ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 2, p.167-179

ON THE FRÉCHET DIFFERENTIABILITY OF LUXEMBURG NORM IN THE SEQUENCE SPACES $l^{(p_n)}$ WITH VARIABLE EXPONENTS

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Abstract

It is shown that the Luxemburg norm in the sequence space $l^{(p_n)}$ with variable exponents is Fréchet - differentiable and a formula expressing the Fréchet derivative of this norm at any nonzero $x \in l^{(p_n)}$ is given.

1 Preliminaries

We consider the discrete analogue of generalized Lebesgue spaces $L^{p(\cdot)}(\Omega)$, namely the sequence spaces $l^{(p_n)}$ with variable exponents. In this section various definitions and basic properties related to the sequence spaces $l^{(p_n)}$ are given. Some interesting properties of these spaces are proved in [7], [5], and [8]. Also a discrete version of Hardy-Littlewood maximal operator on $l^{(p_n)}$ is studied in [9]. Also we mention that the Gâteaux and Fréchet - differentiability of the norm in the generalized Lebesgue spaces $L^{p(\cdot)}(\Omega)$ and corresponding Sobolev spaces was already studied in [3], [4] and [1].

Denote by $\mathcal{P}$ the set of all those real sequences $(p_n)_{n\in\mathbb{N}}$ which satisfy

$$1 < p^- := \inf_{n\in N} p_n \leq p^+ := \sup_{n\in N} p_n < \infty.$$

In this paper we fix $(p_n)_{n\in\mathbb{N}} \in \mathcal{P}$. If $x = (x_n)_{n\in\mathbb{N}}$, $x_n \in \mathbb{R}$ for any $n \in \mathbb{N}$, we define

$$\rho_{(p_n)}(x) := \sum_{n=0}^{\infty} |x_n|^{p_n}.$$

Since for $s > 1$ the function $t \geq 0$, $t \to t^s$, is convex, it follows that $x \longmapsto \rho_{(p_n)}(x)$ is convex and therefore $\rho_{(p_n)}$ is a convex modular in the sense of Musielack [7].

The space $l^{(p_n)}$ is defined as

$$l^{(p_n)} := \{x = (x_n)_{n\in\mathbb{N}}; \rho_{(p_n)}(x) < \infty\}.$$

[0]Mathematics Subject Classification (2010): 46B45, 46A80, 46B20
[0]Keywords: Banach sequence space, smooth space, Fréchet - differentiability of the norm

If $\lambda \in \mathbb{R}$ and $x, y \in l^{(p_n)}$, we get away

$$\begin{array}{c} \rho_{(p_n)}(x+y) \leq 2^{p^+-1}\left(\rho_{(p_n)}(x)+\rho_{(p_n)}(y)\right) \\ \rho_{(p_n)}(\lambda x) \leq \max\left(|\lambda|^{p^-}, |\lambda|^{p^+}\right)\rho_{(p_n)}(x) \end{array}, \tag{1}$$

therefore $l^{(p_n)}$ is a linear space.

The space $l^{(p_n)}$ is a Banach space with the *Luxemburg norm*:

$$\|x\|_{(p_n)} := \inf\{\lambda > 0; \rho_{(p_n)}(\lambda^{-1}x) \leq 1\}.$$

Obviously, if $p_0 = p_1 = \ldots = p_n = \ldots = p = const.$, then the space $l^{(p_n)}$ coincides with the classical sequence space l^p and the norms on these spaces are equal.

Remark 1 From definition we can deduce that

$$\rho_{(p_n)}\left(x/\|x\|_{(p_n)}\right) \leq 1$$

for any $x \in l^{(p_n)}$. Indeed, for any $k \in \mathbb{N}^*$ there exists λ_k, $\|x\|_{(p_n)} \leq \lambda_k \leq \|x\|_{(p_n)} + 1/k$, such that $\rho_{(p_n)}(x/\lambda_k) \leq 1$. Consequently

$$\sum_{n=0}^{m} \frac{|\ x_n\ |^{p_n}}{\left(\|x\|_{(p_n)} + \frac{1}{k}\right)^{p_n}} < \sum_{n=0}^{\infty} \frac{|\ x_n\ |^{p_n}}{\left(\|x\|_{(p_n)} + \frac{1}{k}\right)^{p_n}} < \rho_{(p_n)}(\frac{x}{\lambda_k}) \leq 1.$$

Then

$$\sum_{n=0}^{m} \frac{|\ x_n\ |^{p_n}}{\left(\|x\|_{(p_n)}\right)^{p_n}} = \lim_{k\to\infty} \sum_{n=0}^{m} \frac{|\ x_n\ |^{p_n}}{\left(\|x\|_{(p_n)} + \frac{1}{k}\right)^{p_n}} \leq 1.$$

Therefore

$$\rho_{(p_n)}\left(\frac{x}{\|x\|_{(p_n)}}\right) = \lim_{m\to\infty} \sum_{n=0}^{m} \frac{|\ x_n\ |^{p_n}}{\left(\|x\|_{(p_n)}\right)^{p_n}} \leq 1.$$

Proposition 2 *Considering the given $x \in l^{(p_n)}$, then $\rho_{(p_n)}(x) = 1$ if and only if $\|x\|_{(p_n)} = 1$.*

Proof. Let us suppose that $\rho_{(p_n)}(x) = 1$. By applying the definition of the $\|\cdot\|_{(p_n)}$-norm, the following inequality holds

$$1 = \rho_{(p_n)}(x) = \rho_{(p_n)}(x/1) \geq \|x\|_{(p_n)}.$$

If $\|x\|_{(p_n)} < 1$, then, by taking into account the convexity of $\rho_{(p_n)}$ and Remark 1, we have

$$\rho_{(p_n)}(x) \leq \|x\|_{(p_n)}\rho_{(p_n)}\left(x/\|x\|_{(p_n)}\right) < 1,$$

contradiction.

Reciprocally, if $\|x\|_{(p_n)} = 1$, we can write

$$\rho_{(p_n)}(x) = \rho_{(p_n)}(x/\|x\|_{(p_n)}) \leq 1.$$

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The strict inequality cannot hold. Indeed, if for some x with $\|x\|_{(p_n)} = 1$, we have $\rho_{(p_n)}(x) < 1$, then there exists $\varepsilon > 0$ such that $\rho_{(p_n)}(x) + \varepsilon < 1$. Since the function $x \longmapsto \rho_{(p_n)}(x)$ is convex and upper bounded if $\|x\|_{(p_n)} < 1$, it is therefore continuous, hence $\lim\limits_{\lambda \to 1^+} \rho_{(p_n)}(\lambda x) = \rho_{(p_n)}(x)$. Consequently, there exists $\delta > 0$, such that for each λ with $|\lambda - 1| < \delta$, we have $\left|\rho_{(p_n)}(\lambda x) - \rho_{(p_n)}(x)\right| < \varepsilon$. It results that, for $1 < \lambda < 1 + \delta$, $\rho_{(p_n)}(\lambda x) < \rho_{(p_n)}(x) + \varepsilon < 1$. Since $\rho_{(p_n)}(\lambda x) < 1$, we infer that $\|x\|_{(p_n)} < 1/\lambda < 1$, contradiction. ■

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Corollary 3 *Considering the given* $x \in l^{(p_n)}$. *If* $\|x\|_{(p_n)} < 1$, *then*

$$\|x\|_{(p_n)}^{p^+} \leq \rho_{(p_n)}(x) \leq \|x\|_{(p_n)}^{p^-} . \tag{2}$$

If $\|x\|_{(p_n)} > 1$, *then*

$$\|x\|_{(p_n)}^{p^-} \leq \rho_{(p_n)}(x) \leq \|x\|_{(p_n)}^{p^+} . \tag{3}$$

Proof. Since $p^- \leq p_n \leq p^+$, it follows that

$$\|x\|_{(p_n)}^{p^+} \leq \|x\|_{(p_n)}^{p_n} \leq \|x\|_{(p_n)}^{p^-} . \text{ for any } n \in \mathbb{N}.$$

Then, by using Proposition 2, we obtain

$$\rho_{(p_n)}(x) = \sum_{n=0}^{\infty} \|x\|_{(p_n)}^{p_n} \left(\frac{|\, x_n \,|}{\|x\|_{(p_n)}} \right)^{p_n} \leq \|x\|_{(p_n)}^{p^-} \rho_{(p_n)}\left(\frac{x}{\|x\|_{(p_n)}}\right) = \|x\|_{(p_n)}^{p^-} ,$$

that is the right inequality (2). Similarly one can establish the left inequality (2). If $\|x\|_{(p_n)} > 1$, the proof is the same. ■

A subset $A \subset l^{(p_n)}$ is called *mean bounded* if there exists a positive constant $C > 0$ such that $\rho_{(p_n)}(x) \leq C$ for any $x \in A$.

Remark 4 It follows from Corollary *3* that a set in $l^{(p_n)}$ is norm bounded if and only if it is mean bounded.

Corollary 5 *Let* x *and* $\left(x^{(k)}\right)$, $k = 1, 2, \ldots$ *be in* $l^{(p_n)}$. *Then the following statements are equivalent:*

(a) $\lim\limits_{k\to\infty} \left\|x^{(k)} - x\right\|_{(p_n)} = 0$;

(b) $\lim\limits_{k\to\infty} \rho_{(p_n)}(x^{(k)} - x) = 0$.

Proof. We use Corollary 3. ■

The spaces $l^{(p_n)}$ have various properties in common with their classical counterparts. We give an extension of Hölder's inequality. The *conjugate* $q = (q_n)_{n\in\mathbb{N}}$ of $p = (p_n)_{n\in\mathbb{N}} \in \mathcal{P}$ is defined by

$$p_n^{-1} + q_n^{-1} = 1, \, n \in \mathbb{N}.$$

Obviously, $p \in \mathcal{P}$ implies $q \in \mathcal{P}$.

Proposition 6 *If* $q = (q_n)_{n\in\mathbb{N}}$ *is the conjugate of* $p = (p_n)_{n\in\mathbb{N}} \in \mathcal{P}$, *then for all* $x = (x_n)_{n\in\mathbb{N}} \in l^{(p_n)}$ *and all* $y = (y_n)_{n\in\mathbb{N}} \in l^{(q_n)}$ *we have*

$$\sum_{n=0}^{\infty} |x_n y_n| \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right) \|x\|_{(p_n)} \|y\|_{(q_n)} . \tag{4}$$

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Proof. The inequality (4) is obvious if $\|x\|_{(p_n)} \|y\|_{(q_n)} = 0$. Suppose that $\|x\|_{(p_n)} \|y\|_{(q_n)} \neq 0$. In the inequality

$$ab \leq \frac{a^{p_n}}{p_n} + \frac{b^{q_n}}{q_n}$$

take $a = x_n / \|x\|_{(p_n)}$, $b = y_n / \|y\|_{(q_n)}$, add over n, and use Proposition 2. We obtain

$$\sum_{n=0}^{\infty} \frac{|x_n y_n|}{\|x\|_{(p_n)} \|y\|_{(q_n)}} \leq \frac{1}{p^-} + \frac{1}{q^-},$$

that is (4). ■

Proposition 7 *Let $x = (x_n)_{n\in\mathbb{N}}$ and $x^{(k)} = (x_n^{(k)})_{n\in\mathbb{N}}$, $k = 1, 2, \ldots$ be in $l^{(p_n)}$. Then $\lim_{k\to\infty} \left\|x^{(k)} - x\right\|_{(p_n)} = 0$ if and only if $\lim_{k\to\infty} \rho_{(p_n)}(x^{(k)}) = \rho_{(p_n)}(x)$ and $\lim_{k\to\infty} x_n^{(k)} = x_n$ for any $n \in \mathbb{N}$.*

Proof. Suppose that $\lim_{k\to\infty} \left\|x^{(k)} - x\right\|_{(p_n)} = 0$. For any $n \in \mathbb{N}$, there exists $0 < \theta_n < 1$ such that

$$\left(x_n^{(k)}\right)^{p_n} = x_n^{p_n} + p_n \left(x_n^{(k)} - x_n\right) \left(x_n + \theta_n \left(x_n^{(k)} - x_n\right)\right)^{p_n - 1},$$

therefore

$$\left|x_n^{(k)}\right|^{p_n} < |x_n|^{p_n} + p^+ \left|x_n^{(k)} - x_n\right| \left(|x_n| + \left|x_n^{(k)} - x_n\right|\right)^{p_n - 1}.$$

Consequently

$$\left|\rho_{(p_n)}\left(x^{(k)}\right) - \rho_{(p_n)}(x)\right| \leq p^+ \sum_{n=0}^{\infty} \left|x_n^{(k)} - x_n\right| \left(|x_n| + \left|x_n^{(k)} - x_n\right|\right)^{p_n - 1}$$

$$\leq p^+ \sum_{n=0}^{\infty} \left|x_n^{(k)} - x_n\right| \left(2|x_n| + \left|x_n^{(k)}\right|\right)^{p_n - 1}. \quad (5)$$

Denote $y^{(k)} := \left(\left(2|x_n| + \left|x_n^{(k)}\right|\right)^{p_n - 1}\right)_{n\in\mathbb{N}}$. Since $q_n(p_n - 1) = p_n$, we have

$$\rho_{(q_n)}\left(y^{(k)}\right) = \rho_{(p_n)}\left(z^{(k)}\right),$$

where $z^{(k)} := \left(2|x_n| + \left|x_n^{(k)}\right|\right)_{n\in\mathbb{N}}$. But $z^{(k)} \in l^{(p_n)}$, therefore $\rho_{(q_n)}\left(y^{(k)}\right) < \infty$, that is $y^{(k)} \in l^{(q_n)}$.

Taking into account the generalized Hölder inequality (4), from (5) we obtain:

$$\sum_{n=0}^{\infty} \left|x_n^{(k)} - x_n\right| \left(|x_n| + \left|x_n^{(k)} - x_n\right|\right)^{p_n - 1} \leq M \left\|x^{(k)} - x\right\|_{(p_n)} \left\|y^{(k)}\right\|_{(q_n)}, \quad (6)$$

where $M := \dfrac{1}{p^-} + \dfrac{1}{q^-}$.

On the other hand, it follows :167

$$\rho_{(p_n)}\left(z^{(k)}\right) \leq 2^{p^+-1}\left(2^{p^+}\rho_{(p_n)}(x) + \rho_{(p_n)}\left(x^{(k)}\right)\right).$$

According to Remark 4 the convergent sequence $\left(x^{(k)}\right)$ is mean bounded, therefore the sequence $\left(z^{(k)}\right)$ is mean bounded. and thus it is norm bounded (also Remark 4). Taking into account (5) and (6) it follows that

$$\lim_{k\to\infty}\rho_{(p_n)}(x^{(k)}) = \lim_{k\to\infty}\rho_{(p_n)}(x).$$

Also we have

$$\left|x_n^{(k)} - x_n\right|^{p_n} \leq \rho_{(p_n)}\left(x^{(k)} - x\right) \text{ for any } n \in \mathbb{N}.$$

Since $1 < p^- \leq p_n \leq p^+$, from $\lim_{k\to\infty}\rho_{(p_n)}\left(x^{(k)} - x\right) = 0$ it follows that $\lim_{k\to\infty} x_n^{(k)} = x_n$ for any $n \in \mathbb{N}$.

Reciprocally, let $(x^{(k)})_k$ be a sequence such that

$$\lim_{k\to\infty}\rho_{(p_n)}(x^{(k)}) = \rho_{(p_n)}(x) \text{ and } \lim_{k\to\infty} x_n^{(k)} = x_n \text{ for any } n \in \mathbb{N}. \tag{7}$$

We will show that $\lim_{k\to\infty}\rho_{(p_n)}(x^{(k)} - x) = 0$. Then, from Corollary 5 it follows that $\lim_{k\to\infty}\left\|x^{(k)} - x\right\|_{(p_n)} = 0$ and lemma is proved.

It suffices to show that there exists a subsequence $\left(x^{(j_k)}\right)_k \subset \left(x^{(k)}\right)_k$ such that $\lim_{k\to\infty}\rho_{(p_n)}(x^{(j_k)} - x) = 0$.

Indeed, taking into account (1), we obtain that

$$\rho_{(p_n)}(x^{(k)} - x) \leq 2^{p^+-1}\left(\rho_{(p_n)}(x^{(k)}) + \rho_{(p_n)}(x)\right).$$

Therefore, it follows from (7) that the sequence

$$b_k := \sum_{n=0}^{\infty}\left|x_n^{(k)} - x_n\right|^{p_n} = \rho_{(p_n)}(x^{(k)} - x),\ k \in \mathbb{N},$$

is bounded. Consequently

$$0 \leq \liminf b_k \leq L := \limsup b_k < \infty.$$

Assume that do not have $\lim_{k\to\infty}\rho_{(p_n)}(x^{(k)} - x) = 0$. Then $L > 0$. There exist an ε_0 and a subsequence $\left(x^{(l_k)}\right)_k \subset \left(x^{(k)}\right)_k$ such that

$$\rho_{(p_n)}(x^{(l_k)} - x) \geq \varepsilon_0. \tag{8}$$

For the subsequence $\left(x^{(l_k)}\right)_k$ (7) holds. Therefore there exists another subsequence $\left(x^{\left(j_{l_k}\right)}\right)_k \subset \left(x^{(l_k)}\right)_k$ such that $\lim_{k\to\infty}\rho_{(p_n)}(x^{\left(j_{l_k}\right)} - x) = 0$, which contradicts (8).

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Now, we have

$$\rho_{(p_n)}(x^{(k)} - x) = \sum_{n=0}^{\infty} \left| x_n^{(k)} - x_n \right|^{p_n} = \lim_{m\to\infty} s_m^{(k)},$$

where

$$s_m^{(k)} := \sum_{n=0}^{m} \left| x_n^{(k)} - x_n \right|^{p_n}.$$

Since $\lim_{k\to\infty} x_n^{(k)} = x_n$ for any $n \in \mathbb{N}$, we have

$$\lim_{k\to\infty} s_m^{(k)} = 0,$$

therefore

$$\lim_{m\to\infty} \left(\lim_{k\to\infty} s_m^{(k)} \right) = 0. \tag{9}$$

On the other hand, taking into account that $\lim_{k\to\infty} \rho_{(p_n)}(x^{(k)}) = \lim_{k\to\infty} \rho_{(p_n)}(x)$ and (1), it follows that

$$s_m^{(k)} \leq \sum_{n=0}^{\infty} \left| x_n^{(k)} - x_n \right|^{p_n} \leq 2^{p^+ - 1} \left(\rho_{(p_n)}(x^{(k)}) + \rho_{(p_n)}(x) \right) \text{ for any } m, k \in \mathbb{N}.$$

Consequently, the sequence $\left(s_m^{(k)} \right)_{m,k}$ is bounded. According to a classical result concerning the double sequences, there exists a convergent subsequence $\left(s_{p_m}^{(j_k)} \right)_{m,k}$ such that the iterated limits $\lim_{k\to\infty} \left(\lim_{m\to\infty} s_{p_m}^{(j_k)} \right)$ and $\lim_{m\to\infty} \left(\lim_{k\to\infty} s_{p_m}^{(j_k)} \right)$ exist, and are both equal to the double limit $\lim_{k,m\to\infty} s_{p_m}^{(j_k)}$. Taking into account (9) it follows that

$$\lim_{k\to\infty} \left(\lim_{m\to\infty} s_{p_m}^{(j_k)} \right) = 0$$

or

$$\lim_{k\to\infty} \sum_{n=0}^{\infty} \left| x_n^{(j_k)} - x_n \right|^{p_n} = 0.$$

Consequently

$$\lim_{k\to\infty} \rho_{(p_n)}(x^{(j_k)} - x) = 0.$$

The proof is complete. ■

2 On the Fréchet - differentiability of the norm in the sequence spaces $l^{(p_n)}$

First we will show that if $p^- > 1$, then $\left(l^{(p_n)}, \|x\|_{(p_n)} \right)$ is smooth, that is, given any nonzero element $x \in l^{(p_n)}$, there exists a unique support functional, i.e. there exists a unique element $x^*(x) \in (l^{(p_n)})^*$ for which $\langle x^*(x), x \rangle = \|x\|_{(p_n)}$ and $\|x^*(x)\|_{(l(p_n))^*} = 1$. According to Theorem 1 in Chapter 2 of [2], the proof of the smoothness of $\left(l^{(p_n)}, \|x\|_{(p_n)} \right)$ is demonstrated by equivalently showing that $\|\cdot\|_{(p_n)}$ is Gâteaux differentiable. Moreover, a formula giving expression of the derivative of the $\|\cdot\|_{(p_n)}$ - norm at any $x \neq 0$ is provided as $\|\cdot\|'_{(p_n)}(x)$.

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Theorem 8 ([6]) *If $p^- > 1$, then $\left(l^{(p_n)}, \|x\|_{(p_n)}\right)$ is smooth. At any $x = (x_0, x_1, \ldots, x_n, \ldots) \in l^{(p_n)}$, $x \neq 0$, the gradient of the norm,*

$$\|\cdot\|'_{(p_n)}(x) \in \left(l^{(p_n)}, \|\cdot\|_{(p_n)}\right)^*$$

is given by

$$\left\langle \|\cdot\|'_{(p_n)}(x), h \right\rangle = \frac{\sum\limits_{n=0}^{\infty} \dfrac{p_n |x_n|^{p_n-1} \operatorname{sgn}(x_n) h_n}{\|x\|_{(p_n)}^{p_n-1}}}{\sum\limits_{n=0}^{\infty} \dfrac{p_n |x_n|^{p_n}}{\|x\|_{(p_n)}^{p_n}}} \tag{10}$$

for any $h = (h_0, h_1, \ldots, h_n, \ldots) \in l^{(p_n)}$.

Proof. What we have to prove is that, for a given $x = (x_0, x_1, \ldots, x_n, \ldots) \in l^{(p_n)}$, $x \neq 0$, and any $h = (h_0, h_1, \ldots, h_n, \ldots) \in l^{(p_n)} \setminus \{0\}$, the function $t \in \mathbb{R}$, $t \longmapsto \|x + th\|_{(p_n)}$ is differentiable at $t = 0$. Since $l^{(p_n)} \setminus \{0\}$ is open, there exists $r > 0$ such that $B(x, r) \subset l^{(p_n)} \setminus \{0\}$. Consequently for any $t \in \left(-\frac{r}{\|h\|_{(p_n)}}, \frac{r}{\|h\|_{(p_n)}}\right)$, we have $u_0 + th \in B(x, r)$; therefore $x + th \neq 0$.

Let $k > 1$ be a fixed real number, let $\underline{k} := \min\left(1, \frac{r}{\|h\|_{(p_n)}}\right)$, $D := (-\underline{k}, \underline{k}) \times (\frac{1}{k}\|x\|_{(p_n)}, k\|x\|_{(p_n)})$, and let us consider the following series of functions:

$$\sum_{n=0}^{\infty} \frac{|x_n + th_n|^{p_n}}{\lambda^{p_n}}, \ (t, \lambda) \in D.$$

Since $|t| < 1$ and $\lambda > \frac{1}{k}\|x\|_{(p_n)}$, we can easily deduce that

$$\frac{|x_n + th_n|^{p_n}}{\lambda^{p_n}} \leq \frac{k^{p_n}(|x_n| + |h_n|)^{p_n}}{\|x\|_{(p_n)}^{p_n}} \leq \frac{k^{p^+}}{\min(\|x\|_{(p_n)}^{p^-}, \|x\|_{(p_n)}^{p^+})} (|x_n| + |h_n|)^{p_n}.$$

But $x, h \in l^{(p_n)}$, therefore

$$\sum_{n=0}^{\infty} (|x_n| + |h_n|)^{p_n} < \infty,$$

so, according to a classical result, the series of functions $\sum\limits_{n=0}^{\infty} \frac{|x_n + th_n|^{p_n}}{\lambda^{p_n}}$ is uniformly convergent on D . Consequently, the function $\phi : D \to \mathbb{R}$,

$$\phi(t, \lambda) := \rho_{(p_n)}\left(\frac{x + th}{\lambda}\right) - 1 = \sum_{n=0}^{\infty} \frac{|x_n + th_n|^{p_n}}{\lambda^{p_n}} - 1, \tag{11}$$

is well-defined. We will show that

$$\phi \in \mathcal{C}^1(D), \ \phi\left(0, \|x\|_{(p_n)}\right) = 0, \text{ and } \frac{\partial \phi}{\partial \lambda}(0, \|x\|_{(p_n)}) < 0.$$

Then on the basis of the implicit function theorem, we will obtain that there exist neighborhoods U of 0 and V of $\|x\|_{(p_n)}$such that $U \times V \subset D$ and a unique

$\mathcal{C}^1$-mapping $\lambda : U \to V$ which satisfies $\lambda(0) = \|x\|_{(p_n)}$, $\phi(t, \lambda(t)) = 0$, for any $t \in U$, and

$$\lambda'(t) = -\frac{\frac{\partial\phi}{\partial t}(t, \lambda(t))}{\frac{\partial\phi}{\partial\lambda}(t, \lambda(t))} \text{ for any } t \in U. \tag{12}$$

Taking into account the definition of ϕ (see (11)), $\phi(t, \lambda(t)) = 0$, for any $t \in U$, is equivalent to

$$\rho_{(p_n)}\left(\frac{x + th}{\lambda(t)}\right) = 1 \text{ for any } t \in U.$$

By applying Proposition 2, we deduce from this that

$$\lambda(t) = \|x + th\|_{(p_n)} \text{ for any } t \in U. \tag{13}$$

By combining (12) and (13) we derive, in particular, that $\lambda'(0)$ exists and

$$\lambda'(0) = \lim_{t\to 0} \frac{||x + th||_{(p_n)} - ||x||_{(p_n)}}{t} = -\frac{\dfrac{\partial\phi}{\partial t}(0, ||x||_{(p_n)})}{\dfrac{\partial\phi}{\partial\lambda}(0, ||x||_{(p_n)})}, \tag{14}$$

that is, the $\|\cdot\|_{(p_n)}$- norm is Gâteaux differentiable at x.

To complete the proof, we will prove that the above-defined statements (i), (ii) and (iii) concerning the function ϕ are true. In order to prove that $\phi \in \mathcal{C}^1(D)$, first we will show that we can compute $\dfrac{\partial\phi}{\partial\lambda}$ and $\dfrac{\partial\phi}{\partial t}$. Let us consider $f_n : D \to \mathbb{R}$, $n \in \mathbb{N}$, defined by

$$f_n(t, \lambda) := \frac{|x_n + th_n|^{p_n}}{\lambda^{p_n}}, \; (t, \lambda) \in D. \tag{15}$$

We can observe that the map $(t, \lambda) \in D$, $(t, \lambda) \mapsto f_n(t, \lambda)$, is a $\mathcal{C}^1$ - mapping. Indeed, applying a partial derivative on (15),

$$\frac{\partial f_n}{\partial t}(t, \lambda) = \frac{p_n |x_n + th_n|^{p_n-1} \operatorname{sgn}(x_n + th_n)h_n}{\lambda^{p_n}}, \tag{16}$$

$$\frac{\partial f_n}{\partial\lambda}(t, \lambda) = -\frac{p_n |x_n + th_n|^{p_n}}{\lambda^{p_n+1}} \text{ for any } (t, \lambda) \in D, \tag{17}$$

and, from (16) and (17), we can conclude that the mappings

$$(t, \lambda) \in D \longmapsto \frac{\partial f_n}{\partial t}(t, \lambda)$$

and

$$(t, \lambda) \in D \longmapsto \frac{\partial f_n}{\partial\lambda}(t, \lambda)$$

are continuous.

First, we estimate $\left|\dfrac{\partial f_n}{\partial t}\right|$. Let $(t, \lambda) \in D$. Since $|t| < 1$ and $\lambda > \frac{1}{k}\|x\|_{(p_n)}$, one easily follows that

$$\left|p_n |x_n + th_n|^{p_n-1} \operatorname{sgn}(x_n + th_n)h_n\right| \le p^+ (|\underline{x}_n| + |h_n|)^{p_n}$$

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and

$$\lambda^{p_n} > k^{-p_n} \|x\|_{(p_n)}^{p_n} \geq k^{-p^-} \min(\|x\|_{(p_n)}^{p^-}, \|x\|_{(p_n)}^{p^+}.$$

Consequently, according to (16), one has

$$\left|\frac{\partial f_n}{\partial t}(t,\lambda)\right| < c\,(|x_n| + |h_n|)^{p_n},$$

with $c := p^+ k^{-p^+} / \min(\|x\|_{(p_n)}^{p^-}, \|x\|_{(p_n)}^{p^+})$.

Similarly,

$$\left|\frac{\partial f_n}{\partial t}(t,\lambda)\right| < c_1^{p_n}\,(|x_n| + |h_n|^{p_n}),$$

with $c_1 := p^+ k^{-p^+ +1} / \min(\|x\|_{(p_n)}^{p^- +1}, \|x\|_{(p_n)}^{p^+ +1})$. According to a well - known classical result, the mapping ϕ defined by (11) is a $\mathcal{C}^1$ - mapping and

$$\frac{\partial \phi}{\partial t}(t,\lambda) = \sum_{n=0}^{\infty} \frac{p_n\,|x_n + th_n|^{p_n - 1} \operatorname{sgn}(x_n + th_n) h_n}{\lambda^{p_n}}, \tag{18}$$

$$\frac{\partial \phi}{\partial \lambda}(t,\lambda) = -\sum_{n=0}^{\infty} \frac{p_n\,|x_n + th_n|^{p_n}}{\lambda^{p_n+1}}. \tag{19}$$

The claims (ii) and (iii) are obviously validated. Indeed, by applying Proposition 2,

$$\phi(0, \|x\|_{(p_n)}) = \sum_{n=0}^{\infty} \frac{|x_n|^{p_n}}{\|x\|_{(p_n)}^{p_n}} - 1 = 0.$$

Finally, according to Proposition 2 again

$$\frac{\partial \phi}{\partial \lambda}(0, \|x\|_{(p_n)}) = -\sum_{n=0}^{\infty} \frac{p_n\,|x_n|^{p_n}}{\|x\|_{(p_n)}^{p_n+1}} \leq \frac{-p^-}{\|x\|_{(p_n)}} \sum_{n=0}^{\infty} \frac{|x_n|^{p_n}}{\|x\|_{(p_n)}^{p_n}} = -\frac{p^-}{\|x\|_{(p_n)}} < 0.$$

Clearly, formula (10) is a direct consequence of (14), (18) and (19). ■

Theorem 9 *The norm $\|\cdot\|_{(p_n)}$ is Fréchet-differentiable at any nonzero $x \in l^{(p_n)}$ and the Fréchet-differential of this norm at any nonzero $x \in l^{(p_n)}$ is given for any $h \in l^{(p_n)}$ by (10).*

Proof. We prove that the map

$$x \in l^{(p_n)} \setminus \{0\} \mapsto \|x\|'_{(p_n)}$$

is continuous. The Fréchet - differentiability of the map $x \in l^{(p_n)} \setminus \{0\} \mapsto \|x\|_{(p_n)}$ will then follows. Let $x = (x_0, x_1, \ldots, x_n, \ldots)$ be in $l^{(p_n)} \setminus \{0\}$. Let $\varphi : l^{(p_n)} \setminus \{0\} \to \left(l^{(p_n)} \setminus \{0\}\right)^*$ be defined by

$$\langle \varphi(x), h \rangle := \sum_{n=0}^{\infty} p_n \frac{|x_n|^{p_n - 1} \operatorname{sgn}(x_n)}{\|x\|_{(p_n)}^{p_n - 1}} h_n \text{ for each } h \in l^{(p_n)},$$

$h=(h_0,h_1,\ldots,h_n,\ldots)$ and let $q:l^{(p_n)}\setminus\{0\}\to\mathbb{R}$ be defined by

$$q(x):=\sum_{n=0}^{\infty}p_n\frac{|x_n|^{p_n}}{\|x\|_{(p_n)}^{p_n}}.$$

Since

$$\left\langle \|\cdot\|'_{(p_n)}(x),\cdot\right\rangle=\frac{\langle\varphi(x),\cdot\rangle}{q(x)},\text{ for all } x\in l^{(p_n)}\setminus\{0\},$$

it is sufficient to prove that φ and q are continuous.

Fix $x=(x_0,x_1,\ldots,x_n,\ldots)\in l^{(p_n)}\setminus\{0\}$ and let $\left(x^{(k)}\right)_k\subset l^{(p_n)}\setminus\{0\}$, $x^{(k)}=(x_0^{(k)},x_1^{(k)},\ldots,x_n^{(k)},\ldots)$ be such that $x^{(k)}\to x$ as $k\to\infty$ in the space $\left(l^{(p_n)},\|\cdot\|_{(p_n)}\right)$. It suffices to show that there exists a subsequence $\left(x^{(j_k)}\right)_k\subset\left(x^{(k)}\right)_k$ such that $\varphi(x^{(j_k)})\to\varphi(x)$ and $q(x^{(j_k)})\to q(x)$ as $k\to\infty$. We begin with the map q.

We have

$$\left|q\left(x^{(k)}\right)-q\left(x\right)\right|\le p^+\sum_{n=0}^{\infty}\left|\frac{\left|x_n^{(k)}\right|^{p_n}}{\left\|x^{(k)}\right\|_{(p_n)}^{p_n}}-\frac{|x_n|^{p_n}}{\|x\|_{(p_n)}^{p_n}}\right|.$$

Denote

$$s_m^{(k)}:=\sum_{n=0}^{m}\left|\frac{\left|x_n^{(k)}\right|^{p_n}}{\left\|x^{(k)}\right\|_{(p_n)}^{p_n}}-\frac{|x_n|^{p_n}}{\|x\|_{(p_n)}^{p_n}}\right|.$$

We will show that there exists a subsequence $\left(x^{(j_k)}\right)_k\subset\left(x^{(k)}\right)_k$ such that

$$\sum_{n=0}^{\infty}\left|\frac{\left|x_n^{(j_k)}\right|^{p_n}}{\left\|x^{(j_k)}\right\|_{(p_n)}^{p_n}}-\frac{|x_n|^{p_n}}{\|x\|_{(p_n)}^{p_n}}\right|\to 0\text{ as } k\to\infty. \tag{20}$$

Since $x^{(k)}\to x$ as $k\to\infty$ in $l^{(p_n)}$, according to Proposition 7, we infer that

$$x_n^{(k)}\to x_n\text{ as } k\to\infty,\text{ for any } n\in\mathbb{N}. \tag{21}$$

Hence, for any $n\in\mathbb{N}$, we have

$$\frac{\left|x_n^{(k)}\right|^{p_n}}{\left\|x^{(k)}\right\|_{(p_n)}^{p_n}}\to\frac{|x_n|^{p_n}}{\|x\|_{(p_n)}^{p_n}}\text{ as } k\to\infty. \tag{22}$$

Consequently

$$\lim_{m\to\infty}\left(\lim_{k\to\infty}s_m^{(k)}\right)=0. \tag{23}$$

On the other hand

$$s_m^{(k)}\le\sum_{n=0}^{m}\left(\frac{\left|x_n^{(k)}\right|^{p_n}}{\left\|x^{(k)}\right\|_{(p_n)}^{p_n}}+\frac{|x_n|^{p_n}}{\|x\|_{(p_n)}^{p_n}}\right)\le\rho_{(p_n)}\left(\frac{x^{(k)}}{\left\|x^{(k)}\right\|_{(p_n)}}\right)+\rho_{(p_n)}\left(\frac{x}{\|x\|_{(p_n)}}\right).$$

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Taking into account (22) and Remark 4, it follows that the sequence $\left(\frac{x^{(k)}}{\|x^{(k)}\|_{(p_n)}} \right)_k$ is mean bounded, therefore the double sequence $\left(s_m^{(k)} \right)_{m,k}$ is bounded. According to a classical result, there exists a convergent subsequence $\left(s_{p_m}^{(j_k)} \right)_{m,k}$ such that the iterated limits $\lim_{k\to\infty} \left(\lim_{m\to\infty} s_{p_m}^{(j_k)} \right)$ and $\lim_{m\to\infty} \left(\lim_{k\to\infty} s_{p_m}^{(j_k)} \right)$ exist, and are both equal to the double limit $\lim_{k,m\to\infty} s_{p_m}^{(j_k)}$. Taking into account (23) it follows that

$$\lim_{k\to\infty} \left(\lim_{m\to\infty} s_{p_m}^{(j_k)} \right) = 0$$

or

$$\lim_{k\to\infty} \sum_{n=0}^{\infty} \left| \frac{\left| x_n^{(j_k)} \right|^{p_n}}{\left\| x^{(j_k)} \right\|_{(p_n)}^{p_n}} - \frac{|x_n|^{p_n}}{\|x\|_{(p_n)}^{p_n}} \right| = 0$$

that is (20).

We now show that there exists a subsequence of $\left(x^{(j_k)} \right)_k$, still denoted by $\left(x^{(j_k)} \right)_k$ for convenience, such that

$$\varphi(x^{(j_k)}) \to \varphi(x) \text{ in } \left(l^{(p_n)} \setminus \{0\} \right)^* \text{ as } k \to \infty.$$

But

$$\left\langle \varphi(x^{(j_k)}) - \varphi(x), h \right\rangle = \sum_{n=0}^{\infty} p_n y_n^{(j_k)} h_n, \tag{24}$$

where

$$y_n^{(j_k)} := \frac{\left| x_n^{(j_k)} \right|^{p_n - 1} \operatorname{sgn} x_n^{(j_k)}}{\left\| x^{(j_k)} \right\|_{(p_n)}^{p_n - 1}} - \frac{|x_n|^{p_n - 1} \operatorname{sgn} x_n}{\|x\|_{(p_n)}^{p_n - 1}}, \, n \in \mathbb{N}.$$

Clearly, for any $x = (x_n)_{n\in\mathbb{N}} \in l^{(p_n)} \setminus \{0\}$, the sequence

$$z := \left(\frac{|x_n|^{p_n - 1} \operatorname{sgn} x_n}{\|x\|_{(p_n)}^{p_n - 1}} \right)_{n\in\mathbb{N}} \in l^{(q_n)} \tag{25}$$

because of

$$\left| \frac{|x_n|^{p_n - 1} \operatorname{sgn} x_n}{\|x\|_{(p_n)}^{p_n - 1}} \right| = \left(\frac{|x_n|}{\|x\|_{(p_n)}} \right)^{p_n - 1},$$

and similarly, for any $k \in \mathbb{N}$,

$$z^{(j_k)} := \left(\frac{\left| x_n^{(j_k)} \right|^{p_n - 1} \operatorname{sgn} x_n^{(j_k)}}{\left\| x^{(j_k)} \right\|_{(p_n)}^{p_n - 1}} \right)_{n\in\mathbb{N}} \in l^{(q_n)}. \tag{26}$$

Then $y^{(j_k)} := \left(y_n^{(j_k)} \right)_{n\in\mathbb{N}} \in l^{(q_n)}$. But $h \in l^{(p_n)}$. Therefore, taking (24) and (4) into account , we obtain

$$\left| \left\langle \varphi(x^{(j_k)}) \to \varphi(x), h \right\rangle \right| \le p^+ \sum_{n=0}^{\infty} \left| y_n^{(j_k)} \right| |h_n| \le M \left\| y^{(j_k)} \right\|_{(q_n)} \|h\|_{(p_n)},$$

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where $M = p^+ \left(\frac{1}{p^-} + \frac{1}{(q)^-} \right)$.

Consequently,

$$\left\| \varphi(x^{(j_k)}) \to \varphi(x) \right\| \leq M \left\| y^{(j_k)} \right\|_{(q_n)}. \tag{27}$$

Now, it is clear that, for proving the continuity of φ, it suffices to show that

$$\left\| y^{(j_k)} \right\|_{(q_n)} \to 0 \text{ as } k \to \infty. \tag{28}$$

According to Proposition 7, (28) may be equivalently written as

$$\lim_{k\to\infty} \rho_{(q_n)} \left(z^{(j_k)} \right) = \rho_{(q_n)}(z) \tag{29}$$

and

$$\lim_{k\to\infty} z_n^{(j_k)} = z_n \text{ for any } n \in \mathbb{N}, \tag{30}$$

where $z^{(j_k)}$ and z are given by (25) and (26) respectively.

But

$$\rho_{(q_n)} \left(z^{(j_k)} \right) = \rho_{(p_n)} \left(\frac{x^{(j_k)}}{\left\| x^{(j_k)} \right\|_{(p_n)}} \right) = 1,$$

$$\rho_{(q_n)} (z) = \rho_{(p_n)} \left(\frac{x}{\| x \|_{(p_n)}} \right) = 1,$$

so that (29) holds.

Also (30) is a direct consequence of the fact that

$$\frac{x^{(j_k)}}{\left\| x^{(j_k)} \right\|_{(p_n)}} \to \frac{x}{\| x \|_{(p_n)}} \text{ as } k \to \infty.$$

Hence we conclude that

$$\left\| \varphi(x^{(j_k)}) - \varphi(x) \right\| \to 0 \text{ as } k \to \infty.$$

■

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