SOME APPLICATIONS OF BERNOULLI´S INEQUALITY IN THE APPROXIMATION THEORY.

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Abstract. Two elementary proofs of Weierstrass approximation theorem, respective of Stone-Weierstrass theorem are given. The paper is based on the previous papers [2] and [4]. Besides some simplifications of the proofs given in these two papers, the author also obtained two new results: Theorem 1 and Corollary 3.

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INTRODUCTION.

Weierstrass' theorems of approximating continuous functions on closed intervals by using algebraic or trigonometric polynomials were published in 1885, when he was **70** years old.

Later on, many mathematicians found other proofs for these theorems (see [3]). Among them, we mention:

Picard, Féjér, Landau, de la Vallée Poussin, Runge, Mittag - Leffler, Lebesgue, Lerch, Volterra, Bernstein. All these proofs appeared until 1913.

50 years after, in 1964, H. Kuhn [1] proved the theorem of approximation with algebraic polynomials using Bernoulli's inequality.

This proof is considered by Allan Pinkus "wonderfully simple and elementary".

Kuhn's idea of using Bernoulli's inequality was taken over by other mathematicians, in order to give new "elementary" proofs to other theorems of approximation.

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.1-11

Thus, in 1981, Bruno Brosowski and Frank Deutsch gave an elementary proof to Stone-Weierstrass' theorem**.**

In 1984 T.J.Ransford, using the same technique, found an elementary proof for Bishop's approximation theorem**.**

In 1992, J. Prolla generalizes Von Neumann's variant of Stone-Weierstrass' theorem for continuous functions with range in [0,1], using Ransford's tools.

Finally, in 2005, Rudolf Výborný gave a proof of Weierstrass' theorem of approximation based on Bernoulli's inequality, which turned out to be "more elementary" than Kuhn's proof.

In this line of research, the present paper gives two simple proofs of Weierstrass approximation theorem by polynomials, respective of Stone-Weierstrass theorem , and it is based on the previous papers [2] and [4].

Theorem 1 *.Let X be an arbitrary set , $u: X \to \mathbb{R}$, $0<u(x)<1$, $\forall x\in X$, and let A and B be two disjoint subsets of X. We suppose that there exists $k\in\mathbb{N}^*$, $k>1$, such that $u(x)<\frac{1}{k}, \forall x\in A$ and $u(x)>\frac{1}{k}, \forall x\in B$.*

Then we have:

$$\lim_{n\to\infty}\left(1-u^n(x)\right)^{k^n}=1,\ \forall x\in A$$

$$\lim_{n\to\infty}\left(1-u^n(x)\right)^{k^n}=0,\ \forall x\in B.$$

Proof. The assertion results from Bernoulli´s inequality.

Indeed, if $x\in A$, then $ku(x)<1$ and we have :

$$1\geq(1-u^n(x))^{k^n}\geq 1-(ku(x))^n\to 1$$

If $x\in B$, then $ku(x)>1$ and we have:

$$0\leq(1-u^n(x))^{k^n}\leq\frac{(1-u^n(x))^{k^n}}{(ku(x))^n}\ (1+k^n u^n(x))\leq$$

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.1-11

$$\le \frac{(1-u^n(x))^{k^n}(1+u^n(x))^{k^n}}{(ku(x))^n} = \frac{(1-u^{2n}(x))^{k^n}}{(ku(x))^n} < \frac{1}{(ku(x))^n} \to 0. \qquad \square$$

Corollary 1. *Let* $a,b \in \mathbb{R}$, *such that* $0<a<b<1$. *Then there exists a sequence of polynomials* $\{p_n\}$ *with the properties :*

(i) $0 \le p_n(x) \le 1,\ \forall x \in [0,1]$;

(ii) $\lim\limits_{n\to\infty} p_n(x) = 1,\ \forall x \in [0,a]$;

(iii) $\lim\limits_{n\to\infty} p_n(x) = 0,\ \forall x \in [b,1]$.

Proof. Let $c = \frac{a+b}{2}$, and let $u(x) = \frac{1}{2} + \frac{x-c}{2},\ \forall x \in [0,1]$.

Obviously , u is an increasing polynomial function of degree 1.We observe that:

$$u(0) = \frac{1-c}{2} > 0;\ u(a) = \frac{1}{2} + \frac{a-c}{2} < \frac{1}{2};$$

$$u(b) = \frac{1}{2} + \frac{b-c}{2} > \frac{1}{2};\ u(1) = \frac{2-c}{2} = 1 - \frac{c}{2} < 1.$$

Since u is increasing, it follows that :

$$0 < u(x) < 1,\ \forall x \in [0,1];\ u(x) < \frac{1}{2},\ \forall x \in [0,a];\ u(x) > \frac{1}{2},\ \forall x \in [b,1].$$

If we denote by $p_n(x) = \left(1-u^n(x)\right)^{2^n}$, then $\{p_n\}$ is a sequence of polynomials with the property : $0 \le p_n(x) \le 1,\ \forall x \in [0,1]$.

On the other hand, from Theorem 1, for $k=2$, it follows :

$$\lim_{n\to\infty} p_n(x) = 1,\ x \in [0,a]$$

and

$$\lim_{n\to\infty} p_n(x) = 0,\ x \in [b,1]. \qquad \square$$

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.1-11

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.1-11

Corollary 2. *Let X be a compact Hausdorff space, $\mathscr{A}$ a subalgebra of real valued continuous functions on X containing constants and separating points of X, and $H, K \subset X$ two closed disjoints subsets of X. Then there exists a sequence of functions $\{a_n\}$, $a_n \in \mathscr{A}$, with the properties* :

(i) $0 \le a_n(x) \le 1,\ x \in X$;

(ii) $\lim\limits_{n\to\infty} a_n(x) = 0,\ x \in K$;

(iii) $\lim\limits_{n\to\infty} a_n(x) = 1,\ x \in H$.

Proof. Since $K \cap H = \varnothing$, it results that $U = X \setminus H$ is an open neighborhood of K. Let $x_0 \in K$. As $\mathscr{A}$ separates the points of X it follows that $\forall x \in H$, $\exists\, a_x \in A$ such that :

$$a_x(x_0) \ne a_x(x).$$

If we denote by $b_x = a_x - a_x(x_0)\mathbf{1}_X$, then $b_x \in \mathscr{A}$, $b_x(x_0) = 0$ and $b_x(x_0) \ne 0$.

Let $c_x = \dfrac{b_x^2}{\|b_x\|^2}$. We observe that $c_x \in \mathrm{A}$, $c_x(x_0) = 0$, $c_x(x) > 0$ and $0 \le c_x \le 1$.

Let $V_x = \{t \in X \,|\, c_x(t) > 0\}$. Then V_x is an open neighborhood of x, and $H \subset \bigcup\limits_{x\in H} V_x$.

By the compactness of H, there exists a finite number of points $x_1, x_2, ..., x_m \in H$, such that $H \subset \bigcup\limits_{i=1}^{m} V_{x_i}$.

Finally, if we denote by $u = \dfrac{c_{x_1} + c_{x_2} + ... + c_{x_m}}{m}$, then :

$u \in \mathscr{A},\ 0 \le u \le 1,\ u(x_0) = 0$ şi $u(x) > 0,\ \forall x \in H$.

Using again the compactness of H, it results that there exists $0 < \delta < 1$ such that: $u(x) \ge \delta,\ \forall x \in H$.

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.1-11

Let $W_{x_0} = \left\{ x \in X \middle| u(x) < \frac{\delta}{2} \right\}$. Obviously $x \in W_{x_0}$, W_{x_0} is an open subset of X and $W_{x_0} \subset U = X \setminus H$.

Let $k \in \mathbb{N}^*$ with the property :

$$k - 1 \le \frac{1}{\delta} < k \tag{1}$$

We observe that :

$$k \le \frac{1+\delta}{\delta} < \frac{2}{\delta}$$

and thus:

$$1 < k\delta < 2 \tag{2}$$

If we denote by :

$$a_n(x) = [1 - u(x)]^{k^n}, \ x \in X,$$

then

$$a_n \in \mathscr{A}, \ 0 \le a_n \le 1 \text{ and } a_n(x_0) = 1, \ \forall n \in \mathbb{N}^*.$$

On the other hand, if $x \in W_{x_0}$, then:

$$ku(x) < \frac{k\delta}{2} < 1$$

and so :

$$u(x) < \frac{1}{k}$$

If $x \in H$, then $ku(x) \ge k\delta > 1$, thus $u(x) > \frac{1}{k}$.

By Theorem 1, it follows that:

$$\lim_{n\to\infty} a_n(x) = 1, \text{ if } x \in W_{x_0}$$

and

$$\lim_{n\to\infty} a_n(x) = 0, \text{ if } x \in H.$$

Using now the compactness of K, we infer that there exist $y_1, \ldots y_r \in K$ such that:

$$K \subset \bigcup_{i=1}^{r} W_{y_i}.$$

Let $a_n^i \in \mathscr{A}$, $i = \overline{1,r}$ with the properties :

$$0 \le a_n^i \le 1,$$

$$\lim_{n\to\infty} a_n^i(x) = 1, \ \forall x \in W_{y_i}.$$

and

$$\lim_{n\to\infty} a_n^i(x) = 0, \ \forall x \in H.$$

Finaly, if we denote by $a_n = (1 - a^1{}_n)......(1 - a_n{}^r)$, then:

$$a_n \in \mathscr{A},\ 0 \le a_n \le 1,\ \lim_{n\to\infty} a_n(x) = 0,\ \forall x \in K \text{ and } \lim_{n\to\infty} a_n(x) = 1, \text{ if } \ x \in H. \ \square$$

Corollary 3. *Let X be a compact Hausdorff space, $\mathscr{A}$ a subalgebra of continuous functions on X containing constants and separating the points of X, A_i, B_i, $i = \overline{0,n}$, $2n+2$ closed subsets of X, $A_i \cap B_i = \varnothing$, $\forall i = \overline{0,n}$, $A_i \subset A_{i+1}$, $B_i \supset B_{i+1}$, and $\varepsilon > 0$. Then , $\exists a \in \mathscr{A}$ such that :*

$$a(x) < i\varepsilon + \varepsilon^2, \textit{ if } \ x \in A_i.$$

and

$$a(x) > (j+1)\varepsilon - \varepsilon^2, \textit{ if } \ x \in B_j.$$

Proof. By Corollary 2, it follows that : $\forall i, \exists a_i \in \mathscr{A}$, with the properties :

(i) $0 \le a_i \le 1$;

(ii) $a_i(x) < \dfrac{\varepsilon}{n}$, $x \in A_i$;

(iii) $a_i(x) > 1 - \dfrac{\varepsilon}{n}$, $x \in B_j$, $i = \overline{0,n}$.

Let $a(x) = \varepsilon\left(a_0(x) + a_1(x) + ... + a_n(x)\right)$. Obviously $a \in \mathscr{A}$.

If $x \in A_i$, then:

$$a(x) = \varepsilon\left(a_0(x) + ... + a_{i-1}(x)\right) + \varepsilon\left(a_i(x) + ... + a_n(x)\right) \le i\varepsilon + (n - i + 1)\frac{\varepsilon^2}{n} < i\varepsilon + \varepsilon^2$$

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.1-11

If $x \in B_j$, then :

$$a(x) \geq \varepsilon\left(a_0(x)+...+a_j(x)\right) > \varepsilon(j+1)\left(1-\frac{\varepsilon}{n}\right) > \varepsilon(j+1)-\varepsilon^2. \quad \square$$

Theorem 2 (Weierstrass). *Let* $f:[0,1]\to\mathbb{R}$ *be a continuous function. Then, for* $\forall\varepsilon>0$, *there exists a polynomial* P_ε *such that :*

$$\left|f(x)-P_\varepsilon(x)\right|<\varepsilon,\ \forall x\in[0,1].$$

Proof. We denote by :

$A=\left\{x\in[0,1] \middle| \exists \text{ polynomial } P_\varepsilon \text{ such that } \left|f(x)-P_\varepsilon(t)\right|<\varepsilon,\ t\in[0,x]\right\}.$

First, we observe that the set A is non void . Indeed, since f is continuous in 0, it results that there is a $0<\alpha<1$ such that $\left|f(t)-f(a)\right|<\varepsilon,\ \forall t\in[0,\alpha]$.

As the function $P(t)=f(a),\ \forall t\in[0,\alpha]$, is a polynomial of degree 0 ,we infer that $\alpha\in A$.

Let $s=\sup A$. If $s=1$, the proof is finished.

We suppose now that $0<s<1$.As f is continuous in s, it results that there exists $\delta>0$ such that :

$$\left|f(x)-f(s)\right|<\frac{\varepsilon}{3},\ \forall x\in(s-\delta,x+\delta).$$

On the other hand, since s is the upper bound of A , it follows that there is $c\in A$ such that :

$$s-\delta<c\leq s.$$

By Corollary 1, for $a=s-\delta$, and $b=c$,it results that there exists a sequence of polynomials $\{p_n\}$ with the properties :

$0\leq p_n\leq 1;$

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.1-11

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.1-11

$\lim\limits_{n\to\infty} p_n(x)=1,\ x\in[0,s-\delta];$

$\lim\limits_{n\to\infty} p_n(x)=0,\ x\in[c,1].$

On the other hand, as $c\in A$, it follows that there exists a polynomial Q_ε such that :

$$\left|f(x)-Q_\varepsilon(x)\right|<\varepsilon,\ \forall x\in[0,c].$$

Denotes by:

$$P_\varepsilon(x)=f(s)+\left[Q_\varepsilon(x)-f(s)\right]p_n(x),\ x\in[0,1].$$

We shall show that this is the searching polynomial. Indeed, we observe that:

$$\left|f(x)-P_\varepsilon(x)\right|\le\left|f(x)-Q_\varepsilon(x)\right|p_n(x)+\left|f(x)-f(s)\right|\left(1-p_n(x)\right)\ (*).$$

If $x\in[0,s-\delta]$, then $\lim\limits_{n\to\infty} p_n(x)=1$, and if we pass to the limit after n in $(*)$ it follows that:

$$\left|f(x)-P_\varepsilon(x)\right|\le\left|f(x)-Q_\varepsilon(x)\right|<\varepsilon.$$

If $x\in[s-\delta,c]$, then :

$$\left|f(x)-P_\varepsilon(x)\right|<\varepsilon p_n(x)+\frac{\varepsilon}{3}(1-p_n(x))<\varepsilon.$$

Finally, if $x\in[c,s+\delta]$, then $\lim\limits_{n\to\infty} p_n(x)=0$, and if we pass to the limit in $(*)$, with respect to n, we obtain :

$$\left|f(x)-P_\varepsilon(x)\right|\le\left|f(x)-f(s)\right|<\frac{\varepsilon}{3}<\varepsilon.$$

It results that f may by uniformly approximated by a polynomial on the interval $[0,s+\delta]$, and this contradicts the definition of s. □

Remark 1. *The Weierstrass theorem is valid on any interval* $[a,b]$.

Indeed, if we denote by $\varphi(t)=(1-t)a+tb$, $t\in[0,1]$, then φ is a change of variable, from $[0,1]$ to $[a,b]$,and $t=\varphi^{-1}(x)=\dfrac{x-a}{b-a}$, $x\in[a,b]$.

Let $f:[a,b]\to\mathbb{R}$ be a continuous function, and let $g:[0,1]\to\mathbb{R}$, $g=f\circ\varphi$. Obviously, g is continuous and , according to Theorem 2 ,there exists a polynomial Q_ε such that :

$$\left|g(t)-Q_\varepsilon(t)\right|<\varepsilon,\ \forall t\in[0,1].$$

If we denote by $P_\varepsilon(x)=Q_\varepsilon\left[\varphi^{-1}(x)\right]$, it follows that P_ε is a polynomial and

$\left|g\left[\varphi^{-1}(x)\right]-Q_\varepsilon\left[\varphi^{-1}(x)\right]\right|<\varepsilon,\ x\in[a,b]$ and thus:

$\left|f(x)-P_\varepsilon(x)\right|<\varepsilon,\ \forall x\in[a,b].$

Theorem 2. (Stone-Weierstrass)

Let X be a compact Hausdorff space, and let A be a subalgebra of real valued continuous functions on X containing constants and separating the points of X. Then $\overline{\mathrm{A}}=C(X)$.

Proof. Let $f\in C(X)$, $0<\varepsilon<\frac{1}{3}$ and $n\in\mathbb{N}^*$, such that $\frac{\|f\|}{n-1}<\varepsilon$.

Denote by :

$$A_i=\left\{x\in X\,\middle|\,f(x)\le\left(i-\frac{1}{3}\right)\varepsilon\right\}$$

and by

$$B_i=\left\{x\in X\,\middle|\,f(x)\ge\left(i+\frac{1}{3}\right)\varepsilon\right\}.$$

Obviously , A_i, B_i are closed and $A_i\cap B_i=\varnothing$, $\forall i$.

We observe that :

$A_0=\varnothing\subset A_1\subset A_2\subset\ldots\subset A_n=X$,

$B_0\supset B_1\supset\ldots..\supset B_n=\phi$,

and

$X\setminus A_{i-1}\subset B_{i-2}$.

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.1-11

Indeed, $B_{i-2}=\left\{x\,\middle|\, f(x)\ge\left(i-2+\dfrac{1}{3}\right)\varepsilon\right\}=\left\{x\,\middle|\, f(x)\ge\left(i-\dfrac{5}{3}\right)\varepsilon\right\}$.

If $x\notin A_{i-1}$, then $f(x)>\left(i-1-\dfrac{1}{3}\right)\varepsilon=\left(i-\dfrac{4}{3}\right)\varepsilon>\left(i-\dfrac{5}{3}\right)x$, thus $x\in B_{i-2}$.

On the other hand, we have:

$$X\subset A_1\cup\left(\bigcup_{i=2}^{n}A_i\setminus A_{i-1}\right)\subset A_1\cup\left(\bigcup_{i=1}^{n}\left(A_i\cap B_{i-2}\right)\right).$$

From Corollary 3, we infer that there exists an $a\in\mathscr{A}$ with the properties:

$a(x)<i\varepsilon+\varepsilon^2$ if $x\in A_i$ and

$a(x)>(j+1)\varepsilon-\varepsilon^2$ if $j\in B_j$.

Let $x\in X$ arbitrary.

If $x\in A_1$, then $f(x)\le\dfrac{2}{3}\varepsilon$, and $a(x)\le\varepsilon+\varepsilon^2<\varepsilon+\dfrac{\varepsilon}{3}=\dfrac{4\varepsilon}{3}$.

It follows that:

$\left|f(x)-a(x)\right|<\dfrac{4\varepsilon}{3}<2\varepsilon$.

If $x\in\left(A_i\cap B_{i-2}\right)$, $i\ge2$, then ,according to Theorem 2, it follows that:

$$(i-1)\varepsilon-\varepsilon^2\le a(x)\le i\varepsilon+\varepsilon^2 \text{ and } \left(i-\frac{5}{3}\right)\varepsilon\le f(x)\le\left(i-\frac{1}{3}\right)\varepsilon.$$

As

$$(i-1)\varepsilon-\varepsilon^2<(i-1)\varepsilon-\frac{\varepsilon}{3}=\left(i-\frac{4}{3}\right)\varepsilon, \text{ and } i\varepsilon+\varepsilon^2<i\varepsilon+\frac{\varepsilon}{3}=\left(i+\frac{1}{3}\right)\varepsilon,$$

we have :

$$\left(i-\frac{5}{3}\right)\varepsilon<a(x)<\left(i+\frac{1}{3}\right)\varepsilon,$$

and further :

$$\left|f(x)-a(x)\right|<\left(i+\frac{1}{3}\right)\varepsilon-\left(i-\frac{5}{3}\right)\varepsilon=2\varepsilon. \quad \square$$

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.1-11

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ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.1-11