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DOUBLY WARPED PRODUCTS IN *S*-SPACE FORMS

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Abstract

Recently, the author established a general inequality for doubly warped products in arbitrary Riemannian manifolds [14].

In the present paper, we obtain a similar inequality for doubly warped products isometrically immersed in *S*-space forms. As applications, we derive certain obstructions to the existence of minimal isometric immersions of doubly warped product integral submanifolds in *S*-space forms.

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1. Introduction

Singly warped products or simply warped products were first defined by Bishop and O'Neill in [1]. They used this concept to construct Riemannian manifolds with negative sectional curvature.

A warped product is defined as follows:

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and f a positive differentiable function on M_1. Consider the product manifold $M_1 \times M_2$ with its natural projections $\pi : M_1 \times M_2 \to M_1$ and $\eta : M_1 \times M_2 \to M_2$.

The *warped product* of M_1 and M_2, $M = M_1 \times_f M_2$ is the Riemannian manifold $M_1 \times M_2$ equipped with the Riemannian structure g such that

$$\|X\|^2 = \|\pi_*(X)\|^2 + f^2(\pi(p))\|\eta_*(X)\|^2$$

for any tangent vector $X \in T_pM$, $p \in M$. Thus, we have $g = g_1 + f^2 g_2$. The function f is called the *warping function* of the warped product.

If the warping function is constant, then the manifold M is said to be *trivial*. Let us notice that if $f = 1$, then $M_1 \times_f M_2$ reduces to a Riemannian product manifold.

In general, doubly warped products can be considered as a generalization of singly warped products. Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and let $f_1 : M_1 \to (0, \infty)$ and $f_2 : M_2 \to (0, \infty)$ be differentiable functions.

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The *doubly warped product* $M = {}_{f_2}M_1 \times_{f_1} M_2$ is the product manifold $M_1 \times M_2$ endowed with the metric

$$g = f_2^2 g_1 + f_1^2 g_2 .$$

More precisely, if $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$ are natural projections, the metric g is defined by

$$g = (f_2 \circ \pi_2)^2 \pi_1^* g_1 + (f_1 \circ \pi_1)^2 \pi_2^* g_2 .$$

The function f_1 and f_2 are called *warping functions*. If either $f_1 \equiv 1$ or $f_2 \equiv 1$, but not both, then we obtain a warped product. If both $f_1 \equiv 1$ and $f_2 \equiv 1$, then we have a Riemannian product manifold. If neither f_1 nor f_2 is constant, then we have a *non-trivial* doubly warped product [16].

Let $x : {}_{f_2}M_1 \times_{f_1} M_2 \to \widetilde{M}$ be an isometric immersion of a doubly warped product ${}_{f_2}M_1 \times_{f_1} M_2$ into a Riemannian manifold $\widetilde{M}$. We denote by h the second fundamental form of x and by $H_i = \frac{1}{n_i} \mathit{trace}\, h_i$ the partial mean curvatures, where $\mathit{trace}\ h_i$ is the trace of h restricted to M_i and $n_i = \dim M_i$ $(i = 1,2)$.

The immersion x is said to be *mixed totally geodesic* if $h(X, Z) = 0$, for any vector fields X and Z tangent to D_1 and D_2, respectively, where D_i are the distributions obtained from the vectors tangent to M_i (or more precisely, vectors tangent to the horizontal lifts of M_i).

In [5], B.Y. Chen proved the following general optimal relationship between the warping function f and the extrinsic structures of the warped product $M_1 \times_f M_2$.

Theorem 1.1 *Let* $x : M_1 \times_f M_2 \to \widetilde{M}(c)$ *be an isometric immersion of an n-dimensional warped product* $M_1 \times_f M_2$ *into an m-dimensional Riemannian manifold* $\widetilde{M}(c)$ *of constant sectional curvature c. Then:*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2} \|H\|^2 + n_1 c, \tag{1.1}$$

where $n_i = \dim M_i, \mathrm{i} = 1,2$, *and* Δ *is the Laplacian operator of* M_1. *Moreover, the equality case of* (1.1) *holds if and only if x is a mixed totally geodesic immersion and* $n_1 H_1 = n_2 H_2$ *where* H_i, *i=1,2, are the partial mean curvature vectors.*

The inequality (1.1) was noticed by several authors and they established similar inequalities for different submanifolds in ambient manifolds possessing different kind of structures. For example, in [8], K. Matsumoto and I. Mihai studied warped product submanifolds in Sasakian space forms. In [9] and [10], A. Mihai considered warped product submanifolds in complex space forms and quaternion space forms, respectively. In [11], C. Murathan, K. Arslan, R. Ezentas and I. Mihai studied warped product submanifolds in Kenmotsu space forms.

Later, in 2008, B. Y. Chen and F. Dillen extended this inequality to multiply warped product manifolds in arbitrary Riemannian manifolds (see [6]).

Recently, in [13], the present author studied warped product submanifolds in generalized Sasakian space forms. In [7], M. K. Dwivedi and J.-S. Kim considered warped product submanifolds in *S*-space forms.

In [14], the present author established the following general inequality for arbitrary isometric immersions of doubly warped product manifolds in arbitrary Riemannian manifolds:

Theorem 1.2 *Let x be an isometric immersion of an n-dimensional doubly warped product* $M={}_{f_2}M_1\times_{f_1}M_2$ *into an m-dimensional arbitrary Riemannian manifold* $\widetilde{M}^m$. *Then:*

$$n_2\frac{\Delta_1 f_1}{f_1}+n_1\frac{\Delta_2 f_2}{f_2}\le\frac{n^2}{4}\|H\|^2+n_1n_2\max\widetilde{K}, \tag{1.2}$$

where $n_i=\dim M_i$, $n=n_1+n_2$, Δ_i *is the Laplacian operator of* M_i, $(i=1,2)$ *and* $\max\widetilde{K}(p)$ *denotes the maximum of the sectional curvature function of* $\widetilde{M}^m$ *restricted to* 2*-plane sections of the tangent space* T_pM *of* M *at each point p in M. Moreover, the equality case of* (1.2) *holds if and only if the following two statements hold:*

1. *x is a mixed totally geodesic immersion satisfying* $n_1H_1=n_2H_2$ *where* H_i, *i=1,2, are the partial mean curvature vectors of* M_i.
2. *at each point* $p=(p_1,p_2)\in M$, *the sectional curvature function* $\widetilde{K}$ *of* $\widetilde{M}^m$ *satisfies* $\widetilde{K}(u,v)=\max\widetilde{K}(p)$ *for each unit vector* $u\in T_{p_1}M_1$ *and each unit vector* $v\in T_{p_2}M_2$.

Motivated by the studies of the above authors, we obtain a similar inequality for doubly warped products in *S*-space forms.

On the other hand, the concept of framed metric structure unifies the concepts of almost Hermitian and almost contact metric structures. In particular, an *S*-structure generalizes Kaehler and Sasakian structure. In [2], Blair discusses principal toroidal bundles and generalizes the Hopf fibration to give a canonical example of an *S*-manifold playing the role of complex projective space in Kaehler geometry and the

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odd-dimensional sphere in Sasakian geometry. An S-manifold of constant f-sectional curvature c is called an *S-space form* $\widetilde{M}(c)$ [4], which generalizes the complex space form and Sasakian space form.

2. Preliminaries

In this section, we recall some definitions and basic formulas which we will use later.

Let M be a Riemannian n-manifold isometrically immersed in a Riemannian m-manifold $\widetilde{M}^m$.

We choose a local field of orthonormal frame $e_1,....,e_n,e_{n+1},...,e_m$ in $\widetilde{M}^m$ such that, restricted to M, the vectors $e_1,....,e_n$ are tangent to M and $e_{n+1},...,e_m$ are normal to M.

Let $K(e_i \wedge e_j)$, $1 \leq i < j \leq n$, denote the sectional curvature of the plane section spanned by e_i and e_j. Then the scalar curvature of M is given by

$$\tau = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j). \tag{2.1}$$

Let L be a subspace of T_pM of dimension $r \geq 2$ and $\{e_1,....,e_r\}$ an orthonormal basis of L. The scalar curvature $\tau(L)$ of the r-plane section L is defined by

$$\tau(L) = \sum_{1 \leq \alpha < \beta \leq r} K(e_\alpha \wedge e_\beta). \tag{2.2}$$

Let h be the second fundamental form and R the Riemann curvature tensor of M.

Then the equation of Gauss is given by

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)), \tag{2.3}$$

for any vectors X, Y, Z, W tangent to M.

The mean curvature vector H is defined by

$$H = \frac{1}{n} traceh = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i). \tag{2.4}$$

As is known, M is said to be *minimal* if H vanishes identically.

Also, we set

$$h_{ij}^r = g(h(e_i,e_j),e_r), \quad i,j \in \{1,...,n\},\ r \in \{n+1,....,m\} \tag{2.5}$$

the coefficients of the second fundamental form h with respect to $e_1,....,e_n$, $e_{n+1},...,e_m$, and

$$\|h\|^2 = \sum_{i,j=1}^{n} g\left(h(e_i,e_j), h(e_i,e_j)\right). \tag{2.6}$$

Let M be a Riemannian p-manifold and $\{e_1,....,e_p\}$ be an orthonormal basis of M. For a differentiable function f on M, the Laplacian Δf of f is defined by

$$\Delta f = \sum_{j=1}^{p} \left\{ \left(\nabla_{e_j} e_j\right) f - e_j e_j f \right\}. \tag{2.7}$$

We recall the following general algebraic lemma of Chen for later use.

Lemma 2.1 [5] Let $n \geq 2$ and $a_1, a_2, ..., a_n, b$ real numbers such that

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + b\right) \tag{2.8}$$

Then $2a_1a_2 \geq b$, with equality holding if and only if

$$a_1 + a_2 = a_3 = ... = a_n.$$

3. *S*-space forms

Let $\widetilde{M}$ be a $(2m+s)$-dimensional *framed metric manifold* [19] (also known as *framed f-manifold* [12] or *almost s-contact metric manifold* [17] with a framed structure $(f, \xi_\alpha, \eta^\alpha, \widetilde{g})$, $\alpha \in \{1,...,s\}$, that is, f is a (1,1) tensor field defining a f-structure of rank m; $\xi_1,...,\xi_s$ are vector fields; $\eta^1,...,\eta^s$ are 1-forms and $\widetilde{g}$ is a Riemannian metric on $\widetilde{M}$ such that for all $X, Y \in T\widetilde{M}$ and $\alpha, \beta \in \{1,...,s\}$

$$f^2 = -I + \eta^\alpha \otimes \xi_\alpha,\ \eta^\alpha(\xi_\beta) = \delta^\alpha_\beta,\ f(\xi_\alpha) = 0, \eta^\alpha \circ f = 0, \tag{3.1}$$

$$\langle fX, fY\rangle = \langle X, Y\rangle - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y), \tag{3.2}$$

$$\Omega(X,Y) \equiv \langle X, fY\rangle = -\Omega(Y,X),\ \langle X, \xi_\alpha\rangle = \eta^\alpha(X), \tag{3.3}$$

where $\langle , \rangle$ denotes the inner product of the metric $\widetilde{g}$. A framed metric structure is an S-structure if the Nijenhuis tensor of f equals $-2d\eta^\alpha \otimes \xi_\alpha$ and $\Omega = d\eta^\alpha$, for all $\alpha \in \{1,...,s\}$.

When *s=1*, a framed metric structure is an almost contact metric structure, while an *S*-structure is a Sasakian structure.

When *s=0*, a framed metric structure is an almost Hermitian structure, while an *S*-structure is a Kaehler structure.

If a framed metric structure on $\widetilde{M}$ is an *S*-structure then it is known [2] that

$$\left(\widetilde{\nabla}_X f\right)Y = \sum_{\alpha}\left(\langle fX, fY\rangle \xi_\alpha + \eta^\alpha(Y) f^2 X\right), \tag{3.4}$$

$$\widetilde{\nabla}_X \xi_\alpha = -fX, \quad \alpha \in \{1,...,s\}. \tag{3.5}$$

The converse may also be proved. In case of Sasakian structure (that is s=1), (3.4) implies (3.5). In Kaehler case (that is s=0), we get $\widetilde{\nabla} f = 0$. For s>1, examples of *S*-structures are given in [2] and [3]. Thus, the bundle space of principal toroidal bundles over a Kaehler manifold with certain conditions is an *S*-manifold. Thus, a generalization of the Hopf fibration $\pi^{'} : S^{2m+1} \to PC^m$ is a canonical example of an *S*-manifold playing the role of complex projective space in Kaehler geometry and the odd-dimensional sphere in Sasakian geometry.

A plane section in $T_p\widetilde{M}$ is a *f-section* if there exists a vector $X \in T_p\widetilde{M}$ orthogonal to $\xi_1,...,\xi_s$ such that $\{X, fX\}$ span the section. The sectional curvature of a *f*-section is called a *f-sectional curvature*. It is known that [4] in an *S*-manifold of constant *f*-sectional curvature *c*

$$\begin{aligned}\widetilde{R}(X,Y)Z = \sum_{\alpha,\beta}\{&\eta^\alpha(X)\eta^\beta(Z)f^2Y - \eta^\alpha(Y)\eta^\beta(Z)f^2X - \\ &-\langle fX, fZ\rangle\eta^\alpha(Y)\xi_\beta + \langle fY, fZ\rangle\eta^\alpha(X)\xi_\beta\} + \\ &+\frac{c+3s}{4}\{-\langle fY, fZ\rangle f^2X + \langle fX, fZ\rangle f^2Y\} + \\ &+\frac{c-s}{4}\{\langle X, fZ\rangle fY - \langle Y, fZ\rangle fX + 2\langle X, fY\rangle fZ\},\end{aligned} \tag{3.6}$$

for all $X, Y, Z \in T\widetilde{M}$ where $\widetilde{R}$ is the curvature tensor of $\widetilde{M}$. An *S*-manifold of constant *f*-sectional curvature *c* is called an *S*-space form $\widetilde{M}(c)$.

4. Doubly warped product integral submanifolds in *S*-space forms

Let $\widetilde{M}$ be an *S*-manifold equipped with an *S*-structure $\left(f, \xi_\alpha, \eta^\alpha, \widetilde{g}\right)$. A submanifold *M* of $\widetilde{M}$ is an *integral submanifold* if $\eta^\alpha(X) = 0, \alpha \in \{1,...,s\}$, for every tangent vector *X*. It follows that *f* maps any tangent space to the normal space,

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i.e. $f\left(T_pM\right)\subseteq T_p^{\perp}M$, $\forall p\in M$. In particular case of *s=1*, an integral submanifold M of a Sasakian manifold is a *C-totally real submanifold* [18]. It is known that an *n*-dimensional integral submanifold *M*, of an *S*-manifold $\widetilde{M}$ of dimension *(2n+s)*, is of constant curvature *s* if and only if the normal connection is flat.

Next, we investigate doubly warped product integral submanifolds in an *S*-space form $\widetilde{M}(c)$.

Theorem 4.1 *Let x be an integral isometric immersion of an n-dimensional doubly warped product* ${}_{f_2}M_1\times_{f_1}M_2$ *into a (2m+s)-dimensional S-space form* $\widetilde{M}(c)$*.Then:*

$$n_2\frac{\Delta_1 f_1}{f_1}+n_1\frac{\Delta_2 f_2}{f_2}\leq\frac{n^2}{4}\left\|H\right\|^2+n_1n_2\frac{c+3s}{4}, \tag{4.1}$$

where $n_i=\dim M_i$, $n=n_1+n_2$ *and* Δ_i *is the Laplacian operator of* M_i, $\left(i=1,2\right)$.

Moreover, the equality case of (4.1) *holds if and only if* *x is a mixed totally geodesic immersion and* $n_1H_1=n_2H_2$ *where* H_i, *i=1,2, are the partial mean curvature vectors.*

Proof. Let ${}_{f_2}M_1\times_{f_1}M_2$ be an integral doubly warped product submanifold into an *S*-space form $\widetilde{M}(c)$ of constant *f*-sectional curvature *c*. Since ${}_{f_2}M_1\times_{f_1}M_2$ is a doubly warped product, then

$$\begin{cases}\nabla_XY=\nabla^1_XY-\dfrac{f_2^2}{f_1^2}g_1(X,Y)\nabla^2\left(\ln f_2\right)\\ \nabla_XZ=Z\left(\ln f_2\right)X+X\left(\ln f_1\right)Z\end{cases}, \tag{4.2}$$

for any vector fields *X, Y* tangent to M_1 and *Z* tangent to M_2, respectively, where ∇^1 and ∇^2 are the Levi-Civita connections of the Riemannian metrics g_1 and g_2, respectively. Here, $\nabla^2\left(\ln f_2\right)$ denotes the gradient of $\ln f_2$ with respect to the metric g_2.

If *X* and *Z* are unit fields, *X* tangent to M_1 and *Z* tangent to M_2, it follows from (4.2) that the sectional curvature $K\left(X\wedge Z\right)$ of the plane section spanned by *X* and Z is given by

$$K(X \wedge Z) = \frac{1}{f_1}\left\{(\nabla^1_X X) f_1 - X^2 f_1\right\} + \frac{1}{f_2}\left\{(\nabla^2_Z Z) f_2 - Z^2 f_2\right\}. \tag{4.3}$$

We choose a local orthonormal frame $\{e_1, \ldots, e_{n_1}, e_{n_1+1}, \ldots, e_n\}$ such that $e_1, \ldots, e_{n_1}$ are tangent to M_1, $e_{n_1+1}, \ldots, e_n$ are tangent to M_2. Since it is easily seen that the mean curvature vector H is orthogonal to ξ_α, we may choose e_{n+1} parallel to H and $e_{2m+1} = \xi_1, \ldots, e_{2m+s} = \xi_s$.

Then, using (4.3), we get

$$\sum_{\substack{1 \le j \le n_1 \\ n_1+1 \le s \le n}} K(e_j \wedge e_s) = n_2 \frac{\Delta_1 f_1}{f_1} + n_1 \frac{\Delta_2 f_2}{f_2} . \tag{4.4}$$

From the equation of Gauss (2.3) and taking account of (3.6), we have

$$n^2 \|H\|^2 = 2\tau + \|h\|^2 - n(n-1)\frac{c+3s}{4}, \tag{4.5}$$

where τ denotes the scalar curvature of ${}_{f_2}M_1 \times_{f_1} M_2$, that is,

$$\tau = \sum_{1 \le i < j \le n} K(e_i \wedge e_j).$$

We set

$$\delta = 2\tau - n(n-1)\frac{c+3s}{4} - \frac{n^2}{2}\|H\|^2. \tag{4.6}$$

Then, (4.5) can be written as

$$n^2 \|H\|^2 = 2\left(\delta + \|h\|^2\right). \tag{4.7}$$

Obviously, $h^{2m+\alpha}_{ij} =< h(e_i, e_j), \xi_\alpha) =< \tilde{\nabla}_{e_i} e_j, \xi_\alpha >=< e_j, fe_i >= 0$.

With respect to the above orthonormal frame, (4.7) takes the following form:

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = 2\left[\delta + \sum_{i=1}^{n}\left(h_{ii}^{n+1}\right)^2 + \sum_{i\neq j}\left(h_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}\left(h_{ij}^{r}\right)^2\right].$$

If we put $a_1 = h_{11}^{n+1}$, $a_2 = \sum_{i=2}^{n_1} h_{ii}^{n+1}$ and $a_3 = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}$, the above equation becomes

$$\left(\sum_{i=1}^{3} a_i\right)^2 = 2[\delta + \sum_{i=1}^{3} a_i^{\;2} + \sum_{i\neq j}\left(h_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}\left(h_{ij}^{r}\right)^2 -$$

$$- \sum_{2\leq j\neq k\leq n_1} h_{jj}^{n+1}h_{kk}^{n+1} - \sum_{n_1+1\leq s\neq t\leq n_1} h_{ss}^{n+1}h_{tt}^{n+1}\ ].$$

Thus a_1, a_2, a_3 satisfy the Lemma of Chen (for n=3), i.e.,

$$\left(\sum_{i=1}^{3} a_i\right)^2 = 2\left(b + \sum_{i=1}^{3} a_i^2\right),$$

with

$$b = \delta + \sum_{1\leq i\neq j\leq n}\left(h_{ij}^{n+1}\right)^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}\left(h_{ij}^{r}\right)^2 - \sum_{2\leq j\neq k\leq n_1} h_{jj}^{n+1}h_{kk}^{n+1} - \sum_{n_1+1\leq s\neq t\leq n} h_{ss}^{n+1}h_{tt}^{n+1}.$$

Then $2a_1a_2 \geq b$, with equality holding if and only if $a_1 + a_2 = a_3$.

In the case under consideration, this means

$$\sum_{1\leq j<k\leq n_1} h_{jj}^{n+1}h_{kk}^{n+1} + \sum_{n_1+1\leq s<t\leq n} h_{ss}^{n+1}h_{tt}^{n+1} \geq \frac{\delta}{2} + \sum_{1\leq\alpha<\beta\leq n}\left(h_{\alpha\beta}^{n+1}\right)^2 + \frac{1}{2}\sum_{r=n+2}^{2m}\sum_{\alpha,\beta=1}^{n}\left(h_{\alpha\beta}^{r}\right)^2. \qquad (4.8)$$

Equality holds if and only if

$$\sum_{i=1}^{n_1} h_{ii}^{n+1} = \sum_{t=n_1+1}^{n} h_{tt}^{n+1} \qquad (4.9)$$

Using again the Gauss equation (2.3) and the formulas (4.3) and (2.7), we have

$$n_2\frac{\Delta_1 f_1}{f_1}+n_1\frac{\Delta_2 f_2}{f_2}=\tau-\sum_{1\le j<k\le n_1}K(e_j\wedge e_k)-\sum_{n_1+1\le s<t\le n}K(e_s\wedge e_t)$$

$$=\tau-\frac{n_1(n_1-1)(c+3s)}{8}-\sum_{r=n+1}^{2m}\sum_{1\le j<k\le n_1}\left(h_{jj}^r h_{kk}^r-(h_{jk}^r)^2\right)-$$

$$-\frac{n_2(n_2-1)(c+3s)}{8}-\sum_{r=n+1}^{2m}\sum_{n_1+1\le s<t\le n}\left(h_{ss}^r h_{tt}^r-(h_{st}^r)^2\right). \qquad (4.10)$$

Combining (4.8) and (4.10) and taking account of (4.4), we obtain

$$n_2\frac{\Delta_1 f_1}{f_1}+n_1\frac{\Delta_2 f_2}{f_2}\le\tau-\frac{n(n-1)(c+3s)}{8}+n_1n_2\frac{c+3s}{4}-$$

$$-\frac{\delta}{2}-\sum_{\substack{1\le j\le n_1\\ n_1+1\le t\le n}}(h_{jt}^{n+1})^2-\frac{1}{2}\sum_{r=n+2}^{2m}\sum_{\alpha,\beta=1}^{n}(h_{\alpha\beta}^r)^2+$$

$$+\sum_{r=n+2}^{2m}\sum_{1\le j<k\le n_1}\left((h_{jk}^r)^2-h_{jj}^r h_{kk}^r\right)+\sum_{r=n+2}^{2m}\sum_{n_1+1\le s<t\le n}\left((h_{st}^r)^2-h_{ss}^r h_{tt}^r\right)=$$

$$=\tau-\frac{n(n-1)(c+3s)}{8}+n_1n_2\frac{c+3s}{4}-\frac{\delta}{2}-\sum_{r=n+1}^{2m}\sum_{j=1}^{n_1}\sum_{t=n_1+1}^{n}(h_{jt}^r)^2-$$

$$-\frac{1}{2}\sum_{r=n+2}^{2m}\left(\sum_{j=1}^{n_1}h_{jj}^r\right)^2-\frac{1}{2}\sum_{r=n+2}^{2m}\left(\sum_{tn_1+1}^{n}h_{tt}^r\right)^2\le$$

$$\le\tau-\frac{n(n-1)(c+3s)}{8}+n_1n_2\frac{c+3s}{4}-\frac{\delta}{2}=\frac{n^2}{4}\|H\|^2+n_1n_2\frac{c+3s}{4},$$

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which implies the inequality (4.1).

We see that the equality sign of (4.11) holds if and only if

$$h_{jt}^{r}=0, 1\le j\le n_1, n_1+1\le t\le n,\ n+1\le r\le 2m \tag{4.12}$$

and

$$\sum_{i=1}^{n_1} h_{ii}^{r}=\sum_{t=n_1+1}^{n} h_{tt}^{r}=0,\ n+2\le r\le 2m. \tag{4.13}$$

Obviously (4.12) is equivalent to the mixed totally geodesicness of the doubly warped product ${}_{f_2}M_1\times_{f_1}M_2$ and (4.9) and (4.13) implies $n_1H_1=n_2H_2$.

The converse statement is straightforward.

Remark 4.2 *If either* $f_1\equiv 1$ *or* $f_2\equiv 1$, *then the inequality* (4.1) *is exactly the inequality* (4.15) *from* [7] *for warped products.*

Putting *s=1* in (4.1), we have the following

Corollary 4.3 [14] *Let x be a C-totally real isometric immersion of an n-dimensional doubly warped product* ${}_{f_2}M_1\times_{f_1}M_2$ *into a (2m+1)-dimensional Sasakian space form* $\widetilde{M}(c)$ *. Then*

$$n_2\frac{\Delta_1 f_1}{f_1}+n_1\frac{\Delta_2 f_2}{f_2}\le\frac{n^2}{4}\|H\|^2+n_1n_2\frac{c+3}{4}, \tag{4.14}$$

where $n_i=\dim M_i$ *, and* Δ_i *is the Laplacian operator of* M_i, $(i=1,2)$.

Moreover, the equality case of (4.14) *holds if and only if x is a mixed totally geodesic immersion and* $n_1H_1=n_2H_2$ *where* H_i, *i=1,2, are the partial mean curvature vectors.*

As an application, we obtain certain obstructions to the existence of minimal doubly warped product integral submanifolds in *S*-space forms.

By using the above theorem (Theorem 4.1), we can obtain some important consequences:

Corollary 4.4 *Let* ${}_{f_2}M_1 \times_{f_1} M_2$ *be a doubly warped product whose warping functions are harmonic. Then* ${}_{f_2}M_1 \times_{f_1} M_2$ *admits no minimal integral immersion into an S- space form* $\tilde{M}(c)$ *with c<-3s.*

Proof. Assume f_1 is a harmonic function on M_1, f_2 is a harmonic function on M_2 and ${}_{f_2}M_1 \times_{f_1} M_2$ admits a minimal integral immersion in an *S*-space form $\tilde{M}(c)$. Then, the inequality (4.1) becomes $c \geq -3s$.

Corollary 4.5 *If the warping functions* f_1 *and* f_2 *of a doubly warped product* ${}_{f_2}M_1 \times_{f_1} M_2$ *are eigenfunctions of the Laplacian on* M_1 *and* M_2*, respectively, with corresponding eigenvalues* $\lambda_1 > 0$ *and* $\lambda_2 > 0$*, respectively, then* ${}_{f_2}M_1 \times_{f_1} M_2$ *admits no minimal integral immersion into an S-space form* $\tilde{M}(c)$ *with* $c \leq -3s$.

Corollary 4.6 *Let* ${}_{f_2}M_1 \times_{f_1} M_2$ *be a doubly warped product. If one of the warping functions is harmonic and the other one is an eigenfunction of the Laplacian with corresponding eigenvalue* $\lambda > 0$*, then* ${}_{f_2}M_1 \times_{f_1} M_2$ *admits no minimal integral immersion into an S-space form* $\tilde{M}(c)$ *with* $c \leq -3s$.

Using s=1 in the above corollaries, we immediately get the following results from [14], [15]:

Corollary 4.7 [14], [15] *Let* ${}_{f_2}M_1 \times_{f_1} M_2$ *be a doubly warped product whose warping functions are harmonic. Then* ${}_{f_2}M_1 \times_{f_1} M_2$ *admits no minimal C-totally real immersion into a Sasakian space form* $\tilde{M}(c)$ *with c<-3.*

Corollary 4.8 [14], [15] *If the warping functions* f_1 *and* f_2 *of a doubly warped product* ${}_{f_2}M_1 \times_{f_1} M_2$ *are eigenfunctions of the Laplacian on* M_1 *and* M_2*, respectively, with corresponding eigenvalues* $\lambda_1 > 0$ *and* $\lambda_2 > 0$*, respectively, then* ${}_{f_2}M_1 \times_{f_1} M_2$ *admits no minimal C-totally real immersion in a Sasakian space form* $\tilde{M}(c)$ *with* $c \leq -3$.

Corollary 4.9 [14], [15] *Let* ${}_{f_2}M_1 \times_{f_1} M_2$ *be a doubly warped product. If one of the warping functions is harmonic and the other one is an eigenfunction of the Laplacian with corresponding eigenvalue* $\lambda > 0$*, then* ${}_{f_2}M_1 \times_{f_1} M_2$ *admits no minimal C-totally real immersion into a Sasakian space form* $\tilde{M}(c)$ *with* $c \leq -3$.

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