k}, \quad t \in J,
\end{aligned}$$

for some $0<\lambda<1$. Then, by assumption, we have

$$
\begin{aligned}
E\|x^\lambda(t)\|^2 \le\ & 9M_1Tr(Q)\int_0^t p(s)\Omega(\|(\overline{x^\lambda})_s\|^2)ds + 9M_1\int_0^t 16M_2M_3\Big[\|x_1\|^2 + M_1\|\varphi(0)\|^2\\
& + M_1Tr(Q)\int_0^b p(s)\Omega(\|(\overline{x^\lambda})_s\|^2)ds + M_1\sum_{k=1}^m c_k^1\|(\overline{x^\lambda})_{t_k}\|^2 + M_1\sum_{k=1}^m c_k^2\Big]d\eta\\
& + 9b^2M_1\sum_{k=1}^m c_k^1\|(\overline{x^\lambda})_{t_k}\|^2 + 9M_1\sum_{k=1}^m c_k^2\\
\le\ & 144b^2M_1M_2M_3\Big[\|x_1\|^2 + M_1\|\varphi(0)\|^2\\
& + M_1Tr(Q)\int_0^b p(s)\Omega((M_b+J_0^\varphi+K_bM_1H)\|\varphi\|_{\mathcal{B}}^2 + K_b\alpha^\lambda(s))ds\\
& + M_1\sum_{k=1}^m c_k^1((M_b+J_0^\varphi+K_bM_1H)\|\varphi\|_{\mathcal{B}}^2 + K_b\alpha^\lambda(s)) + M_1\sum_{k=1}^m c_k^2\Big]\\
& + 9M_1Tr(Q)\int_0^t p(s)\Omega((M_b+J_0^\varphi+K_bM_1H)\|\varphi\|_{\mathcal{B}}^2 + K_b\alpha^\lambda(s))ds\\
& + 9M_1\sum_{k=1}^m c_k^2 + 9M_1\sum_{0<t_k<t} c_k^1((M_b+J_0^\varphi+K_bM_1H)\|\varphi\|_{\mathcal{B}}^2 + K_b\alpha^\lambda(t)).
\end{aligned}
$$

Now, we consider the function $\zeta^\lambda(t)$ defined by

$$\zeta^\lambda(t) = E\|x^\lambda(t)\|^2, \quad 0\le t\le b.$$

Since $\rho(s,\bar{x}_s)\le s, s\in[0,t], t\in J$ and the above inequality, we have

$$\|x^\lambda(t)\|^2 \le 2[(M_b+J_0^\varphi+K_bM_1H)\|\varphi\|_{\mathcal{B}}]^2 + 2K_b\sup_{0\le s\le b}E\|x^\lambda(s)\|^2.$$

If $\zeta^\lambda(t)\le 2(M_b+J_0^\varphi+K_bM_1H)\|\varphi\|_{\mathcal{B}}^2 + 2K_b\alpha(t)$, Therefore, we obtain that

$$
\begin{aligned}
\zeta^\lambda(t) \le\ & 2(M_b+J_0^\varphi+K_bM_1H)\|\varphi\|_{\mathcal{B}}^2 + 2K_b\Big[\overline{M} + 144b^2M_1M_2M_3\Big(M_1Tr(Q)\\
& \int_0^b p(s)\Omega(\zeta^\lambda(s))ds + M_1\sum_{k=1}^m c_k^1(\zeta^\lambda(t))\Big)\\
& + 9K_bM_1Tr(Q)\int_0^t p(s)\Omega(\zeta^\lambda(s))ds + 9K_bM_1\sum_{k=1}^m c_k^1\zeta^\lambda(t)\Big]\\
\le\ & N_1 + 288K_bb^2M_1M_2M_3\Big[M_1Tr(Q)\int_0^b p(s)\Omega(\zeta^\lambda(s))ds\\
& + M_1\sum_{k=1}^m c_k^1(\zeta^\lambda(t))\Big] + 18K_bM_1Tr(Q)\int_0^t p(s)\Omega(\zeta^\lambda(s))ds\\
& + 18K_bM_1\sum_{k=1}^m c_k^1\zeta^\lambda(t)\\
\le\ & \frac{N_1}{\gamma} + \frac{N_2}{\gamma}\int_0^t p(s)\Omega(\zeta^\lambda(s))ds.
\end{aligned}
$$

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Since

$$\zeta^\lambda(t) \le v_\lambda(t), \quad t \in J,$$

$$\begin{aligned} v_\lambda(t) &\le \frac{N_1}{\gamma} + \frac{N_2}{\gamma}\int_0^t p(s)\Omega(v^\lambda(s))ds, \\ v'_\lambda(t) &\le \frac{N_2}{\gamma}p(s)\Omega(v^\lambda(s)), \end{aligned}$$

and hence

$$\int_{v_\lambda(0)=C}^{v_\lambda(t)} \frac{ds}{\Omega(s)} \le \frac{N_2}{\gamma}\int_0^b p(s)ds < \int_C^\infty \frac{ds}{\Omega(s)},$$

which implies that the set of functions $\{v_\lambda(\cdot) : \lambda \in (0,1)\}$ is bounded in $C(J,R)$. Thus $\{x^\lambda(\cdot) : \lambda \in (0,1)\}$ is bounded in $\mathcal{BPC}$.

To prove that Ψ is completely continuous, we introduce the decomposition $\Psi x = \Psi_1 x + \Psi_2 x$ where $(\Psi_i x)_0 = 0, i = 1,2$ and

$$\begin{aligned} \Psi_1 x(t) &= \int_0^t T(t-s)F(s,\bar{x}_{\rho(s,\bar{x}_s)})dw(s) + \int_0^t T(t-s)Bu(s)ds, && t \in J, \\ \Psi_2 x(t) &= \sum_{0<t_k<t} T(t-t_k)I_k(\bar{x}_{t_k}), && t \in J. \end{aligned}$$

From the proof of Theorem 3.1, we deduce that Ψ_1 is completely continuous. Next, by using Lemma 2.1, we prove that Ψ_2 is also completely continuous. The continuity of Ψ_2 can be prove by using the phase space axioms. From the definition of Ψ_2, for $r > 0, t \in [t_k, t_{k+1}] \cap (0,b], k \ge 1$, and $m \in B_r = B_r(0, \mathcal{BPC})$, we find that

$$\widetilde{\Psi_2 m(t)} \in \begin{cases} \sum_{j=1}^{k} T(t-t_j)I_j(B_{r^*}(0;X)), & t \in (t_k, t_{k+1}), \\ \sum_{j=0}^{k} T(t_{k+1}-t_j)I_j(B_{r^*}(0;X)), & t = t_{k+1}, \\ \sum_{j=1}^{k-1} T(t_k-t_j)I_j(B_{r^*}(0;X)) + I_k(B_{r^*}(0;X)), & t = t_k, \end{cases}$$

where $r^* = (M_b + HM_1)\|\varphi\|_{\mathcal{B}}^2 + K_b r$, which proves that $[\widetilde{\Psi_2(B_r)}]_i(t)$ is relatively compact in X, for every $t \in [t_k, t_{k+1}]$, since the maps I_k are completely continuous. Moreover, using the compactness of the operator I_k and the strong continuity of $(T(t))_{t\ge0}$, we can prove that $[\widetilde{\Psi_2(B_r)}]_i$ is equicontinuous at t, for every $t \in [t_k, t_{k+1}]$. Now, from the Lemma 2.1 we conclude that Ψ_2 is completely continuous.

Finally, from Lemma 2.3 shows that the controllability of mild solution for problem (1.1)-(1.3) is controllable on J_1. The proof is complete.

4 Controllability Results For First Order Neutral Impulsive Stochastic Systems

In this section, we prove the controllability result for nonlinear systems with state-dependent delay. Consider the impulsive neutral stochastic control systems of the form

$$d[x(t) - g(t, x_t)] = [Ax(t) + Bu(t)]dt + F(t, x_{\rho(t,x_t)})dw(t), \qquad t \in J = [0, b], \tag{4.1}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{4.2}$$

$$\Delta x(t_k) = I_k(x_{t_k}), \quad k = 1, 2, \ldots, m, \tag{4.3}$$

where A, B, ρ, F, I_k are defined in equations (1.1)-(1.3). Here $g : J \times \mathcal{B} \to H$ is an appropriate function. Furthermore, we assume the following conditions:

(H7) For every $y \in Y$, the function $t \to T(t)y$ is continuous from $[0, \infty)$ into Y. Moreover, $T(t)(Y) \subset D(A)$ for every $t > 0$ and there exists a positive function $\beta \in L^1([0, b])$ such that $\|AT(t)\|_{\mathcal{L}(Y,X)} \leq \beta(t)$, for every $t \in J$.

(H8) The function $g : J \times \mathcal{B} \to H$ is completely continuous and there exists $M_g > 0$ such that

$$\|g(t, \psi_1) - g(t, \psi_2)\|^2 \leq M_g\|\psi_1 - \psi_2\|^2, \quad (t, \psi_m) \in J \times \mathcal{B}, m = 1, 2.$$

(H9) There exists positive constants θ_1, θ_2 such that $\|g(t, \psi)\|^2 \leq \theta_1\|\psi\|^2 + \theta_2$, for every $(t, \psi) \in J \times \mathcal{B}$.

Definition 4.3 An $\mathcal{F}_t$-adapted stochastic process $x : (-\infty, b] \to H$ is called mild solution of the system (4.1)-(4.3) if $x_0 = \varphi \in \mathcal{B}$ on J_0 satisfying $\|\varphi\|^2_{\mathcal{B}} < \infty$; the restrictions of $x(\cdot)$ to the interval $[0, b)$ is a continuous stochastic process, for each $s \in [0, t)$ the function $AT(t - s)g(s, x_s)$ is integrable and $\Delta x(t_k) = I_k(x(t_k)), k = 1, 2, \ldots, m$ such that

$$x(t) = T(t)[\varphi(0) - g(0, \varphi)] + g(t, x_t) + \int_0^t AT(t - s)g(s, x_s)ds + \int_0^t T(t - s)Bu(s)ds$$
$$+ \int_0^t T(t - s)F(s, x_{\rho(s,x_s)})dw(s) + \sum_{0<t_k<t} T(t - t_k)I_k(x_{t_k}), \quad t \in J.$$

Definition 4.4 The nonlinear stochastic differential equations (4.1)-(4.3) is said to be controllable on the interval J_1, if for every continuous initial stochastic process $\varphi \in \mathcal{B}$ defined on J_0, there exists a stochastic control $u \in L_2(J, U)$ which is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that the solution $x(\cdot)$ of (4.1)-(4.3) satisfies $x(b) = x_1$ and b are preassigned terminal state and time, respectively.

Remark 4.1 Let $x(\cdot)$ be function as in axiom(**A**). Let us mention that the conditions $(H_7)(H_8)(H_9)$ are linked to the integrability of the function $s \to AT(t - s)g(s, x_s)$. In general, except for the trivial case in which A is a bounded linear operator, the operator function $t \to AT(t)$ is not integrable over J. However, if condition H_7 holds and g satisfies either assumption H_8 or H_9, then it follows from Bochner's criterion and the estimate

$$\|AT(t - s)g(, x_s)\|^2 \leq \|AT(t - s)\|^2_{\mathcal{L}(Y,X)}\|g(s, x_s)\|^2_Y$$
$$\leq \beta(t - s) \sup_{s\in J} \|g(s, x_s)\|^2_Y,$$

that $s \to AT(t - s)g(s, x_s)$ is integrable over $[0, t)$, for every $t \in J$.

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Theorem 4.3 *Assume that the assumptions* (H_φ), *(H1)-(H5) and (H7)-(H9) hold. Then the system (4.1)-(4.3) is controllable on* $(-\infty, b]$ *provided that*

$$\left(36 + 36^2 b^2 M_1 M_2 M_3\right) K_b \Bigg[M_g \left(1 + \int_0^b \beta(s) ds\right) + M_1 Tr(Q) \liminf_{\xi\to\infty} \frac{\Omega(\xi)}{\xi} \int_0^b p(s)ds + M_1 \sum_{k=1}^{n} M_{I_k} \Bigg] \leq 1.$$

Proof: Consider the space $Y = \{x \in \mathcal{PC} : u(0) = \varphi(0)\}$ endowed with the uniform convergence topology. Using the assumption (H2), for an arbitrary function $x(\cdot)$, we define the control

$$u(t) = W^{-1}\Bigg\{ x_1 - T(b)[\varphi(0) - g(0,\varphi)] - g(t, x_t) - \int_0^b AT(b-s)g(s, x_s) - \int_0^b T(b-s)F(s, \bar{x}_{\rho(s,\bar{x}_s)})dw(s) - \sum_{k=1}^{n} T(b-t_k)I_k(x_{t_k}) \Bigg\}(t).$$

Using this control, we shall show that the operator $\Psi : Y \to Y$ defined by

$$\begin{aligned}\Psi x(t) &= T(t)[\varphi(0) - g(0,\varphi)] + g(t, \bar{x}_t) + \int_0^t AT(t-s)g(s, \bar{x}_s) \\ &+ \int_0^t T(t-s)F(s, \bar{x}_{\rho(s,\bar{x}_s)})dw(s) + \sum_{0<t_k<t} T(t-t_k)I_k(\bar{x}_{t_k}) \\ &+ \int_0^t T(t-\eta)BW^{-1}\Bigg\{ x_1 - T(b)[\varphi(0) - g(0,\varphi)] - g(t, \bar{x}_t) - \int_0^b AT(b-s)g(s, \bar{x}_s) \\ &- \int_0^b T(b-s)F(s, \bar{x}_{\rho(s,\bar{x}_s)})dw(s) - \sum_{k=1}^{n} T(b-t_k)I_k(\bar{x}_{t_k}) \Bigg\}(\eta)d\eta, t \in J,\end{aligned}$$

has a fixed point $x(\cdot)$. This fixed point $x(\cdot)$ is then a mild solution of the system (4.1)-(4.3). Clearly, $(\Psi x)(b) = x_1$, which means that the control u steers the systems from the initial state φ to x_1 in time b, provided we can obtain a fixed point of the operator Ψ which implies that the systems is controllable. Here $\bar{x} : (-\infty, b] \to H$ is such that $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on J. From the axiom (A) and our assumption on φ, it is easy to see that $\Psi x \in \mathcal{PC}$.

Next we claim that there exists $r > 0$ such that $\Psi(B_r(y_{|_J}, Y)) \subset (B_r(y_{|_J}, Y))$. If this assume this property is false, then for every $r > \|\varphi\|^2$ there exist $x^r \in (B_r(y_{|_J}, Y))$ and $t^r \in J$ such that $r < E\|\Psi x^r(t^r)\|^2$. Then by using Lemma 3.4, we get

$$\begin{aligned} r &< E\|\Psi x^r(t^r)\|^2 \\ &\leq E\|T(t)[\varphi(0) - g(0,\varphi)] + g(t, \bar{x}_t) + \int_0^{t^r} AT(t-s)g(s, \bar{x}_s) \\ &\quad + \int_0^{t^r} T(t-s)F(s, \bar{x}_{\rho(s,\bar{x}_s)})dw(s) + \sum_{0<t_k<t} T(t-t_k)I_k(\bar{x}_{t_k}) \end{aligned}$$

$$
\begin{aligned}
&+ M_1M_2M_3 \int_0^{t^r} \Big[x_1 - T(b)[\varphi(0) - g(0,\varphi)] - g(t,\bar{x}_t) - \int_0^b AT(t-s)g(s,\bar{x}_s) \\
&- \int_0^b T(b-s)F(s,\bar{x}_{\rho(s,\bar{x}_s)})dw(s) - \sum_{k=1}^n T(b-t_k)I_k(\bar{x}_{t_k})\Big](\eta)d\eta\|^2 \\
\le\ & 72M_1H\|\varphi\|_{\mathcal{B}}^2 + 72M_1\|g(0,\varphi) - g(t^r,\varphi)\|^2 + 36\|g(t^r,(\overline{x^r})_{t^r}) - g(t^r,\varphi)\|^2 \\
&+ 36\int_0^{t_r} \|AT(t^r-s)\|_{\mathcal{L}(Y,X)}^2 \|g(s,(\overline{x^r})_s) - g(s,\varphi)\|^2 ds \\
&+ 36\int_0^{t_r} \|AT(t^r-s)\|_{\mathcal{L}(Y,X)}^2 \|g(s,\varphi)\|^2 ds \\
&+ 36M_1Tr(Q)\int_0^{t_r} p(s)\Omega(\|\overline{x^r}_{\rho(s,\bar{x}_s^r)}\|_{\mathcal{B}}^2)ds + 36^2M_1M_2M_3\int_0^{t_r}\Big[\|x_1\|^2 \\
&- 2M_1H\|\varphi\|_{\mathcal{B}}^2 - 2M_1\|g(0,\varphi) - g(t^r,\varphi)\|^2 - \|g(t^r,(\overline{x^r})_{t^r}) - g(t^r,\varphi)\|^2 \\
&- \int_0^b \|AT(t^r-s)\|_{\mathcal{L}(Y,X)}^2 \|g(s,(\overline{x^r})_s) - g(s,\varphi)\|^2 ds \\
&- \int_0^b \|AT(t^r-s)\|_{\mathcal{L}(Y,X)}^2 \|g(s,\varphi)\|^2 ds \\
&- M_1Tr(Q)\int_0^b p(s)\Omega(\|\overline{x^r}_{\rho(s,\bar{x}_s^r)}\|_{\mathcal{B}}^2)ds - M_1\sum_{k=1}^n (M_{I_k}\|\bar{x}_{t_k}\|_{\mathcal{B}}^2 + \|I_k(0)\|^2)\Big]d\eta \\
&+ 36M_1\sum_{k=1}^n (M_{I_k}\|\bar{x}_{t_k}\|_{\mathcal{B}}^2 + \|I_k(0)\|^2) \\
\le\ & 72M_1H\|\varphi\|_{\mathcal{B}}^2 + 72M_1\|g(0,\varphi) - g(t^r,\varphi)\|^2 + 36M_g(K_br + (M_b+1)\|\varphi\|^2) \\
&+ 36M_g(K_ar + (M_b+1)\|\varphi\|^2)\int_0^b \beta(s)ds + 49\|g(s,\varphi)\|^2\int_0^b \beta(s)ds \\
&+ 36M_1Tr(Q)((M_b + J_0^{\varphi})\|\varphi\|_{\mathcal{B}}^2 + K_br)\int_0^{t_r} p(s)ds + 36^2b^2M_1M_2M_3\Big[\|x_1\|^2 \\
&+ 2M_1H\|\varphi\|_{\mathcal{B}}^2 + 2M_1\|g(0,\varphi) - g(t^r,\varphi)\|^2 + M_g(K_br + (M_b+1)\|\varphi\|^2) \\
&+ M_g(K_ar + (M_b+1)\|\varphi\|^2)\int_0^b \beta(s)ds + \|g(s,\varphi)\|^2\int_0^b \beta(s)ds \\
&+ M_1Tr(Q)((M_b + J_0^{\varphi})\|\varphi\|_{\mathcal{B}}^2 + K_ar)\int_0^b p(s)ds \\
&+ M_1\sum_{k=1}^n (M_{I_k}((M_b + J_0^{\varphi})\|\varphi\|_{\mathcal{B}}^2 + K_br) + \|I_k(0)\|^2)\Big] \\
&+ 36M_1\sum_{k=1}^n (M_{I_k}((M_b + J_0^{\varphi})\|\varphi\|_{\mathcal{B}}^2 + K_br) + \|I_k(0)\|^2),
\end{aligned}
$$

and hence

$$
\begin{aligned}
1 \le \left(36 + 36^2b^2M_1M_2M_3\right) K_b\Big[& M_g\left(1 + \int_0^b \beta(s)ds\right) \\
& + M_1Tr(Q)\liminf_{\xi\to\infty}\frac{\Omega(\xi)}{\xi}\int_0^b p(s)ds + M_1\sum_{k=1}^n M_{I_k}\Big],
\end{aligned}
$$

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which is the contrary to the our assumption.

Let $r > 0$ be such that $\Psi(B_r(y_{|_J}, Y)) \subset (B_r(y_{|_J}, Y))$. In order to prove that Ψ is a condensing map on $\Psi(B_r(y_{|_J}, Y))$ into $(B_r(y_{|_J}, Y))$. We decompose Ψ as Ψ_1 and Ψ_2 (i.e) $\Psi = \Psi_1 + \Psi_2$ where

$$\begin{aligned}
\Psi_1 x(t) &= T(t)[\varphi(0) - g(0,\varphi)] + g(t,\bar{x}_t) + \int_0^t AT(t-s)g(s,\bar{x}_s)ds \\
&\quad + \sum_{0<t_k<t} T(t-t_k)I_k(\bar{x}_{t_k}), \qquad\qquad t \in J, \\
\Psi_2 x(t) &= \int_0^t T(t-s)F(t,\bar{x}_{\rho(s,\bar{x}_s)})dw(s) + \int_0^t T(t-s)Bu(s)ds, \qquad t \in J.
\end{aligned}$$

Similarly, same as in the proof of Theorem 3.1. we can conclude that Ψ is continuous and that Ψ_2 is completely continuous. Moreover, from estimate

$$\|\Psi_1 u - \Psi_1 v\|^2 \leq 16K_b \left[M_g \left(1 + \int_0^b \beta(s)ds \right) + M_1 \sum_{k=1}^m M_{I_k} \right] \|u - v\|^2_{\mathcal{PC}},$$

it follows that Ψ_1 is a contraction on $(B_r(y_{|_J}, Y))$ which implies that Ψ is a condensing operator on $(B_r(y_{|_J}, Y))$ into $(B_r(y_{|_J}, Y))$.

Finally, from Lemma 2.3, Ψ has a fixed point in Y which implies that any fixed point $\Psi(\cdot)$ is a mild solution of the problem (4.1)-(4.3). This completes the proof.

Theorem 4.4 *Assume that the hypotheses (H_φ), (H1)-(H9) satisfied. Further assume that $\rho(t,\psi) \leq t$ for every $(t,\psi) \in J \times \mathcal{B}$ and that $g : J \times \mathcal{B} \to H$ the maps I_k are completely continuous. If* $\mu = \left[1 - 72(1 + 49b^2 M_1 M_2 M_3) \left(\theta_1 K_b (1 + \int_0^b \beta(s)ds) + M_1 K_b \sum_{k=1}^m c_k^1 \right) \right] > 0,$ *and*

$$\frac{72K_b M_1 Tr(Q)}{\mu} \int_0^b p(s)ds < \int_N^\infty \frac{ds}{\Omega(s)},$$

where

$$\begin{aligned}
N &= v(0) = \frac{M_1}{\mu}, \\
\overline{R} &= 36M_1\|g(0,\varphi)\|^2 + 36\theta_2(1 + \int_0^a \beta(s)ds) \\
&\quad + 1764b^2 M_1 M_2 M_3 \Big[\|x_1\|^2 + M_1\|\varphi(0)\|^2 + M_1\|g(0,\varphi)\|^2 \\
&\quad + \theta_2(1 + \int_0^a \beta(s)ds) + M_1 \sum_{k=1}^m c_k^2 \Big] + 36M_1 \sum_{k=1}^m c_k^2, \\
M_1 &= 2(M_b + J_0^\varphi + K_b M_1 H)\|\varphi\|^2_{\mathcal{B}} + 2K_b \overline{R}, \\
M_2 &= 72K_b M_1 Tr(Q)(1 + 49K_b b^2 M_1 M_2 M_3).
\end{aligned}$$

Then there exists a mild solutions of (4.1)-(4.3) and the systems (4.1)-(4.3) is controllable on J_1.

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Proof: On the space $\mathcal{BPC} = \{u : (-\infty, b] \to H, u_0 = 0, u|_J \in \mathcal{PC}\}$ endowed with the norm $\|\cdot\|_{\mathcal{PC}}$, we define the operator $\Psi : \mathcal{BPC} \to \mathcal{BPC}$ by $(\Psi u)_0 = 0$ and

$$\begin{aligned}\Psi x(t) = T(t)g(0,\varphi) + g(t,\bar{x}_t) + \int_0^t AT(t-s)g(s,\bar{x}_s)ds + \int_0^t T(t-\eta)BW^{-1}\Big[x_1 \\ - T(b)[\varphi(0) - g(0,\varphi)] - g(t,\bar{x}_t) - \int_0^t AT(b-s)g(s,\bar{x}_s)ds \\ - \int_0^b T(b-s)F(s,\bar{x}_{\rho(s,\bar{x}_s)})dw(s) - \sum_{k=1}^m T(b-t_k)I_k(\bar{x}_{t_k})\Big](\eta)d\eta \\ + \int_0^t T(t-s)F(s,\bar{x}_{\rho(s,\bar{x}_s)})dw(s) + \sum_{k=1}^m T(t-t_k)I_k(\bar{x}_{t_k}),\end{aligned}$$

where $\bar{x} = x + y$ on $(0, b]$ and $y(\cdot)$ is the function introduced in (H9). In order to use Lemma 2.3, we establish a priori estimates for the solutions of the integral equation $z = \lambda\Psi z, \lambda \in (0, 1)$. By using Lemma 3.4, the notation $\alpha^\lambda(s) = \sup_{\theta\in[0,s]} E\|x^\lambda(\theta)\|^2$ and the fact that $\rho(s, (\bar{x})_s) \le s$, for each $s \in J$, we have that

$$\begin{aligned}x^\lambda(t) = T(t)g(0,\varphi) + g(t,\overline{x^\lambda}_t) + \lambda\int_0^t AT(t-s)g(s,x^\lambda_s)ds + \int_0^t T(t-\eta)BW^{-1}\Big[x_1 \\ - T(b)[\varphi(0) - g(0,\varphi)] - g(t,\overline{x^\lambda}_t) - \int_0^t AT(b-s)g(s,\overline{x^\lambda}_s)ds \\ - \int_0^b T(b-s)F(s,\overline{x^\lambda}_{\rho(s,\bar{x}^\lambda_s)})dw(s) - \sum_{k=1}^m T(b-t_k)I_k(\overline{x^\lambda})_{t_k}\Big](\eta)d\eta \\ + \int_0^t T(t-s)F(s,\overline{x^\lambda}_{\rho(s,\bar{x}^\lambda_s)})dw(s) + \sum_{k=1}^m T(t-t_k)I_k(\overline{x^\lambda})_{t_k}\end{aligned}$$

$$\begin{aligned}E\|x^\lambda(t)\|^2 &\le 36M_1\|g(0,\varphi)\|^2 + 36\theta_1\|(\overline{x^\lambda})_t\|^2_{\mathcal{B}} + 36\theta_2 + 36\int_0^b \varsigma(b-s)(\theta_1\|(\overline{x^\lambda})_t\|^2_{\mathcal{B}} + \theta_2)ds \\ &+ 1764M_1\int_0^t M_2M_3\Big[\|x_1\|^2 + M_1\|\varphi(0)\|^2 + M_1\|g(0,\varphi)\|^2 + \theta_1\|(\overline{x^\lambda})_t\|^2_{\mathcal{B}} + \theta_2 \\ &+ \int_0^t \varsigma(t-s)(\theta_1\|(\overline{x^\lambda})_s\|^2_{\mathcal{B}} + \theta_2)ds + M_1Tr(Q)\int_0^b p(s)\Omega(\|(\overline{x^\lambda})_s\|)ds \\ &+ M_1\sum_{k=1}^m c^1_k\|(\overline{x^\lambda})_{t_k}\|^2 + M_1\sum_{k=1}^m c^2_k\Big]d\eta + 36M_1Tr(Q)\int_0^t p(s)\Omega(\|(\overline{x^\lambda})_s\|)ds \\ &+ 36M_1\sum_{k=1}^m c^1_k\|\overline{(x^\lambda)}_{t_k}\|^2 + 36M_1\sum_{k=1}^m c^2_k \\ &\le 36M_1\|g(0,\varphi)\|^2 + 36\theta_1((M_b + J^\varphi_0 + K_bM_1H)\|\varphi\|^2_{\mathcal{B}} + \theta_1K_b\alpha^\lambda(t)) \\ &(1 + \int_0^a \beta(s)ds) + 36\theta_2(1 + \int_0^a \beta(s)ds) \\ &+ 1764M_1\int_0^t M_2M_3\Big[\|x_1\|^2 + M_1\|\varphi(0)\|^2 + M_1\|g(0,\varphi)\|^2\end{aligned}$$

$$+36\theta_1((M_b+J_0^{\varphi}+K_bM_1H)\|\varphi\|_{\mathcal{B}}^2+\theta_1K_b\alpha^{\lambda}(t))(1+\int_0^a\beta(s)ds)$$
$$+\theta_2(1+\int_0^a\beta(s)ds)+M_1Tr(Q)\int_0^t p(s)\Omega((M_b+J_0^{\varphi}+K_bM_1H)\|\varphi\|_{\mathcal{B}}^2$$
$$+M_1\sum_{k=1}^{m}c_k^1((M_b+J_0^{\varphi}+K_bM_1H)\|\varphi\|_{\mathcal{B}}^2+K_b\alpha^{\lambda}(t))+M_1\sum_{k=1}^{m}c_k^2\Big]d\eta$$
$$+36M_1Tr(Q)\int_0^t p(s)\Omega((M_b+J_0^{\varphi}+K_bM_1H)\|\varphi\|_{\mathcal{B}}^2$$
$$+K_b\alpha^{\lambda}(s))ds+36M_1\sum_{k=1}^{m}c_k^2$$
$$+36M_1\sum_{0<t_k<t}c_k^1((M_b+J_0^{\varphi}+K_bM_1H)\|\varphi\|_{\mathcal{B}}^2+K_b\alpha^{\lambda}(t)).$$

Now, we consider the function $\zeta^{\lambda}(t)$ defined by

$$\zeta^{\lambda}(t)=E\|x^{\lambda}(s)\|^2,\quad 0\le t\le b.$$

Since $\rho(s,\bar{x}_s)\le s, s\in[0,t], t\in J$ and the above inequality, we have

$$E\|x^{\lambda}(t)\|^2\le 2[(M_b+J_0^{\varphi}+K_bM_1H)\|\varphi\|_{\mathcal{B}}]^2+2K_b\sup_{0\le s\le b}E\|x^{\lambda}(s)\|^2.$$

If $\zeta^{\lambda}(t)\le 2(M_b+J_0^{\varphi}+K_bM_1H)\|\varphi\|_{\mathcal{B}}^2+2K_b\alpha^{\lambda}(t)$. Therefore, we obtain that

$$\zeta^{\lambda}(t)\le 2(M_b+J_0^{\varphi}+K_bM_1H)\|\varphi\|_{\mathcal{B}}^2+2K_b\Big[\overline{R}+1764b^2M_1M_2M_3\Big(M_1Tr(Q)$$
$$\int_0^b p(s)\Omega(\zeta^{\lambda}(s))ds+M_1\sum_{k=1}^{m}c_k^1(\zeta^{\lambda}(t))+\theta_1\zeta^{\lambda}(t)(1+\int_0^b\beta(s)ds)\Big)$$
$$+36M_1Tr(Q)\int_0^t p(s)\Omega(\zeta^{\lambda}(s))ds+36\theta_1\zeta^{\lambda}(t)(1+\int_0^b\beta(s)ds)$$
$$+36M_1\sum_{k=1}^{m}c_k^1\zeta^{\lambda}(t)\Big]$$
$$\le M_1+3528K_bb^2M_1M_2M_3\Big(M_1Tr(Q)\int_0^b p(s)\Omega(\zeta^{\lambda}(s))ds+M_1\sum_{k=1}^{m}c_k^1(\zeta^{\lambda}(t))\Big)$$
$$+72K_bM_1Tr(Q)\int_0^t p(s)\Omega(\zeta^{\lambda}(s))ds+\theta_1\zeta^{\lambda}(t)(1+\int_0^b\beta(s)ds)+72K_bM_1\sum_{k=1}^{m}c_k^1\zeta^{\lambda}(t)$$
$$\le\frac{M_1}{\gamma}+\frac{M_2}{\gamma}\int_0^t p(s)\Omega(\zeta^{\lambda}(s))ds.$$

Denoting by $v_{\lambda}(t)$ the right-hand side of the above inequality. Here $\zeta^{\lambda}(t)\le v_{\lambda}(t), t\in J$,

$$v_{\lambda}'(t)\le\frac{M_2}{\mu}p(t)\Omega(v_{\lambda}(t)),$$

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and hence

$$\int_{v_\lambda(0)=N}^{v_\lambda(t)} \frac{ds}{\Omega(s)} \le \frac{M_2}{\mu}\int_0^b p(s)ds < \int_N^\infty \frac{ds}{\Omega(s)}, \quad t \in J,$$

which implies that the set of functions $\{v_\lambda(\cdot) : \lambda \in (0,1)\}$ is bounded in $C(J,R)$. Thus $\{x^\lambda(\cdot) : \lambda \in (0,1)\}$ is bounded in $\mathcal{BPC}$.

To prove that Ψ is completely continuous, we introduce the decomposition $\Psi x = \Psi_1 x + \Psi_2 x + \Psi_3 x$ where $(\Psi_i x)_0 = 0, i = 1,2$ and

$$\Psi_1 x(t) = T(t)g(0,\varphi) + g(s,\bar{x}_s) + \int_0^t T(t-s)F(s,\bar{x}_{\rho(s,\bar{x}_s)})dw(s) + \int_0^t T(t-s)Bu(s)ds, \quad t \in J,$$
$$\Psi_2 x(t) = \int_0^t AT(t-s)g(s,\bar{x}_s)ds, \qquad t \in J,$$
$$\Psi_3 x(t) = \sum_{0<t_k<t} T(t-t_k)I_k(\bar{x}_{t_k}), \qquad t \in J.$$

From the proof of Theorem 3.1, and our assumption on g we can say that Ψ_1 is completely continuous and we can prove that Φ_2 is continuous. It remains to show that Ψ_2 is compact and Ψ_3 is also completely continuous. Now, by using the proof of [[26] Theorem 3.2] together with Arzela-Ascoli theorem we conclude that Ψ_2 is completely continuous. Next, by using Lemma 2.1, the continuity of Ψ_3 can be proved by using phase space axioms. On the other hand for $r > 0, t \in [t_k, t_{k+1}] \cap (0,b], k \ge 1$, and $m \in B_r = B_r(0, \mathcal{BPC})$, we find that

$$\widetilde{\Psi_3 u(t)} \in \begin{cases} \sum_{j=1}^{k} T(t-t_j)I_j(B_{r^*}(0;X)), & t \in (t_k, t_{k+1}), \\ \sum_{j=0}^{k} T(t_{k+1}-t_j)I_j(B_{r^*}(0;X)), & t = t_{k+1}, \\ \sum_{j=1}^{k-1} T(t_k-t_j)I_j(B_{r^*}(0;X)) + I_k(B_{r^*}(0;X)), & t = t_k, \end{cases}$$

where $r^* = (M_a + HM_1)\|\varphi\|_{\mathcal{B}}^2 + K_a r$, which proves that $[\widetilde{\Psi_3(B_r)}]_i(t)$ is relatively compact in H, for every $t \in [t_k, t_{k+1}]$, since the maps M_{I_k} are completely continuous. Moreover, using the compactness of the operator M_{I_k} and the strong continuity of $(T(t))_{t\ge 0}$, we can prove that $[\widetilde{\Psi_2(B_r)}]_i$ is equicontinuous at t, for every $t \in [t_k, t_{k+1}]$. Now, from the Lemma 2.1 we conclude that Ψ_3 is completely continuous.

Finally, these remarks and Lemma 2.3 shows that the controllability of mild solutions for problem (4.1)-(4.3) is controllable on J_1. The proof is complete.

5 Example

Example 1 Consider the following impulsive stochastic partial differential equation with state-dependent delay of the form

$$\frac{\partial}{\partial t}w(t,y)=\frac{\partial^2}{\partial y^2}w(t,y)+z(y)u(t)+\int_{-\infty}^{0}b(s-t)w(s-\rho_1(t)\rho_2(\|w(t)\|),y)d\beta(s), \tag{5.1}$$

$$w(t,0)=w(t,\pi)=0,\quad t\in J=[0,b], \tag{5.2}$$

$$w(\tau,y)=\varphi(\tau,y),\quad \tau\in(-\infty,0],\quad y\in[0,\pi], \tag{5.3}$$

$$\Delta w(t_k,y)=\int_{-\infty}^{t_k}a_k(t_k-s)w(s,y)ds,\quad y\in[0,\pi],\quad k=1,2,\dots,m, \tag{5.4}$$

where $\varphi\in\mathcal{B}=\mathcal{PC}_0\times L^2(g,H)$ and $0<t_1<\cdots<t_m<b$ are prefixed numbers, then $\rho_i:[0,\infty)\to(0,\infty]$ is continuous, and $\beta(s)$ is a one-dimensional standard Wiener process. Define $A:H\to H$ by $Az=z''$ with domain $D(A)=\{z(\cdot)\in H: z,'$, are absolutely continuous, $z''\in H, z(0)=z(\pi)=0\}$.

Then

$$Az=\sum_{n=1}^{\infty}n^2(z,e_n),\quad z\in D(A)$$

where $e_n(y)=\sqrt{\frac{2}{\pi}}sin(ny), n=1,2,\dots$ is a orthonormal set of eigenvectors in A. Then A is the infinitesimal generator of a compact C_0-semigroup of bounded linear operator $(T(t))_{t\geq 0}$ in H

$$T(t)z=\sum_{n=1}^{\infty}e^{-n^2}(z,e_n)e_n,\quad z\in H.$$

Let $\alpha<0$, define the phase space

$$\mathcal{B}=\Big\{\phi\in C((-\infty,0],H):\lim_{s\to-\infty}e^{\alpha s}\phi(s)\quad\text{exist in}\quad H\Big\},$$

and let $\|\phi\|_{\mathcal{B}}=\sup_{-\infty<s<0}\{e^{\alpha s}\,\|\phi(s)\|_{L_2}\}$. Then, $(\mathcal{B},\|\phi\|_{\mathcal{B}})$ is a Banach space which satisfies the Axioms from (i)-(iii) with $L=1, K_b=\max\{1,e^{-\alpha t}\}, M_b=e^{-\alpha t}$. Hence for $(t,\phi)\in[0,b]\times\mathcal{B}$, where $\phi(s)(y)=\phi(\theta,y),(s,y)\in(-\infty,0]\times[0,\pi]$, let $z(t)(y)=z(t,y)$.

To study the above systems, we impose the following conditions hold:

(i) The function $b:R\to R,\rho_i:[0,\infty)\to(0,\infty], i=1,2$ are continuous, bounded

$$M_F=\Big(\int_{-\infty}^{0}\frac{(b^2(s))}{g(s)}ds\Big)^{\frac{1}{2}}<\infty.$$

(ii) The function $a_k:R\to R$ are continuous such that

$$M_{I_k}=\left(\int_{-\infty}^{0}\frac{(a_k^2(s))}{g(s)}ds\right)^{\frac{1}{2}},\quad k=1,2,\dots,m.$$

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Assume that the bounded linear operator $B \in L(R, H)$ is defined by

$$Bu(t) = z(y)u, \quad 0 \leq y \leq \pi, \quad u \in R \quad z(y) \in L^2([0, \pi]).$$

By defining the operator $\rho, F : J \times \mathcal{B} \to H$ and $I_k : \mathcal{B} \to H$ by

$$F(t, \phi)(y) = \int_{-\infty}^{0} b(s)\phi(s, y)ds,$$

$$\rho(t, \phi) = s - \rho_1(s)\rho_2(\|\phi(0)\|),$$

$$I_k(\phi)(y) = \int_{-\infty}^{0} a_k(-s)\phi(s, y)ds, \quad k = 1, 2, \ldots, m$$

we can transform the systems (5.1)-(5.4) into the abstract impulsive Cauchy problem (1.1)-(1.3). Now the linear operator W is given by

$$Wu = \int_0^b T(b - s)Bu(s)ds.$$

Assume that this operator has a bounded inverse W^{-1} in $L^2(J, U)/KerW$. Moreover the function $F, I_k, k = 1, 2, \ldots, m$ are bounded linear operators with $\|F(t, \cdot)\|^2_{\mathcal{L}(\mathcal{B},\mathcal{H})} \leq M_F$, $\|I_k\|^2_{\mathcal{L}(\mathcal{B},\mathcal{H})} \leq M_{I_k}$. Hence all the conditions of Theorem 3.1 have been satisfied for the system (5.1)-(5.4), and so system is controllable on J_1.

Example 2 Consider the following impulsive neutral stochastic partial differential equation with state-dependent delay of the form

$$\frac{\partial}{\partial t}\left[w(t, y) + \int_{-\infty}^{t}\int_0^{\pi} a(t - s, \eta, y)w(s, \eta)d\eta ds\right] = \frac{\partial^2}{\partial y^2}w(t, y)$$

$$+ z(y)u(t) + \int_{-\infty}^{t} b(s - t)w(s - \rho_1(t)\rho_2(\|w(t)\|), y)d\beta(s) \tag{5.5}$$

$$w(t, 0) = w(t, \pi) = 0, \quad t \in J, \tag{5.6}$$

$$w(\tau, y) = \varphi(\tau, y), \quad \tau \in (-\infty, 0], \quad y \in [0, \pi], \tag{5.7}$$

$$\Delta w(t_k, y) = \int_{-\infty}^{t_k} a_k(t_k - s)w(s, y)ds, \quad y \in [0, \pi], \quad k = 1, 2, \ldots, m, \tag{5.8}$$

where $\varphi, B, b, \rho_i, i = 1, 2$ and M_F are defined in Example 1. Assume that the conditions (ii)of the previous example holds and that

(ii) The function $a(s, \eta, y)$, $\frac{\partial a(s,\eta,y)}{\partial y}$ are continuous and measurable, $a(s, \eta, \pi) = a(s, \eta, 0) = 0$ and

$$M_g = \max\left[\left(\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi}\frac{1}{g(s)}\left(\frac{\partial^j a(s, \eta, y)}{\partial y^j}\right)d\eta dsdy\right)^{\frac{1}{2}} : j = 0, 1\right] < \infty.$$

Define the function A, B, F, ρ, I_k, and W as in Example 1 and the operator $g : J \times \mathcal{B} \to H$ by

$$g(\phi)(y) = \int_{-\infty}^{0}\int_0^{\pi} a(s, \upsilon, y)\phi(s, \upsilon)d\upsilon ds,$$

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we can transform the systems (5.5)-(5.8) into the abstract Cauchy problem (4.1)-(4.3). Moreover, the function g is a bonded linear operator with $\|g(t,\cdot)\|_{\mathcal{L}(\mathcal{B},\mathcal{H})} \leq M_g$. Hence all the conditions of Theorem 4.3 have been satisfied for the system (5.5)-(5.8), and so system is controllable on J_1.

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