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IMPROVED CHEN'S INEQUALITIES FOR LAGRANGIAN SUBMANIFOLDS IN QUATERNIONIC SPACE FORMS

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Abstract. Riemannian invariants (in particular Chen invariants) play an important role in the theory of submanifolds. They are very useful in providing relationships between the extrinsic and intrinsic invariants of a submanifold. On the other hand, Lagrangian submanifolds are one of the most studied, with important roles in other fields. In this paper, an improved inequality for the Chen invariant δ_M and an inequality for the invariant $\delta(n_1, \ldots, n_k)$, both in the case of a Lagrangian submanifold in a quaternionic space form, by using an optimization method, are obtained.

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1. Preliminaries

Let $\tilde{M}$ be a differentiable manifold and we assume that there is a rank 3-subbundle σ of $\mathrm{End}(T\tilde{M})$ such that a local basis $\{J_1, J_2, J_3\}$ exists on sections of σ satisfying for all $\alpha \in \{1, 2, 3\}$

$$J_\alpha^2 = -\,\mathrm{Id}, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2}, \tag{1.1}$$

where Id denotes the identity field of type $(1, 1)$ on M and the indices are taken from $\{1, 2, 3\}$ modulo 3. The bundle σ is called an *almost quaternionic structure* on M and $\{J_1, J_2, J_3\}$ is called a canonical basis of σ. $(\tilde{M}, \sigma)$ is said to be an *almost quaternionic manifold*. It's easy to see that any almost quaternionic manifold is of dimension $4m,\ m \geq 1$.

A Riemannian metric $\tilde{g}$ on $\tilde{M}$ is said to be *adapted to the almost quaternionic structure* σ if it satisfies

$$\tilde{g}(J_\alpha X, J_\alpha Y) = \tilde{g}(X, Y), \quad \forall \alpha \in \{1, 2, 3\}, \tag{1.2}$$

for all vector fields $X,\ Y$ on $\tilde{M}$ and any canonical basis $\{J_1, J_2, J_3\}$ on σ. $(\tilde{M}, \sigma, \tilde{g})$ is said to be an *almost quaternionic Hermitian manifold.*

$(\tilde{M}, \sigma, \tilde{g})$ is said to be a *quaternionic Kaehler manifold* [9] if the bundle σ is parallel with respect to the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{g}$, i.e., locally defined 1-forms $\omega_1,\ \omega_2,\ \omega_3$ exist such that we have

$$\tilde{\nabla}_X J_\alpha = \omega_{\alpha+2}(X) J_{\alpha+1} - \omega_{\alpha+1}(X) J_{\alpha+2}, \tag{1.3}$$

for all $\alpha \in \{1, 2, 3\}$ and for any vector field X on $\tilde{M}$, where the indices are taken from $\{1, 2, 3\}$ modulo 3.

Let $(\tilde{M}, \sigma, \tilde{g})$ be a quaternionic Kaehler manifold and let X be a non-null vector on $\tilde{M}$. The 4-plane spanned by $\{X, J_1X, J_2X, J_3X\}$ is called a *quaternionic plane* and is denoted by $Q(X)$. Any 2-plane in $Q(X)$ is called a *quaternionic plane.* The sectional curvature of a quaternionic plane is called a *quaternionic sectional curvature.* A quaternionic Kaehler manifold is a *quaternionic space form* if its quaternionic sectional curvature are equal to a constant, say c, i.e., its curvature tensor is given by

(1.4)
$$\tilde{R}(X,Y)Z = \frac{c}{4}\left\{\tilde{g}(Z,Y)X - \tilde{g}(X,Z)Y + \sum_{\alpha=1}^{3}[\tilde{g}(Z,J_\alpha Y)J_\alpha X - \tilde{g}(Z,J_\alpha X)J_\alpha Y + 2\tilde{g}(X,J_\alpha y)J_\alpha Z]\right\},$$

for all vector fields $X,\ Y,\ Z$ on $\tilde{M}$ and any local basis $\{J_1, J_2, J_3\}$ on σ.

For a submanifold M of a quaternionic Kaehler manifold $(\tilde{M}, \sigma, \tilde{g})$ we denote by g the metric tensor induced on M. If ∇ is the covariant differentiation induced on M, the Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{1.5}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{1.6}$$

for $X, Y \in \Gamma(TM),\ N \in \Gamma(TM^{\perp})$, where h is the second fundamental form of M, $\nabla^{\perp}$ is the connection on the normal bundle and A_N is the shape operator of M with respect to N. The relation between the second fundamental form h and shape operator A_N is

$$\tilde{g}(h(X,Y),N) = g(A_N X, Y), \tag{1.7}$$

for all $X,\ Y \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$.

For $\{e_1, \dots, e_n\}$ an orthonormal basis of T_pM and $\{e_{n+1}, \dots, e_{4m}\}$ an orthonormal basis of $T_p^{\perp}M$, where $p \in M$, we denote by H the mean curvature vector, i.e.,

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i).$$

Also,

$$h_{ij}^r = g(h(e_i, e_j), e_r),\ i,j \in \overline{1,n},\ r \in \overline{n+1, 4m}$$

and

$$||h||^2(p) = \sum_{i,j=1}^{n} g(h(e_i,e_j), h(e_i,e_j)).$$

Let $f : M \to \tilde{M}(c)$ an isometric immersion of the n dimensional Riemannian manifold M in the $4n$ dimensional quaternionic space form $\tilde{M}(c)$. M is said to be a *Lagrangian submanifold* if

$$J_\alpha(T_pM) \subset T_p^{\perp}M, \forall p \in M, \forall \alpha \in \{1,2,3\}.$$

We choose an orthonormal frame field in $\tilde{M}(c)$

$$\{e_1, e_2, \dots, e_n;\quad e_{\phi_1(1)} = J_1(e_1), \dots,\ e_{\phi_1(n)} = J_1(e_n);$$
$$e_{\phi_2(1)} = J_2(e_1), \dots,\ e_{\phi_2(n)} = J_2(e_n);\quad e_{\phi_3(1)} = J_3(e_1), \dots,\ e_{\phi_3(n)} = J_3(e_n)\},$$

such that, restricted to M, $e_1, e_2, \dots, e_n$ are tangent to M.

We set

$$h_{ij}^{\alpha} = g(h(e_i,e_j), e_\alpha),\quad \alpha \in \{\phi_1(1), \dots, \phi_1(n), \phi_2(1), \dots, \phi_2(n), \phi_3(1), \dots, \phi_3(n)\}$$

and then, for any $r = 1, 2, 3$, we have (see (2.9.) in [7])

$$h_{ij}^{\phi_r(k)} = h_{ki}^{\phi_r(j)} = h_{jk}^{\phi_r(i)}. \tag{1.8}$$

Let (M,g) be a Riemannian submanifold of a Riemannian manifold $(\tilde{M}, \tilde{g})$ and $f \in \mathcal{C}^\infty(\tilde{M})$. We attach the following optimum problem

$$\min_{x \in M} f(x). \tag{1.9}$$

We recall the following result.

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Theorem 1.1. [10] *If $x_0 \in M$ is a solution of the problem 1.9, then*

a) $(\mathrm{grad})(x_0) \in T^{\perp}_{x_0}M$;

b) The bilinear form $\alpha : T_{x_0}M \times T_{x_0}M \to \mathbb{R}$, $\alpha(X,Y) = \mathrm{Hess}_f(X,Y) + \tilde{g}(h(X,Y), (\mathrm{grad})(x_0))$ *is semipositive definite, where h is the second fundamental form of the submanifold M in $\tilde{M}$.*

2. Chen Invariants

In this section we recall the basic definitions and standard notations (see, for example, [3]).

Let M be an n dimensional Riemannian manifold and $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM$, $p \in M$. For any orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of the tangent space T_pM, the scalar curvature τ at p is defined by

$$\tau(p) = \sum_{1 \le i < j \le 1} K(e_i \wedge e_j). \tag{2.1}$$

One denotes (see [2])

$$(\inf K)(p) = \inf\{K(\pi) | \pi \subset T_pM, \ \dim \pi = 2\}$$

and the *Chen first invariant* is defined by

$$\delta_M(p) = \tau(p) - (\inf K)(p). \tag{2.2}$$

If L is a subspace of T_pM of dimension $r \ge 2$ and $\{e_1, \ldots, e_r\}$ an orthonormal basis of L, the scalar curvature $\tau(L)$ of the r-plane section L is defined by

$$\tau(L) = \sum_{1 \le \alpha < \beta \le r} K(e_\alpha \wedge e_\beta).$$

For given integers $n \ge 3$ and $k \ge 1$, denote by $S(n,k)$ the finite set of all $k-$tuples $(n_1, \ n_2, \ \ldots, \ n_k)$ of integers satisfying

$$2 \le n_1, \ n_2, \ \ldots, \ n_k < n \text{ and } n_1 + \cdots + n_k \le n.$$

Let $S(n)$ be the union $\bigcup_{k \ge 1} S(n,k)$.

For each $(n_1, \ n_2, \ \ldots, \ n_k) \in S(n)$ and each point $p \in M$, B.-Y. Chen introduced a Riemannian invariant

$$\delta(n_1, \ \ldots, \ n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \ldots + \tau(L_k)\},$$

where $L_1, \ \ldots, \ L_k$ run over all k mutually ortoghonal subspaces of T_pM such that $\dim L_j = n_j$, $j = 1, \ldots, k$.

3. Improved δ_M-Inequality for Lagrangian submanifolds in Quaternionic Space Forms

B.-Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken [6] showed that every totally real submanifold M of a real dimension n in a complex space form $\tilde{M}(c)$ of real dimension $2m$ satisfies Chen's inequality

$$\delta_M \le \frac{n-2}{n} \left\{ \frac{n^2}{n-1} \|H\|^2 + (n+1)\frac{c}{4} \right\}.$$

J. Bolton et al. [1] established an improved inequality for this invariant, in the case of a Lagrangian submanifold in a complex space form:

$$\delta_M \le \frac{(n-2)(n+1)}{2} \frac{c}{4} + \frac{n^2}{2} \frac{2n-3}{2n+3} \|H\|^2$$

(see also Oprea [10]).

In this section we prove a similar inequality for Lagrangian submanifold of a quaternionic space form.

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Theorem 3.1. *Let M be an $n-$dimensional Lagrangian submanifold of a quaternionic space form $\tilde{M}(c)$. Then, we have*

$$\delta_M \leq \frac{(n-2)(n+1)}{2}\cdot\frac{c}{4}+\frac{n^2}{2}\cdot\frac{2n-3}{2n+3}\cdot||H||^2. \tag{3.1}$$

The equality sign holds at a point $p \in M$ if and only if there is an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ at p such that with respect to this basis the second fundamental form h satisfies the conditions

$$h_{ij}^{\phi_r(k)}=0\ ,\ i=\overline{1,n}\ ,\ j=\overline{3,n}\ ,\ i\neq j,\ k\in\{1,\dots,n\}\diagdown\{i,j\}.$$

Proof. From the Gauss equation we have

$$R(X,Y,Z,W)=\frac{c}{4}\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\}+$$
$$+g(h(X,W),h(Y,Z))-g(h(X,Z),h(Y,W)),$$

for any X, Y, Z, W tangent to M.

We get

$$\tau=\frac{c}{4}\cdot\frac{n(n-1)}{2}+\frac{n^2}{2}||H||^2-\frac{1}{2}||h||^2\ . \tag{3.2}$$

Let e_1, e_2 be tangent to M. Then

$$\tilde{R}(e_1,e_2,e_1,e_2)=R(e_1,e_2,e_1,e_2)-\sum_{r=1}^{3}\sum_{k=1}^{n}\left[h_{11}^{\phi_r(k)}h_{22}^{\phi_r(k)}-\left(h_{12}^{\phi_r(k)}\right)^2\right].$$

From this, we obtain

$$K(e_1\wedge e_2)=\frac{c}{4}+\sum_{r=1}^{3}\sum_{k=1}^{n}\left[h_{11}^{\phi_r(k)}h_{22}^{\phi_r(k)}-\left(h_{12}^{\phi_r(k)}\right)^2\right]. \tag{3.3}$$

$$\delta_M=\tau-K(e_1\wedge e_2)=\frac{c}{4}\left\{\frac{n(n-1)}{2}-1\right\}+\frac{n^2||H||^2}{2}-\frac{||h||^2}{2}-\sum_{r=1}^{3}\sum_{k=1}^{n}\left[h_{11}^{\phi_r(k)}h_{22}^{\phi_r(k)}-(h_{12}^{\phi_r(k)})^2\right],$$

equivalent with

$$\delta_M=\frac{c}{4}\left(\frac{n^2-n-2}{2}\right)+\sum_{r=1}^{3}\sum_{k=1}^{n}\sum_{1\leq i<j\leq n}\left[h_{ii}^{\phi_r(k)}h_{jj}^{\phi_r(k)}-(h_{ij}^{\phi_r(k)})^2\right]-$$
$$-\sum_{r=1}^{3}\sum_{k=1}^{n}\left[h_{11}^{\phi_r(k)}h_{22}^{\phi_r(k)}-(h_{12}^{\phi_r(k)})^2\right].$$

It follows that

$$\delta_M=\frac{(n+1)(n-2)}{2}\frac{c}{4}+\sum_{r=1}^{3}\sum_{k=1}^{n}\left[\sum_{j=3}^{n}\left(h_{11}^{\phi_r(k)}+h_{22}^{\phi_r(k)}\right)h_{jj}^{\phi_r(k)}+\sum_{3\leq i<j\leq n}h_{ii}^{\phi_r(k)}h_{jj}^{\phi_r(k)}\right]-$$
$$-\sum_{r=1}^{3}\sum_{k=1}^{n}\left[\sum_{j=3}^{n}\left(h_{1j}^{\phi_r(k)}\right)^2+\left(h_{2j}^{\phi_r(k)}\right)^2\right]-\sum_{r=1}^{3}\sum_{k=1}^{n}\sum_{3\leq i<j\leq n}\left(h_{ij}^{\phi_r(k)}\right)^2,$$

and then

$$\delta_M=\frac{(n+1)(n-2)}{2}\frac{c}{4}+\sum_{r=1}^{3}\sum_{k=1}^{n}\left[\sum_{j=3}^{n}\left(h_{11}^{\phi_r(k)}+h_{22}^{\phi_r(k)}\right)h_{jj}^{\phi_r(k)}+\sum_{3\leq i<j\leq n}h_{ii}^{\phi_r(k)}h_{jj}^{\phi_r(k)}\right]-$$

$$-\sum_{r=1}^{3}\left\{\sum_{j=3}^{n}\left[\left(h_{1j}^{\phi_r(1)}\right)^2+\left(h_{2j}^{\phi_r(2)}\right)^2+\left(h_{1j}^{\phi_r(j)}\right)^2+\left(h_{2j}^{\phi_r(j)}\right)^2\right]+\right.$$

$$\left.+\sum_{1\le k\le n}^{k\ne 1}\sum_{3\le j\le n}^{k\ne j}\left(h_{1j}^{\phi_r(k)}\right)^2+\sum_{1\le k\le n}^{k\ne 2}\sum_{3\le j\le n}^{k\ne j}\left(h_{2j}^{\phi_r(k)}\right)^2\right\}-$$

$$-\sum_{r=1}^{3}\left[\sum_{3\le i<j\le n}\left(h_{ij}^{\phi_r(i)}\right)^2+\sum_{3\le i<j\le n}\left(h_{ij}^{\phi_r(j)}\right)^2+\sum_{k=1}^{n}\sum_{3\le i<j\le n}^{k\ne i,j}\left(h_{ij}^{\phi_r(k)}\right)^2\right].$$

This implies

$$\delta_M=\frac{(n+1)(n-2)}{2}\frac{c}{4}+\sum_{r=1}^{3}\sum_{k=1}^{n}\left[\sum_{j=3}^{n}\left(h_{11}^{\phi_r(k)}+h_{22}^{\phi_r(k)}\right)h_{jj}^{\phi_r(k)}+\sum_{3\le i<j\le n}h_{ii}^{\phi_r(k)}h_{jj}^{\phi_r(k)}\right]-$$

$$-\sum_{r=1}^{3}\left\{\sum_{j=3}^{n}\left[\left(h_{11}^{\phi_r(j)}\right)^2+\left(h_{22}^{\phi_r(j)}\right)^2+\left(h_{jj}^{\phi_r(1)}\right)^2+\left(h_{jj}^{\phi_r(2)}\right)^2\right]+\right.$$

$$\left.+\sum_{1\le k\le n}^{k\ne 1}\sum_{3\le j\le n}^{k\ne j}\left(h_{1j}^{\phi_r(k)}\right)^2+\sum_{1\le k\le n}^{k\ne 2}\sum_{3\le j\le n}^{k\ne j}\left(h_{2j}^{\phi_r(k)}\right)^2\right\}-$$

$$-\sum_{r=1}^{3}\left\{\sum_{3\le i<j\le n}\left[\left(h_{ii}^{\phi_r(j)}\right)^2+\left(h_{jj}^{\phi_r(i)}\right)^2\right]+\sum_{k=1}^{n}\sum_{3\le i<j\le n}^{k\ne i,j}\left(h_{ij}^{\phi_r(k)}\right)^2\right\}.$$

From the previous relation, it follows that

$$\delta_M\le\frac{(n+1)(n-2)}{2}\frac{c}{4}+\sum_{r=1}^{3}\sum_{k=1}^{n}\left[\sum_{j=3}^{n}\left(h_{11}^{\phi_r(k)}+h_{22}^{\phi_r(k)}\right)h_{jj}^{\phi_r(k)}+\sum_{3\le i<j\le n}h_{ii}^{\phi_r(k)}h_{jj}^{\phi_r(k)}\right]-\tag{3.4}$$

$$-\sum_{r=1}^{3}\sum_{j=3}^{n}\left[\left(h_{11}^{\phi_r(j)}\right)^2+\left(h_{22}^{\phi_r(j)}\right)^2+\left(h_{jj}^{\phi_r(1)}\right)^2+\left(h_{jj}^{\phi_r(2)}\right)^2\right]-$$

$$-\sum_{r=1}\sum_{3\le i<j\le n}\left[\left(h_{ii}^{\phi_r(j)}\right)^2+\left(h_{jj}^{\phi_r(i)}\right)^2\right].$$

For $r\in\{1,2,3\}$, let us consider the quadratic forms $f_{\phi_r(k)}:\mathbb{R}^n\to\mathbb{R},\ k=\overline{1,n}$ defined by

$$f_{\phi_r(1)}(h_{11}^{\phi_r(1)},h_{22}^{\phi_r(1)},\dots,h_{nn}^{\phi_r(1)})=\sum_{j=3}^{n}\left(h_{11}^{\phi_r(1)}+h_{22}^{\phi_r(1)}\right)h_{jj}^{\phi_r(1)}+\tag{3.5}$$

$$+\sum_{3\le i<j\le n}h_{ii}^{\phi_r(1)}h_{jj}^{\phi_r(1)}-\sum_{j=3}^{n}\left(h_{jj}^{\phi_r(1)}\right)^2,$$

$$f_{\phi_r(2)}(h_{11}^{\phi_r(2)},h_{22}^{\phi_r(2)},\dots,h_{nn}^{\phi_r(2)})=\sum_{j=3}^{n}\left(h_{11}^{\phi_r(2)}+h_{22}^{\phi_r(2)}\right)h_{jj}^{\phi_r(2)}+\tag{3.6}$$

$$+\sum_{3\le i<j\le n}h_{ii}^{\phi_r(2)}h_{jj}^{\phi_r(2)}-\sum_{j=3}^{n}\left(h_{jj}^{\phi_r(2)}\right)^2,$$

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$$f_{\phi_r(k)}(h_{11}^{\phi_r(k)}, h_{22}^{\phi_r(k)}, \ldots, h_{nn}^{\phi_r(k)}) = \sum_{j=3}^{n} \left(h_{11}^{\phi_r(k)} + h_{22}^{\phi_r(k)}\right) h_{jj}^{\phi_r(k)} + \tag{3.7}$$

$$+ \sum_{3 \le i < j \le n} h_{ii}^{\phi_r(k)} h_{jj}^{\phi_r(k)} - \left(h_{11}^{\phi_r(k)}\right)^2 - \left(h_{22}^{\phi_r(k)}\right)^2 - \sum_{3 \le i \le n}^{i \neq k} \left(h_{ii}^{\phi_r(k)}\right)^2,$$

where

$$k = \overline{3, n},\ r = 1, 2, 3,\ \phi_1 = I,\ \phi_2 = J,\ \phi_3 = K.$$

For $r \in \{1, 2, 3\}$, we must find an upper bound for $f_{\phi_r(1)}$, subject to

$$P : h_{11}^{\phi_r(1)} + h_{22}^{\phi_r(1)} + \ldots + h_{nn}^{\phi_r(1)} = c^{\phi_r(1)}, \tag{3.8}$$

where $c^{\phi_r(1)}$ is a real number.

The bilinear form $\alpha : T_qP \times T_qP \to \mathbb{R}$ has the expression

$$\alpha(X, Y) = \mathrm{Hess}(f_r)(X, Y) + \langle h'(X, Y), \operatorname{grad} f_r(q) \rangle,$$

where h' is the second fundamental form of P in $\mathbb{R}^n$ and $\langle\ ,\ \rangle$ is the standard inner-product on $\mathbb{R}^n$.

Searching for the partial derivatives of the function $f_{\phi_r(1)}$, we get

$$\frac{\partial f_{\phi_r(1)}}{\partial h_{11}^{\phi_r(1)}} = \sum_{j=3}^{n} h_{jj}^{\phi_r(1)},$$

$$\frac{\partial f_{\phi_r(1)}}{\partial h_{22}^{\phi_r(1)}} = \sum_{j=3}^{n} h_{jj}^{\phi_r(1)},$$

$$\frac{\partial f_{\phi_r(1)}}{\partial h_{tt}^{\phi_r(1)}} = h_{11}^{\phi_r(1)} + h_{22}^{\phi_r(1)} + \sum_{3 \le j \le n}^{t \neq j} h_{jj}^{\phi_r(1)} - 2h_{tt}^{\phi_r(1)},\ t = \overline{3, n}.$$

In the standard frame of $\mathbb{R}^n$, the Hessian of $f_{\phi_r(1)}$ has the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & \ldots & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & -2 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & -2 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & -2 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & -2 & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ 1 & 1 & 1 & 1 & 1 & 1 & \ldots & -2 \end{pmatrix}.$$

As P is totally geodesic in $\mathbb{R}^n$ (P a hyperplane; $h' = 0$), we get

$$\alpha(X, X) = \sum_{j=3}^{n} \left(X_{\phi_r(1)} + X_{\phi_r(2)}\right) X_{\phi_r(j)} + \sum_{j=3}^{n} \sum_{1 \le k \le n}^{k \neq j} X_{\phi_r(j)} X_{\phi_r(k)} - 2 \sum_{j=3}^{n} \left(X_{\phi_r(j)}\right)^2 =$$

$$= \left(\sum_{j=1}^{n} X_{\phi_r(j)}\right)^2 - 2X_{\phi_r(1)} X_{\phi_r(2)} - \left(X_{\phi_r(1)}\right)^2 - \left(X_{\phi_r(2)}\right)^2 - 3 \sum_{j=3}^{n} \left(X_{\phi_r(j)}\right)^2 =$$

$$= -\left(X_{\phi_r(1)} + X_{\phi_r(2)}\right)^2 - 3 \sum_{j=3}^{n} \left(X_{\phi_r(j)}\right)^2 \le 0,$$

so the Hessian of $f_{\phi_r(1)}$ is negative semidefinite.

Searching for the critical points $h_{11}^{\phi_r(1)}, h_{22}^{\phi_r(1)}, \ldots, h_{nn}^{\phi_r(1)}$ of $f_{\phi_r(1)}$, we find

$$h_{33}^{\phi_r(1)} = h_{44}^{\phi_r(1)} = \ldots = h_{nn}^{\phi_r(1)} = \lambda,$$

$$\frac{\partial f_{\phi_r(1)}}{\partial h_{22}^{\phi_r(1)}} = \frac{\partial f_{\phi_r(1)}}{\partial h_{33}^{\phi_r(1)}}.$$

It follows

$$(n-2)\lambda = h_{11}^{\phi_r(1)} + h_{22}^{\phi_r(1)} + (n-3)\lambda - 2\lambda,$$

which implies

$$h_{11}^{\phi_r(1)} + h_{22}^{\phi_r(1)} = 3\lambda.$$

From (3.5) we obtain

$$3\lambda + (n-2)\lambda = c^{\phi_r(1)}.$$

Then

$$\lambda = \frac{c^{\phi_r(1)}}{n+1}$$

and

$$h_{11}^{\phi_r(1)} + h_{22}^{\phi_r(1)} = \frac{3c^{\phi_r(1)}}{n+1},$$

$$h_{33}^{\phi_r(1)} = h_{44}^{\phi_r(1)} = \ldots = h_{nn}^{\phi_r(1)} = \frac{c^{\phi_r(1)}}{n+1}.$$

Again, from (3.5), we find

$$f_{\phi_r(1)} \leq \frac{3c^{\phi_r(1)}}{n+1}(n-2)\frac{c^{\phi_r(1)}}{n+1} + C_{n-2}^2\left(\frac{c^{\phi_r(1)}}{n+1}\right)^2 - (n-2)\left(\frac{c^{\phi_r(1)}}{n+1}\right)^2 =$$

$$= \frac{3(n-2)}{(n+1)^2}\cdot\left(c^{\phi_r(1)}\right)^2 + \frac{(n-2)(n-3)}{2(n+1)^2}\cdot\left(c^{\phi_r(1)}\right)^2 - \frac{n-2}{(n+1)^2}\cdot\left(c^{\phi_r(1)}\right)^2 =$$

$$= \frac{1}{2(n+1)^2}\cdot\left(c^{\phi_r(1)}\right)^2\cdot[6(n-2) + (n-2)(n-3) - 2(n-2)] =$$

$$= \frac{1}{2(n+1)^2}\cdot\left(c^{\phi_r(1)}\right)^2\cdot(n+1)(n-2),$$

which implies

$$f_{\phi_r(1)} \leq \frac{n^2}{2}\cdot\frac{n-2}{n+1}\cdot\left(H^{\phi_r(1)}\right)^2. \tag{3.9}$$

In a similar manner, we find for $f_{\phi_r(2)}$

$$f_{\phi_r(2)} \leq \frac{n^2}{2}\cdot\frac{n-2}{n+1}\cdot\left(H^{\phi_r(2)}\right)^2. \tag{3.10}$$

Next, we must find an upper bound for $f_{\phi_r(k)}$, $k = \overline{3,n}$, $r \in \{1,2,3\}$, subject to

$$P : h_{11}^{\phi_r(k)} + h_{22}^{\phi_r(k)} + \ldots + h_{nn}^{\phi_r(k)} = c^{\phi_r(k)}. \tag{3.11}$$

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For $k = 3$ we have

$$f_{\phi_r(3)} = \left(h_{11}^{\phi_r(3)} + h_{22}^{\phi_r(3)}\right)\sum_{j=3}^{n} h_{jj}^{\phi_r(3)} + \sum_{3\le i<j\le n} h_{ii}^{\phi_r(3)} h_{jj}^{\phi_r(3)} - \tag{3.12}$$

$$- \left(h_{11}^{\phi_r(3)}\right)^2 - \left(h_{22}^{\phi_r(3)}\right)^2 - \sum_{3\le i\le n}^{i\neq 3} \left(h_{ii}^{\phi_r(3)}\right)^2.$$

We calculate the partial derivatives of $f_{\phi_r(3)}$:

$$\frac{\partial f_{\phi_r(3)}}{\partial h_{11}^{\phi_r(3)}} = \sum_{j=3}^{n} h_{jj}^{\phi_r(3)} - 2h_{11}^{\phi_r(3)},$$

$$\frac{\partial f_{\phi_r(3)}}{\partial h_{22}^{\phi_r(3)}} = \sum_{j=3}^{n} h_{jj}^{\phi_r(3)} - 2h_{22}^{\phi_r(3)},$$

$$\frac{\partial f_{\phi_r(3)}}{\partial h_{33}^{\phi_r(3)}} = h_{11}^{\phi_r(3)} + h_{22}^{\phi_r(3)} + \sum_{3\le j\le n}^{j\neq 3} h_{jj}^{\phi_r(3)},$$

$$\frac{\partial f_{\phi_r(3)}}{\partial h_{tt}^{\phi_r(3)}} = h_{11}^{\phi_r(3)} + h_{22}^{\phi_r(3)} + \sum_{3\le j\le n}^{j\neq t} h_{jj}^{\phi_r(3)} - 2h_{tt}^{\phi_r(3)},\ t = \overline{4,n}.$$

From the above relations, we find

$$h_{11}^{\phi_r(3)} = h_{22}^{\phi_r(3)} = 3\lambda,$$
$$h_{33}^{\phi_r(3)} = 4h_{11}^{\phi_r(3)} = 12\lambda,$$
$$h_{44}^{\phi_r(3)} = \ldots = h_{nn}^{\phi_r(3)} = 4\lambda,$$

so these relations and (3.8) implies

$$6\lambda + 12\lambda + (n-3)\cdot 4\lambda = c^{\phi_r(3)},$$

equivalent with

$$\lambda = \frac{c^{\phi_r(3)}}{2(2n+3)}.$$

Then

$$h_{11}^{\phi_r(3)} = h_{22}^{\phi_r(3)} = \frac{3c^{\phi_r(3)}}{2(2n+3)}, \tag{3.13}$$

$$h_{33}^{\phi_r(3)} = 4h_{11}^{\phi_r(3)} = \frac{6c^{\phi_r(3)}}{2n+3}, \tag{3.14}$$

$$h_{44}^{\phi_r(3)} = \ldots = h_{nn}^{\phi_r(3)} = \frac{2c^{\phi_r(3)}}{2n+3}. \tag{3.15}$$

In the standard frame of $\mathbb{R}^n$, the Hessian of $f_{\phi_r(3)}$ has the matrix

$$\begin{pmatrix} -2 & 0 & 1 & 1 & 1 & 1 & \ldots & 1 \\ 0 & -2 & 1 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & -2 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & -2 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & 1 & -2 & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & & & \\ 1 & 1 & 1 & 1 & 1 & 1 & \ldots & -2 \end{pmatrix}.$$

As P is totally geodesic in $\mathbb{R}^n$, we get

$$\alpha(X,X) = -2\sum_{1\le j\le n}^{j\ne 3}(X_j)^2 + 2\sum_{j=3}^{n}(X_1+X_2)X_j + 2\sum_{3\le i<j\le n} X_iX_j =$$

$$= \left(\sum_{j=1}^{n} X_j\right)^2 - 2X_1X_2 - \sum_{j=1}^{n}(X_j)^2 - 2\sum_{1\le j\le n}^{j\ne 3}(X_i)^2 =$$

$$= \left(\sum_{j=1}^{n} X_j\right)^2 - (X_1+X_2)^2 - (X_3)^2 - 2(X_1)^2 - 2(X_2)^2 - 3\sum_{j=4}^{n}(X_j)^2 < 0,$$

so the Hessian of $f_{\phi_r(3)}$ is negative semidefinite.

From (3.12), (3.13), (3.14) and (3.15), we get

$$f_{\phi_r(3)} \le \frac{3c^{\phi_r(3)}}{2n+3}\cdot\left[\frac{6c^{\phi_r(3)}}{2n+3} + (n-3)\cdot\frac{2c^{\phi_r(3)}}{2n+3}\right] +$$

$$+\frac{6c^{\phi_r(3)}}{2n+3}\cdot(n-3)\frac{2c^{\phi_r(3)}}{2n+3} + C_{n-3}^2\cdot\left(\frac{2c^{\phi_r(3)}}{2n+3}\right)^2 - 2\cdot\left[\frac{3c^{\phi_r(3)}}{2(2n+3)}\right]^2 - (n-3)\left(\frac{2c^{\phi_r(3)}}{2n+3}\right)^2 =$$

$$= \frac{18(c^{\phi_r(3)})^2}{(2n+3)^2} + \frac{6(n-3)}{(2n+3)^2}\cdot(c^{\phi_r(3)})^2 + \frac{12(n-3)}{(2n+3)^2}\cdot(c^{\phi_r(3)})^2 + \frac{(n-3)(n-4)}{2}\cdot\frac{4(c^{\phi_r(3)})^2}{(2n+3)^2} -$$

$$-2\cdot\frac{9(c^{\phi_r(3)})^2}{4(2n+3)^2} - \frac{4(n-3)(c^{\phi_r(3)})^2}{(2n+3)^2} =$$

$$= \frac{(c^{\phi_r(3)})^2}{2(2n+3)^2}\left[36 + 12(n-3) + 24(n-3) + 4(n-3)(n-4) - 9 - 8(n-3)\right] =$$

$$= \frac{(c^{\phi_r(3)})^2}{2(2n+3)^2}(4n^2-9) = \frac{(c^{\phi_r(3)})^2}{2(2n+3)^2}(2n-3)(2n+3),$$

so we obtain

$$f_{\phi_r(3)} \le \frac{2n-3}{2(2n+3)}(c^{\phi_r(3)})^2.$$

From previous relation and (3.11), we find that

$$f_{\phi_r(3)} \le \frac{n^2}{2}\cdot\frac{2n-3}{2n+3}\cdot(H^{\phi_r(3)})^2. \tag{3.16}$$

In a similar manner, $\forall r\in\{1,2,3\}$, $\forall k\ge 3$, we get

$$f_{\phi_r(k)} \le \frac{n^2}{2}\cdot\frac{2n-3}{2n+3}\cdot(H^{\phi_r(k)})^2. \tag{3.17}$$

Since $\frac{n-2}{n+1} < \frac{2n-3}{2n+3}$, then we have

$$\delta_M \le \frac{(n+1)(n-2)}{2}\cdot\frac{c}{4} + \frac{n^2}{2}\cdot\frac{2n-3}{2n+3}\cdot||H||^2,$$

which is an improvement of the result of Y. Hong, and C.S. Houh [8].

□

4. $\delta(n_1, \ldots, n_k)$-Inequality for Lagrangian submanifold of a quaternionic space form

B.-Y. Chen established the following inequalities for Chen invariant $\delta(n_1, \ldots, n_k)$ of Lagrangian submanifolds in complex space forms.

Theorem 4.1. [4] *Let M be a Lagrangian submanifold of a complex space form $\tilde{M}(c)$. For a given k-tuple $(n_1, n_2, \ldots, n_k) \in S(n)$, we put $N = n_1 + n_2 + \ldots + n_k$ and $Q = \sum_{i=1}^{k}(2+n_i)^{-1}$. If $Q \le \dfrac{1}{3}$ and $N < n$, we have*

$$\delta(n_1, n_2, \ldots, n_k) \le \frac{n^2\{n - N + 3k - 1 - 6Q\}}{2\{n - N + 3k - 1 - 6Q\}}\|H\|^2 + \tag{4.1}$$

$$+\frac{1}{2}\left\{n(n-1) - \sum_{i=1}^{k} n_i(n_1 - 1)\right\}\frac{c}{4}.$$

The equality sign holds at a point $p \in M$ if and only if there is an orthonormal basis $\{e_1, \ldots, e_n\}$ at p such that with respect to this basis the second funfamental form h takes the following form

$$h(e_{\alpha_i}, e_{\beta_i}) = \sum_{\gamma_i \in \Delta_i} h^{\gamma_i}_{\alpha_i\beta_i} Je_{\gamma_i} + \frac{3\delta_{\alpha_i\beta_i}}{2+n_i}\lambda Je_{N+1},$$

$$h(e_{\alpha_i}, e_{\alpha_j}) = 0, \quad \sum_{\alpha_i \in \Delta_i} h^{\gamma_i}_{\alpha_i\alpha_i} = 0,$$

$$h(e_{\alpha_i}, e_{N+1}) = \frac{3\lambda}{2+n_i} Je_{\alpha_i}, \ \ h(e_{\alpha_i}, e_u) = 0,$$

$$h(e_{N+1}, e_{N+1}) = 3\lambda Je_{N+1}, \ \ h(e_{N+1}, e_u) = \lambda Je_u,$$

$$h(e_u, e_v) = \lambda\delta_{uv} Je_{N+1},$$

for distinct $i, j \in \{1, \ldots, k\}, u, v \in \{N+2, \ldots, n\}$ and $\lambda = \frac{1}{3}h^{N+1}_{N+1N+1}$.

Theorem 4.2. [5] *Let M be a Lagrangian submanifold of a complex space form $\tilde{M}(c)$. For a given k-tuple $(n_1, n_2, \ldots, n_k) \in S(n)$, we put $N = n_1 + n_2 + \ldots + n_k$ and $Q = \sum_{i=1}^{k}(2+n_i)^{-1}$. If $Q > \frac{1}{3}$ and $N < n$, we have*

$$\delta(n_1, n_2, \ldots, n_k) \le \frac{n^2(n - N + 3k - 3)}{2(n - N + 3k)}\|H\|^2 + \tag{4.2}$$

$$+\frac{1}{2}\left\{n(n-1) - \sum_{i=1}^{k} n_i(n_1 - 1)\right\}\frac{c}{4}.$$

The equality sign holds at a point $p \in M$ if and only if there is an orthonormal basis $\{e_1, \ldots, e_n\}$ at p such that

$$h(e_{\alpha_i}, e_{\beta_i}) = \sum_{\gamma_i \in \Delta_i} h^{\gamma_i}_{\alpha_i\beta_i} Je_{\gamma_i}, \quad \sum_{\alpha_i \in \Delta_i} h^{\gamma_i}_{\alpha_i\alpha_i} = 0,$$

$$h(e_A, e_B) = 0 \ \textit{otherwise},$$

for $\alpha_i, \beta_i, \gamma_i \in \Delta_i, i \in \{1, \ldots, k\}$, and $A, B, C \in \{1, \ldots, n\}$.

By using the method of constrained maximum, we obtain a similiar inequality in the case of Lagrangian submanifolds in quaternionic space forms in the next Theorem.

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Theorem 4.3. *Let M be an $n-$dimensional Lagrangian submanifold of a quaternionic space form $\tilde{M}(c)$. For a given $k-$tuple $(n_1, n_2, \ldots, n_k) \in S(n)$, we put $N = n_1 + n_2 + \ldots + n_k$ and $Q = \sum_{i=1}^{k}(2+n_i)^{-1}$. If $N < n$ then we have*

a) if $Q \le \dfrac{1}{3}$,

$$\delta(n_1, n_2, \ldots, n_k) \le \frac{n^2\{n - N + 3k - 1 - 6Q\}}{2\{n - N + 3k + 2 - 6Q\}}\|H\|^2 + \\ + \frac{1}{2}\left\{n(n-1) - \sum_{i=1}^{k} n_i(n_i - 1)\right\}\frac{c}{4}. \tag{4.3}$$

The equality sign holds at a point $p \in M$ if and only if there is an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ at p such that with respect to this basis the second fundamental form h takes the following form

$$h(e_{\alpha_i}, e_{\beta_i}) = \sum_{r=1}^{3}\left(\sum_{\gamma_i \in \Delta_i} h_{\alpha_i\beta_i}^{\phi_r(\gamma_i)}\phi_r e_{\gamma_i} + \frac{3\delta_{\alpha_i\beta_i}}{2+n_i}\lambda\phi_r(e_{N+1})\right), \ \alpha_i, \beta_i \in \Delta_i, \ i = \overline{1,k},$$

$$h(e_{\alpha_i}, e_{\alpha_j}) = 0, \ \sum_{\alpha_i \in \Delta_i} h_{\alpha_i\alpha_i}^{\phi_r(\gamma_i)} = 0, \ r = \overline{1,3}, \ \alpha_i \in \Delta_i, \ \alpha_j \in \Delta_j, \ i \ne j, \ i, j \in \{1, 2, \ldots, k\},$$

$$h(e_{\alpha_i}, e_{N+1}) = \frac{3\lambda}{2+n_i}\sum_{r=1}^{3}\phi_r(e_{\alpha_i}), \ h(e_{\alpha_i}, e_u) = 0, \ u \in \{N+2, \ldots, n\},$$

$$h(e_{N+1}, e_{N+1}) = 3\lambda\sum_{r=1}^{3}\phi_r(e_{N+1}),$$

$$h(e_{N+1}, e_u) = \lambda\sum_{r=1}^{3}\phi_r(e_u), \ u \in \{N+2, \ldots, n\},$$

$$h(e_u, e_v) = \lambda\delta_{uv}\sum_{r=1}^{3}\phi_r(e_{N+1}), \ u, v \in \{N+2, \ldots, n\},$$

for $\lambda = \dfrac{1}{3}h_{e_{N+1}e_{N+1}}^{N+1}$.

b) if $Q > \dfrac{1}{3}$,

$$\delta(n_1, n_2, \ldots, n_k) \le \frac{n^2\{n - N + 3k - 3\}}{2\{n - N + 3k\}}\|H\|^2 + \\ + \frac{1}{2}\left\{n(n-1) - \sum_{i=1}^{k} n_i(n_i - 1)\right\}\frac{c}{4}. \tag{4.4}$$

The equality sign holds at a point $p \in M$ if and only if there is an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ at p such that with respect to this basis the second fundamental form h takes the following form

$$h(e_{\alpha_i}, e_{\beta_i}) = \sum_{r=1}^{3}\sum_{\gamma_i \in \Delta_i} h_{\alpha_i\beta_i}^{\phi_r(\gamma_i)}\phi_r(e_{\gamma_i}),$$

$$\sum_{r=1}^{3}\sum_{\alpha_i \in \Delta_i} h_{\alpha_i\alpha_i}^{\phi_r(\gamma_i)}\phi_r(e_{\gamma_i}) = 0,$$

$$h(e_A, e_B) = 0 \quad \textit{otherwise},$$

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for $\alpha_i,\ \beta_i,\ \gamma_i \in \Delta_i,\ i = \overline{1,k},\ A,\ B,\ C = \overline{1,n}.$

Proof. Let $\{e_1, e_2, \dots, e_n\}$ an orthonormal basis in $p \in M$, $(n_1, n_2, \dots, n_k) \in S(n)$ and $L_1, L_2, \dots, L_k$ be k mutual orthogonal subspaces of T_pM, $\dim L_j = n_j$, $j = \overline{1,k}$, $L_j = \text{Span}\{e_{n_1+\dots+n_{j-1}+1}, \dots, e_{n_1+\dots+n_j}\}$, $j = \overline{1,k}$.

We choose

$$\Delta_1 = \{1, \dots, n_1\},$$
$$\Delta_2 = \{n_1+1, n_1+2, \dots, n_1+n_2\},$$
$$\dots$$
$$\Delta_k = \{n_1+\dots+n_{k-1}+1, \dots, n_1+\dots+n_k\},$$
$$\Delta_{k+1} = \{n_1+\dots+n_k+1, \dots, n\},$$

and

$$N = n_1 + \dots + n_k.$$

By using Gauss equation, we have

$$\tau = \sum_{r=1}^{3}\sum_{k=1}^{n}\sum_{1\le A<B\le n}\left[h_{AA}^{\phi_r(k)}h_{BB}^{\phi_r(k)} - \left(h_{AB}^{\phi_r(k)}\right)^2\right] + \frac{n(n-1)}{2}\frac{c}{4},$$

$$\tau(L_i) = \sum_{r=1}^{3}\sum_{k=1}^{n}\sum_{A,B\in\Delta_i}^{A<B}\left[h_{AA}^{\phi_r(k)}h_{BB}^{\phi_r(k)} - \left(h_{AB}^{\phi_r(k)}\right)^2\right] + \frac{n_i(n_i-1)}{2}\frac{c}{4},\ i = \overline{1,k}.$$

Using the following convention concerning indices

$$\alpha_i, \beta_i, \gamma_i \in \Delta_i,\ i, j \in \{1, \dots, k\},$$
$$r, s, t \in \Delta_{k+1},\ u, v \in \{N+2, \dots, n\},$$
$$A, B, C \in \{1, \dots, n\},$$

we get

$$\tau = \sum_{r=1}^{3}\sum_{A=1}^{n}\sum_{B<C}\left[h_{BB}^{\phi_r(A)}h_{CC}^{\phi_r(A)} - \left(h_{BC}^{\phi_r(A)}\right)^2\right] + \frac{n(n-1)}{2}\frac{c}{4}, \tag{4.5}$$

$$\tau(L_i) = \sum_{r=1}^{3}\sum_{A=1}^{n}\sum_{\alpha_i<\beta_i}\left[h_{\alpha_i\alpha_i}^{\phi_r(A)}h_{\beta_i\beta_i}^{\phi_r(A)} - \left(h_{\alpha_i\beta_i}^{\phi_r(A)}\right)^2\right] + \frac{n_i(n_i-1)}{2}\frac{c}{4},\ i = \overline{1,k}. \tag{4.6}$$

From the relations (4.5) and (4.6), we obtain

$$\tau - \sum_{i=1}^{k}\tau(L_i) = \sum_{r=1}^{3}\sum_{A=1}^{n}\left[\sum_{1\le B<C\le n}h_{BB}^{\phi_r(A)}h_{CC}^{\phi_r(A)} - \sum_{\alpha_i<\beta_i, i=\overline{1,k}}^{\alpha_i,\beta_i\in\Delta_i}h_{\alpha_i\alpha_i}^{\phi_r(A)}h_{\beta_i\beta_i}^{\phi_r(A)}\right] -$$

$$-\sum_{r=1}^{3}\sum_{A=1}^{n}\left[\sum_{1\le B<C\le n}\left(h_{BC}^{\phi_r(A)}\right)^2 - \sum_{\alpha_i<\beta_i, i=\overline{1,k}}^{\alpha_i,\beta_i\in\Delta_i}\left(h_{\alpha_i\beta_i}^{\phi_r(A)}\right)^2\right] + \frac{1}{2}\left[n(n-1) - \sum_{i=1}^{k}n_i(n_i-1)\right]\frac{c}{4} =$$

$$= \sum_{r=1}^{3}\sum_{A=1}^{n}\left[\sum_{t,s\in\Delta_{k+1}}^{t<s}h_{tt}^{\phi_r(A)}h_{ss}^{\phi_r(A)} + \sum_{\alpha_i\in\Delta_i,\alpha_j\in\Delta_j}^{1\le i<j\le k}h_{\alpha_i\alpha_i}^{\phi_r(A)}h_{\alpha_j\alpha_j}^{\phi_r(A)} + \sum_{\alpha_i\in\Delta_i,t\in\Delta_{k+1}}^{1\le i\le k}h_{\alpha_i\alpha_i}^{\phi_r(A)}h_{tt}^{\phi_r(A)}\right] -$$

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$$-\sum_{r=1}^{3}\sum_{A=1}^{n}\left[\sum_{t,s\in\Delta_{k+1}}^{t<s}\left(h_{ts}^{\phi_r(A)}\right)^2+\sum_{\alpha_i\in\Delta_i,\alpha_j\in\Delta_j}^{1\le i<j\le k}\left(h_{\alpha_i\alpha_j}^{\phi_r(A)}\right)^2+\sum_{\alpha_i\in\Delta_i,t\in\Delta_{k+1}}^{1\le i\le k}\left(h_{\alpha_i t}^{\phi_r(A)}\right)^2\right]+$$

$$+\frac{1}{2}\left[n(n-1)-\sum_{i=1}^{k}n_i(n_i-1)\right]\frac{c}{4}\le$$

$$\le\sum_{r=1}^{3}\sum_{A=1}^{n}\left[\sum_{t,s\in\Delta_{k+1}}^{t<s}h_{tt}^{\phi_r(A)}h_{ss}^{\phi_r(A)}+\sum_{\alpha_i\in\Delta_i,\alpha_j\in\Delta_j}^{1\le i<j\le k}h_{\alpha_i\alpha_i}^{\phi_r(A)}h_{\alpha_j\alpha_j}^{\phi_r(A)}+\sum_{\alpha_i\in\Delta_i,t\in\Delta_{k+1}}^{1\le i\le k}h_{\alpha_i\alpha_i}^{\phi_r(A)}h_{tt}^{\phi_r(A)}\right]-$$

$$-\sum_{r=1}^{3}\sum_{A=1}^{n}\sum_{\alpha_i\in\Delta_i}^{s\in\Delta_{k+1}}\left(h_{ss}^{\phi_r(\alpha_i)}\right)^2-\sum_{r=1}^{3}\sum_{t=N+1}^{n}\sum_{1\le A\le n}^{A\ne t}\left(h_{AA}^{\phi_r(t)}\right)^2+\frac{1}{2}\left[n(n-1)-\sum_{i=1}^{k}n_i(n_i-1)\right]\frac{c}{4}\ .$$

Thus, we find

$$\tau-\tau(L_i)\le\sum_{r=1}^{3}\sum_{A=1}^{n}\left[\sum_{t,s\in\Delta_{k+1}}^{t<s}h_{tt}^{\phi_r(A)}h_{ss}^{\phi_r(A)}+\sum_{\alpha_i\in\Delta_i,\alpha_j\in\Delta_j}^{1\le i<j\le k}h_{\alpha_i\alpha_i}^{\phi_r(A)}h_{\alpha_j\alpha_j}^{\phi_r(A)}+\sum_{\alpha_i\in\Delta_i,t\in\Delta_{k+1}}^{1\le i\le k}h_{\alpha_i\alpha_i}^{\phi_r(A)}h_{tt}^{\phi_r(A)}\right]-$$

$$-\sum_{r=1}^{3}\sum_{A=1}^{n}\sum_{\alpha_i\in\Delta_i}^{s\in\Delta_{k+1}}\left(h_{ss}^{\phi_r(\alpha_i)}\right)^2-\sum_{r=1}^{3}\sum_{t=N+1}^{n}\sum_{1\le A\le n}^{A\ne t}\left(h_{AA}^{\phi_r(t)}\right)^2+\frac{1}{2}\left[n(n-1)-\sum_{i=1}^{k}n_i(n_i-1)\right]\frac{c}{4}. \tag{4.7}$$

For $A=\overline{1,n}$ we consider the quadratic forms $f_{\phi_r(A)}:\mathbb{R}^n\to\mathbb{R}$ defined by

$$f_{\phi_r(\alpha_i)}(h_{11}^{\phi_r(\alpha_i)},h_{22}^{\phi_r(\alpha_i)},\dots,h_{nn}^{\phi_r(\alpha_i)})=\sum_{t,s\in\Delta_{k+1}}^{t<s}h_{tt}^{\phi_r(\alpha_i)}h_{ss}^{\phi_r(\alpha_i)}+ \tag{4.8}$$

$$+\sum_{\alpha_j\in\Delta_j,\alpha_h\in\Delta_h}^{1\le j<h\le k}h_{\alpha_j\alpha_j}^{\phi_r(\alpha_i)}h_{\alpha_h\alpha_h}^{\phi_r(\alpha_i)}+\sum_{\alpha_j\in\Delta_j,t\in\Delta_{k+1}}^{1\le j\le k}h_{\alpha_j\alpha_j}^{\phi_r(\alpha_i)}h_{tt}^{\phi_r(\alpha_i)}-\sum_{s\in\Delta_{k+1}}\left(h_{ss}^{\phi_r(\alpha_i)}\right)^2,$$

$$f_{\phi_r(t)}(h_{11}^{\phi_r(t)},h_{22}^{\phi_r(t)},\dots,h_{nn}^{\phi_r(t)})=\sum_{s,q\in\Delta_{k+1}}^{s<q}h_{ss}^{\phi_r(t)}h_{qq}^{\phi_r(t)}+ \tag{4.9}$$

$$+\sum_{\alpha_i\in\Delta_i,\alpha_j\in\Delta_j}^{1\le i<j\le k}h_{\alpha_i\alpha_i}^{\phi_r(t)}h_{\alpha_j\alpha_j}^{\phi_r(t)}+\sum_{\alpha_i\in\Delta_i,s\in\Delta_{k+1}}^{1\le i\le k}h_{\alpha_i\alpha_i}^{\phi_r(t)}h_{ss}^{\phi_r(t)}-\sum_{1\le A\le n}^{A\ne t}\left(h_{AA}^{\phi_r(t)}\right)^2,$$

for $\alpha_i\in\Delta_i,\ i=\overline{1,k},\ t\in\Delta_{k+1},\ r\in\{1,2,3\}$.

For $r\in\{1,2,3\}$

$$f_{\phi_r(1)}(h_{11}^{\phi_r(1)},h_{22}^{\phi_r(1)},\dots,h_{nn}^{\phi_r(1)})=\sum_{t,s\in\Delta_{k+1}}^{t<s}h_{tt}^{\phi_r(1)}h_{ss}^{\phi_r(1)}+ \tag{4.10}$$

$$+\sum_{\alpha_j\in\Delta_j,\alpha_h\in\Delta_h}^{1\le j<h\le k}h_{\alpha_j\alpha_j}^{\phi_r(1)}h_{\alpha_h\alpha_h}^{\phi_r(1)}+\sum_{\alpha_j\in\Delta_j,t\in\Delta_{k+1}}^{1\le j\le k}h_{\alpha_j\alpha_j}^{\phi_r(1)}h_{tt}^{\phi_r(1)}-\sum_{t\in\Delta_{k+1}}\left(h_{tt}^{\phi_r(1)}\right)^2.$$

We want to find an upper bound for $f_{\phi_r(1)}$, subject to

$$P:h_{11}^{\phi_r(1)}+h_{22}^{\phi_r(1)}+\dots+h_{nn}^{\phi_r(1)}=c^{\phi_r(1)}, \tag{4.11}$$

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where $c^{\phi_r(1)}$ is a real number.

The bilinear form $\alpha : T_qP \times T_qP \to \mathbb{R}$ has the expression

$$\alpha(X,Y) = \mathrm{Hess}(f_r)(X,Y) + \langle h'(X,Y), \mathrm{grad}\, f_r(q)\rangle,$$

where h' is the second fundamental form of P in $\mathbb{R}^n$ and $\langle\ ,\ \rangle$ is the standard inner-product on $\mathbb{R}^n$.

A vector $X \in T_qP$ satisfies $\sum_{i=1}^{n} X^i = 0$.

Searching for the partial derivatives of the function $f_{\phi_r(1)}$, we get

$$\frac{\partial f_{\phi_r(1)}}{\partial h_{11}^{\phi_r(1)}} = \sum_{\alpha_h \in \Delta_h}^{2\le h\le k} h_{\alpha_h\alpha_h}^{\phi_r(1)} + \sum_{t\in\Delta_{k+1}} h_{tt}^{\phi_r(1)},$$

$$\frac{\partial f_{\phi_r(1)}}{\partial h_{22}^{\phi_r(1)}} = \sum_{\alpha_h \in \Delta_h}^{2\le h\le k} h_{\alpha_h\alpha_h}^{\phi_r(1)} + \sum_{t\in\Delta_{k+1}} h_{tt}^{\phi_r(1)},$$

$$\frac{\partial f_{\phi_r(1)}}{\partial h_{\alpha_1\alpha_1}^{\phi_r(1)}} = \sum_{\alpha_h \in \Delta_h, h\neq 1}^{1\le h\le k} h_{\alpha_h\alpha_h}^{\phi_r(1)} + \sum_{t\in\Delta_{k+1}} h_{tt}^{\phi_r(1)},\ \alpha_1 \in \Delta_1.$$

In a similar manner, we find

$$\frac{\partial f_{\phi_r(1)}}{\partial h_{\alpha_2\alpha_2}^{\phi_r(1)}} = \sum_{\alpha_h \in \Delta_h, h\neq 2}^{1\le h\le k} h_{\alpha_h\alpha_h}^{\phi_r(1)} + \sum_{t\in\Delta_{k+1}} h_{tt}^{\phi_r(1)},\ \alpha_2 \in \Delta_2,$$

so, in general,

$$\frac{\partial f_{\phi_r(1)}}{\partial h_{\alpha_i\alpha_i}^{\phi_r(1)}} = \sum_{\alpha_j \in \Delta_j, j\neq i}^{1\le j\le k} h_{\alpha_j\alpha_j}^{\phi_r(1)} + \sum_{t\in\Delta_{k+1}} h_{tt}^{\phi_r(1)},\ \alpha_i \in \Delta_i,\ i=\overline{1,k}. \tag{4.12}$$

For $s \in \Delta_{k+1}$, we also find

$$\frac{\partial f_{\phi_r(1)}}{\partial h_{ss}^{\phi_r(1)}} = \sum_{t\in\Delta_{k+1}}^{t\neq s} h_{tt}^{\phi_r(1)} + \sum_{\alpha_j\in\Delta_j}^{1\le j\le k} h_{\alpha_j\alpha_j}^{\phi_r(1)} - 2h_{ss}^{\phi_r(1)}. \tag{4.13}$$

In the standard frame of $\mathbb{R}^n$, the Hessian of $f_{\phi_r(1)}$ has the matrix

$$\begin{pmatrix} O_1 & A_{12} & A_{13} & A_{14} & \dots & A_{1k} & B_1 \\ A_{21} & O_2 & A_{23} & A_{24} & \dots & A_{2k} & B_2 \\ A_{31} & A_{32} & O_3 & A_{34} & \dots & A_{3k} & B_3 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ A_{k1} & A_{2k} & A_{k3} & A_{k4} & \dots & O_k & B_k \\ B_1^t & B_2^t & B_3^t & B_4^t & \dots & B_k^t & A \end{pmatrix},$$

where $O_i \in \mathcal{M}_{n_i}(\mathbb{R})$, with all the elements equals to 0, $i=\overline{1,k}$, $A_{ij} \in \mathcal{M}_{n_i,n_j}(\mathbb{R})$, $i\neq j$, $i,j=\overline{1,k}$, with all the lements equals to 1, $B_i \in \mathcal{M}_{n_i,n-N}(\mathbb{R})$, with all the elements equals to 1 and A is the matrix

$$A = \begin{pmatrix} -2 & 1 & 1 & \dots & 1 \\ 1 & -2 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & -2 \end{pmatrix}, \quad A \in \mathcal{M}_{n-N,n-N}(\mathbb{R}).$$

As P is totally geodesic in $\mathbb{R}^n$ (P a hyperplane; $h'=0$), we get

$$\alpha(X,X) = \left[X_1 \sum_{\alpha_j\in\Delta_j, j\neq 1}^{1\le j\le k} X_{\alpha_j} + \dots + X_{n_1} \sum_{\alpha_j\in\Delta_j, j\neq 1}^{1\le j\le k} X_{\alpha_j} \right] +$$

$$+\left[X_{n_1+1}\sum_{\alpha_j\in\Delta_j,j\neq 2}^{1\leq j\leq k}X_{\alpha_j}+\ldots+X_{n_1+n_2}\sum_{\alpha_j\in\Delta_j,j\neq 2}^{1\leq j\leq k}X_{\alpha_j}\right]+\ldots+$$

$$+\left[X_{n_1+\ldots+n_{k-1}+1}\sum_{\alpha_j\in\Delta_j,j\neq k}^{1\leq j\leq k}X_{\alpha_j}+\ldots+X_{n_1+\ldots+n_k}\sum_{\alpha_j\in\Delta_j,j\neq k}^{1\leq j\leq k}X_{\alpha_j}\right]+$$

$$+\left(\sum_{\alpha_i\in\Delta_i}^{1\leq i\leq k}X_{\alpha_i}\right)\cdot\sum_{s\in\Delta_{k+1}}X_s+X_{N+1}\cdot\left[\sum_{\alpha_i\in\Delta_i}^{1\leq i\leq k}X_{\alpha_i}+\sum_{s\in\Delta_{k+1}}^{s\neq N+1}X_s\right]+$$

$$+X_{N+2}\cdot\left[\sum_{\alpha_i\in\Delta_i}^{1\leq i\leq k}X_{\alpha_i}+\sum_{s\in\Delta_{k+1}}^{s\neq N+2}X_s\right]+\ldots+X_n\cdot\left[\sum_{\alpha_i\in\Delta_i}^{1\leq i\leq k}X_{\alpha_i}+\sum_{s\in\Delta_{k+1}}^{s\neq n}X_s\right]-2\sum_{s\in\Delta_{k+1}}(X_s)^2.$$

Thus, we get

$$\alpha(X,X)=2\sum_{\alpha_i\in\Delta_i,\alpha_j\in\Delta_j}^{1\leq i<j\leq k}X_{\alpha_i}X_{\alpha_j}+2\sum_{\alpha_i\in\Delta_i,s\in\Delta_{k+1}}^{1\leq i\leq k}X_{\alpha_i}X_s+$$

$$+2\sum_{s,t\in\Delta_{k+1}}^{s\neq t}X_sX_t-2\sum_{s\in\Delta_{k+1}}(X)_s)^2=$$

$$=\left(\sum_{i=1}^{n}X_i\right)^2-\sum_{\alpha_i\in\Delta_i}^{1\leq i\leq k}(X_{\alpha_i})^2-\sum_{s\in\Delta_{k+1}}(X_s)^2-2\sum_{\alpha_i\in\Delta_i,\beta_i\in\Delta_i}^{\alpha_i\neq\beta_i}X_{\alpha_i}X_{\beta_i}-2\sum_{s\in\Delta_{k+1}}(X_s)^2=$$

$$=\left(\sum_{i=1}^{n}X_i\right)^2-\sum_{i=1}^{k}\left(\sum_{\alpha_i\in\Delta_i}X_{\alpha_i}\right)^2-3\sum_{s\in\Delta_{k+1}}(X_s)^2<0,$$

so the Hessian of $f_{\phi_r(1)}$ is negative semidefinite.

Searching for the critical point $(h_{11}^{\phi_r(1)},h_{22}^{\phi_r(1)},\ldots,h_{nn}^{\phi_r(1)})$ of $f_{\phi_r(1)}$, we find

$$\frac{\partial f_{\phi_r(1)}}{\partial h_{\alpha_i\alpha_i}^{\phi_r(1)}}=\frac{\partial f_{\phi_r(1)}}{\partial h_{\alpha_j\alpha_j}^{\phi_r(1)}}\ ,\ i\neq j\ ,\ \alpha_i\in\Delta_i,\alpha_j\in\Delta_j,$$

then

$$\sum_{\alpha_h\in\Delta_h,h\neq i}^{1\leq h\leq k}h_{\alpha_h\alpha_h}^{\phi_r(1)}=\sum_{\alpha_h\in\Delta_h,h\neq j}^{1\leq h\leq k}h_{\alpha_h\alpha_h}^{\phi_r(1)},$$

$$\sum_{\alpha_i\in\Delta_i}h_{\alpha_i\alpha_i}^{\phi_r(1)}=\sum_{\alpha_j\in\Delta_j}h_{\alpha_j\alpha_j}^{\phi_r(1)}\quad i\neq j. \tag{4.14}$$

From

$$\frac{\partial f_{\phi_r(1)}}{\partial h_{ss}^{\phi_r(1)}}=\frac{\partial f_{\phi_r(1)}}{\partial h_{qq}^{\phi_r(1)}}\quad s,q\in\Delta_{k+1},\ s\neq q$$

we get

$$\sum_{t\in\Delta_{k+1}}^{t\neq s}h_{tt}^{\phi_r(1)}-2h_{ss}^{\phi_r(1)}=\sum_{t\in\Delta_{k+1}}^{t\neq q}h_{tt}^{\phi_r(1)}-2h_{qq}^{\phi_r(1)}\implies h_{ss}^{\phi_r(1)}=h_{qq}^{\phi_r(1)},$$

$$h_{N+1N+1}^{\phi_r(1)}=\ldots=h_{nn}^{\phi_r(1)}. \tag{4.15}$$

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$$\frac{\partial f_{\phi_r(1)}}{\partial h_{ss}^{\phi_r(1)}} = \frac{\partial f_{\phi_r(1)}}{\partial h_{\alpha_i\alpha_i}^{\phi_r(1)}}.$$

Then

$$\sum_{\alpha_j\in\Delta_j, j\neq i}^{1\leq j\leq k} h_{\alpha_i\alpha_i}^{\phi_r(1)} + \sum_{t\in\Delta_{k+1}} h_{tt}^{\phi_r(1)} = \sum_{t\in\Delta_{k+1}}^{t\neq s} h_{tt}^{\phi_r(1)} + \sum_{\alpha_j\in\Delta_j}^{1\leq j\leq k} h_{\alpha_i\alpha_i}^{\phi_r(1)} - 2h_{ss}^{\phi_r(1)} \Longrightarrow$$

$$\Longrightarrow h_{ss}^{\phi_r(1)} = \sum_{\alpha_i\in\Delta_i} h_{\alpha_i\alpha_i}^{\phi_r(1)} - 2h_{ss}^{\phi_r(1)}$$

and from this, we get

$$h_{ss}^{\phi_r(1)} = \frac{1}{3}\cdot \sum_{\alpha_i\in\Delta_i} h_{\alpha_i\alpha_i}^{\phi_r(1)}\ ,\ s\in\Delta_{k+1}\ . \tag{4.16}$$

Let

$$\sum_{\alpha_1\in\Delta_1} h_{\alpha_1\alpha_1}^{\phi_r(1)} = \sum_{\alpha_2\in\Delta_2} h_{\alpha_2\alpha_2}^{\phi_r(1)} = \ldots = \sum_{\alpha_k\in\Delta_k} h_{\alpha_k\alpha_k}^{\phi_r(1)} = 3a^1, \tag{4.17}$$

$$h_{ss}^{\phi_r(1)} = \frac{1}{3}\cdot \sum_{\alpha_i\in\Delta_i} h_{\alpha_i\alpha_i}^{\phi_r(1)} = a^1,\ s\in\Delta_{k+1}, \tag{4.18}$$

where a^1 is a real number.

From the relations (4.11), (4.17) and (4.18), we get

$$k\cdot 3a^1 + (n-N)a^1 = c^{\phi_r(1)} \Longleftrightarrow a^1 = \frac{c^{\phi_r(1)}}{n-N+3k}.$$

One gets

$$\sum_{\alpha_1\in\Delta_1} h_{\alpha_1\alpha_1}^{\phi_r(1)} = \sum_{\alpha_2\in\Delta_2} h_{\alpha_2\alpha_2}^{\phi_r(1)} = \ldots = \sum_{\alpha_k\in\Delta_k} h_{\alpha_k\alpha_k}^{\phi_r(1)} = \frac{3c^{\phi_r(1)}}{n-N+3k}, \tag{4.19}$$

$$h_{ss}^{\phi_r(1)} = \frac{c^{\phi_r(1)}}{n-N+3k},\ s\in\Delta_{k+1}. \tag{4.20}$$

Thus, from (4.10), (4.19) and (4.20), we have

$$f_{\phi_r(1)} \leq C_{n-N}^2\cdot\left(\frac{c^{\phi_r(1)}}{n-N+3k}\right)^2 + \left(\frac{3c^{\phi_r(1)}}{n-N+3k}\right)^2 +$$

$$+\frac{3k(n-N)(c^{\phi_r(1)})^2}{(n-N+3k)^2} - (n-N)\cdot\left(\frac{c^{\phi_r(1)}}{n-N+3k}\right)^2 =$$

$$= \left(\frac{c^{\phi_r(1)}}{n-N+3k}\right)^2\cdot\left[C_{n-N}^2 + 9\cdot C_k^2 + 3k(n-N) - (n-N)\right] =$$

$$= \left(\frac{c^{\phi_r(1)}}{n-N+3k}\right)^2\cdot\left[\frac{(n-N)(n-N-1)}{2} + \frac{9k(k-1)}{2} + 3k(n-N) - (n-N)\right] =$$

$$= \frac{(c^{\phi_r(1)})^2}{2(n-N+3k)^2}\left[(n-N)(n-N-1) + 9k(k-1) + 6k(n-N) - 2(n-N)\right] =$$

$$= \frac{(c^{\phi_r(1)})^2}{2(n-N+3k)^2}(n^2 - nN - n - nN + N^2 + N + 9k^2 - 9k + 6kn - 6kN - 2n + 2N) =$$

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$$= \frac{(c^{\phi_r(1)})^2}{2(n-N+3k)^2}(n^2+N^2-2nN-3n+3N+9k^2-9k+6kn-6kN) =$$

$$= \frac{(c^{\phi_r(1)})^2}{2(n-N+3k)^2}\left[(n-N+3k)^2-3n+3N-9k\right] =$$

$$= \frac{(c^{\phi_r(1)})^2}{2(n-N+3k)^2}\left[(n-N+3k)^2-3(n-N+3k)\right] =$$

$$= \frac{(c^{\phi_r(1)})^2}{2(n-N+3k)^2}(n-N+3k)(n-N+3k-3) =$$

$$= \frac{(c^{\phi_r(1)})^2}{2(n-N+3k)} \cdot (n-N+3k+3),$$

which implies

$$f_{\phi_r(1)} \leq \frac{n^2}{2} \cdot \left(\frac{n-N+3k-3}{n-N+3k}\right) \cdot \left(H^{\phi_r(1)}\right)^2. \tag{4.21}$$

In a similar manner, we find

$$f_{\phi_r(\alpha_i)} \leq \frac{n^2}{2} \cdot \left(\frac{n-N+3k-3}{n-N+3k}\right) \cdot \left(H^{\phi_r(\alpha_i)}\right)^2, \ i=\overline{1,k}, \ \alpha_i \in \Delta_i. \tag{4.22}$$

Let $r \in \{1,2,3\}, \ t \in \Delta_{k+1}$,

$$f_{\phi_r(t)}(h_{11}^{\phi_r(t)}, h_{22}^{\phi_r(t)}, \ldots, h_{nn}^{\phi_r(t)}) = \sum_{s,q\in\Delta_{k+1}}^{s<q} h_{ss}^{\phi_r(t)} h_{qq}^{\phi_r(t)} + \tag{4.23}$$

$$+ \sum_{\alpha_i\in\Delta_i,\alpha_j\in\Delta_j}^{1\leq i<j\leq k} h_{\alpha_i\alpha_i}^{\phi_r(t)} h_{\alpha_j\alpha_j}^{\phi_r(t)} + \sum_{\alpha_i\in\Delta_i,s\in\Delta_{k+1}}^{1\leq i\leq k} h_{\alpha_i\alpha_i}^{\phi_r(t)} h_{ss}^{\phi_r(t)} - \sum_{1\leq A\leq n}^{A\neq t} \left(h_{AA}^{\phi_r(t)}\right)^2.$$

Searching for the partial derivatives of $f_{\phi_r(t)}$, we have

$$\frac{\partial f_{\phi_r(t)}}{h_{\alpha_i\alpha_i}^{\phi_r(t)}} = \sum_{\alpha_j\in\Delta_j,j\neq i}^{1\leq j\leq k} h_{\alpha_j\alpha_j}^{\phi_r(t)} + \sum_{s\in\Delta_{k+1}} h_{ss}^{\phi_r(t)} - 2h_{\alpha_i\alpha_i}^{\phi_r(t)}, \ \alpha_i \in \Delta_i, \ i=\overline{1,k}, \tag{4.24}$$

$$\frac{\partial f_{\phi_r(t)}}{h_{tt}^{\phi_r(t)}} = \sum_{q\in\Delta_{k+1}}^{q\neq t} h_{qq}^{\phi_r(t)} + \sum_{\alpha_i\in\Delta_i}^{1\leq i\leq k} h_{\alpha_i\alpha_i}^{\phi_r(t)}, \tag{4.25}$$

$$\frac{\partial f_{\phi_r(t)}}{h_{ss}^{\phi_r(t)}} = \sum_{q\in\Delta_{k+1}}^{q\neq s} h_{qq}^{\phi_r(t)} + \sum_{\alpha_i\in\Delta_i}^{1\leq i\leq k} h_{\alpha_i\alpha_i}^{\phi_r(t)} - 2h_{ss}^{\phi_r(t)}, \ s \neq t. \tag{4.26}$$

In the standard frame of $\mathbb{R}^n$, the Hessian of $f_{\phi_r(t)}$ has the matrix

$$\begin{pmatrix} I_1 & A_{12} & A_{13} & A_{14} & \ldots & A_{1k} & B_1 \\ A_{21} & I_2 & A_{23} & A_{24} & \ldots & A_{2k} & B_2 \\ A_{31} & A_{32} & I_3 & A_{34} & \ldots & A_{3k} & B_3 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ A_{k1} & A_{2k} & A_{k3} & A_{k4} & \ldots & O_I & B_k \\ B_1^t & B_2^t & B_3^t & B_4^t & \ldots & B_k^t & A_t \end{pmatrix},$$

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where $I_i \in \mathcal{M}_{n_i}(\mathbb{R})$, with all the elements equals to 0,except those on the first diagonal that are equals to -2, $i=\overline{1,k}$, $A_{ij} \in \mathcal{M}_{n_i,n_j}(\mathbb{R})$, $i \neq j$, $i,j=\overline{1,k}$, with all the lements equals to 1, $B_i \in \mathcal{M}_{n_i,n-N}(\mathbb{R})$, with all the elements equals to 1 and A_t is the matrix :

$$A_t = (a_{ij})_{i,j=\overline{1,n-N}}\,, \quad A \in \mathcal{M}_{n-N,n-N}(\mathbb{R}),$$

$$a_{ii} = -2,\ i=\overline{1,n-N},\ i \neq t,$$

$$a_{ij} = 1,\ i,j=\overline{1,n-N},\ i \neq j,$$

$$a_{tt} = 0.$$

As P is totally geodesic in $\mathbb{R}^n$, we get

$$\alpha(X,X) = -2\sum_{\alpha_i\in\Delta_i}^{1\le i\le k}(X_{\alpha_i})^2 + 2\sum_{\alpha_i\in\Delta_i,\alpha_j\in\Delta_j}^{1\le i\neq j\le k} X_{\alpha_i}X_{\alpha_j} +$$

$$+2\sum_{\alpha_i\in\Delta_i,s\in\Delta_{k+1}}^{1\le i\le k} X_{\alpha_i}X_s - 2\sum_{s\in\Delta_{k+1}}^{s\neq t}(X_s)^2 + 2\sum_{s,q\in\Delta k+1}^{s\neq q} X_sX_q =$$

$$= \left(\sum_{i=1}^{n} X_i\right)^2 - \sum_{\alpha_i\in\Delta_i}^{1\le i\le k}(X_{\alpha_i})^2 - \sum_{s\in\Delta_{k+1}}(X_s)^2 - 2\sum_{\alpha_i,\beta_i\in\Delta_i,\alpha_i\neq\beta_i}^{1\le i\le k} X_{\alpha_i}X_{\beta_i} -$$

$$-2\sum_{\alpha_i\in\Delta_i}^{1\le i\le k}(X_{\alpha_i})^2 - 2\sum_{s\in\Delta_{k+1}}^{s\neq t}(X_s)^2 =$$

$$= \left(\sum_{i=1}^{n} X_i\right)^2 - \sum_{i=1}^{k}\left(\sum_{\alpha_i\in\Delta_i} X_{\alpha_i}\right)^2 - 2\sum_{\alpha_i\in\Delta_i}^{1\le i\le k}(X_{\alpha_i})^2 - 3\sum_{s\in\Delta_{k+1}}^{s\neq t}(X_s)^2 - (X_t)^2 < 0,$$

so the Hessian of $f_{\phi_r(t)}$ is negative semidefinite.

Searching for the critical point $(h_{11}^{\phi_r(t)}, h_{22}^{\phi_r(t)}, \ldots, h_{nn}^{\phi_r(t)})$ of $f_{\phi_r(t)}$, we have

$$\frac{\partial f_{\phi_r(t)}}{h_{\alpha_i\alpha_i}^{\phi_r(t)}} = \frac{\partial f_{\phi_r(t)}}{h_{\beta_i\beta_i}^{\phi_r(t)}}\ ,\ \alpha_i,\beta_i \in \Delta_i \implies h_{\alpha_i\alpha_i}^{\phi_r(t)} = h_{\beta_i\beta_i}^{\phi_r(t)}\ ,\ \alpha_i,\beta_i\in\Delta_i \implies$$

$$h_{11}^{\phi_r(t)} = \ldots = h_{n_1n_1}^{\phi_r(t)},$$

$$h_{n_1+1n_1+1}^{\phi_r(t)} = \ldots = h_{n_1+n_2n_1+n_2}^{\phi_r(t)},\ \ldots.$$

Thus, we get

$$\forall i=\overline{1,k},\ \forall\alpha_i,\beta_i\in\Delta_i,\ h_{\alpha_i\alpha_i}^{\phi_r(t)} = h_{\beta_i\beta_i}^{\phi_r(t)}. \tag{4.27}$$

In the same way, we find

$$\frac{\partial f_{\phi_r(t)}}{h_{ss}^{\phi_r(t)}} = \frac{\partial f_{\phi_r(t)}}{h_{vv}^{\phi_r(t)}},\ v,s\in\Delta_{k+1},\ t\notin\{v,s\} \implies$$

$$\sum_{q\in\Delta_{k+1}}^{q\neq s} h_{qq}^{\phi_r(t)} + \sum_{\alpha_i\in\Delta_i}^{1\le i\le k} h_{\alpha_i\alpha_i}^{\phi_r(t)} - 2h_{ss}^{\phi_r(t)} = \sum_{q\in\Delta_{k+1}}^{q\neq v} h_{qq}^{\phi_r(t)} + \sum_{\alpha_i\in\Delta_i}^{1\le i\le k} h_{\alpha_i\alpha_i}^{\phi_r(t)} - 2h_{vv}^{\phi_r(t)},\ t\notin\{v,s\}$$

$$\implies h_{vv}^{\phi_r(t)} - 2h_{ss}^{\phi_r(t)} = h_{ss}^{\phi_r(t)} - 2h_{vv}^{\phi_r(t)},\ \text{which is}$$

(4.28) $$h_{ss}^{\phi_r(t)} = h_{vv}^{\phi_r(t)},\ \forall s, v \in \Delta_{k+1},\ t \notin \{s, v\}.$$

Also, for $s \neq t$, we have

$$\frac{\partial f_{\phi_r(t)}}{h_{tt}^{\phi_r(t)}} = \frac{\partial f_{\phi_r(t)}}{h_{ss}^{\phi_r(t)}},$$

which implies

$$\sum_{q\in\Delta_{k+1}}^{q\neq t} h_{qq}^{\phi_r(t)} + \sum_{\alpha_i\in\Delta_i}^{1\leq i\leq k} h_{\alpha_i\alpha_i}^{\phi_r(t)} = \sum_{q\in\Delta_{k+1}}^{q\neq s} h_{qq}^{\phi_r(t)} + \sum_{\alpha_i\in\Delta_i}^{1\leq i\leq k} h_{\alpha_i\alpha_i}^{\phi_r(t)} - 2h_{ss}^{\phi_r(t)} \implies$$

$$h_{ss}^{\phi_r(t)} = h_{tt}^{\phi_r(t)} - 2h_{ss}^{\phi_r(t)},$$

from which we have

(4.29) $$h_{tt}^{\phi_r(t)} = 3h_{ss}^{\phi_r(t)},\ s \neq t,\ s \in \Delta_{k+1}.$$

From $\dfrac{\partial f_{\phi_r(t)}}{h_{\alpha_i\alpha_i}^{\phi_r(t)}} = \dfrac{\partial f_{\phi_r(t)}}{h_{\alpha_j\alpha_j}^{\phi_r(t)}}$, $i \neq j$, we get

$$\sum_{\alpha_h\in\Delta_h, h\neq i}^{1\leq h\leq k} h_{\alpha_h\alpha_h}^{\phi_r(t)} + \sum_{s\in\Delta_{k+1}} h_{ss}^{\phi_r(t)} - 2h_{\alpha_i\alpha_i}^{\phi_r(t)} = \sum_{\alpha_h\in\Delta_h, h\neq j}^{1\leq h\leq k} h_{\alpha_h\alpha_h}^{\phi_r(t)} + \sum_{s\in\Delta_{k+1}} h_{ss}^{\phi_r(t)} - 2h_{\alpha_j\alpha_j}^{\phi_r(t)}.$$

Thus

$$\sum_{\alpha_j\in\Delta_j} h_{\alpha_j\alpha_j}^{\phi_r(t)} - 2h_{\alpha_i\alpha_i}^{\phi_r(t)} = \sum_{\alpha_i\in\Delta_i} h_{\alpha_i\alpha_i}^{\phi_r(t)} - 2h_{\alpha_j\alpha_j}^{\phi_r(t)} \implies$$

(4.30) $$\sum_{\alpha_i\in\Delta_i} h_{\alpha_i\alpha_i}^{\phi_r(t)} + 2h_{\alpha_i\alpha_i}^{\phi_r(t)} = \sum_{\alpha_j\in\Delta_j} h_{\alpha_j\alpha_j}^{\phi_r(t)} + 2h_{\alpha_j\alpha_j}^{\phi_r(t)}.$$

From (4.30), using the relation (4.27), we get

(4.31) $$(n_i+2)h_{\alpha_i\alpha_i}^{\phi_r(t)} = (n_j+2)h_{\alpha_j\alpha_j}^{\phi_r(t)},\ \alpha_i \in \Delta_i,\ \alpha_j \in \Delta_j,\ i \neq j.$$

Also, from $\dfrac{\partial f_{\phi_r(t)}}{h_{\alpha_i\alpha_i}^{\phi_r(t)}} = \dfrac{\partial f_{\phi_r(t)}}{h_{tt}^{\phi_r(t)}}$, for some $i = \overline{1,k}$, $\alpha_i \in \Delta_i$ we get

$$\sum_{\alpha_j\in\Delta_j, j\neq i}^{1\leq j\leq k} h_{\alpha_j\alpha_j}^{\phi_r(t)} + \sum_{s\in\Delta_{k+1}} h_{ss}^{\phi_r(t)} - 2h_{\alpha_i\alpha_i}^{\phi_r(t)} = \sum_{s\in\Delta_{k+1}}^{s\neq t} + \sum_{\alpha_i\in\Delta_i}^{1\leq i\leq k} h_{\alpha_i\alpha_i}^{\phi_r(t)} \implies$$

$$\implies h_{tt}^{\phi_r(t)} - 2h_{\alpha_i\alpha_i}^{\phi_r(t)} = \sum_{\alpha_i\in\Delta_i} h_{\alpha_i\alpha_i}^{\phi_r(t)},\ \text{and we get}$$

(4.32) $$h_{tt}^{\phi_r(t)} = (n_i+2)h_{\alpha_i\alpha_i}^{\phi_r(t)},\ i = \overline{1,k},\ \alpha_i \in \Delta_i.$$

From $\dfrac{\partial f_{\phi_r(t)}}{h_{\alpha_i\alpha_i}^{\phi_r(t)}} = \dfrac{\partial f_{\phi_r(t)}}{h_{ss}^{\phi_r(t)}}$, for some $i = \overline{1,k}$, $\alpha_i \in \Delta_i$, $s \neq t$, we get

$$\sum_{\alpha_j\in\Delta_j, j\neq i}^{1\leq j\leq k} h_{\alpha_j\alpha_j}^{\phi_r(t)} + \sum_{q\in\Delta_{k+1}} h_{qq}^{\phi_r(t)} - 2h_{\alpha_i\alpha_i}^{\phi_r(t)} = \sum_{q\in\Delta_{k+1}}^{q\neq s} h_{qq}^{\phi_r(t)} + \sum_{\alpha_i\in\Delta_i}^{1\leq i\leq k} h_{\alpha_i\alpha_i}^{\phi_r(t)} - 2h_{ss}^{\phi_r(t)} \implies$$

$$\implies h_{ss}^{\phi_r(t)} - 2h_{\alpha_i\alpha_i}^{\phi_r(t)} = \sum_{\alpha_i\in\Delta_i} h_{\alpha_i\alpha_i}^{\phi_r(t)} - 2h_{ss}^{\phi_r(t)}.$$

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Then

$$3h^{\phi_r(t)}_{ss} = \sum_{\alpha_i \in \Delta_i} h^{\phi_r(t)}_{\alpha_i\alpha_i} + 2h^{\phi_r(t)}_{\alpha_i\alpha_i} \; .$$

Using this relation and (4.25), we get

$$3h^{\phi_r(t)}_{ss} = (n_i+2)h^{\phi_r(t)}_{\alpha_i\alpha_i}, \;\; s \neq t, \;\; \alpha_i \in \Delta_i, \;\; i = \overline{1,k}. \tag{4.33}$$

From the relations (4.27), (4.28), (4.29), (4.31), (4.32) and (4.33), we find

$$h^{\phi_r(t)}_{\alpha_i\alpha_i} = h^{\phi_r(t)}_{\beta_i\beta_i} = a^i, \;\; \forall i = \overline{1,k}, \;\; \alpha_i, \beta_i \in \Delta_i, \tag{4.34}$$

$$h^{\phi_r(t)}_{ss} = h^{\phi_r(t)}_{vv} = a^s, \;\; s, v \in \Delta_{k+1}, \;\; t \notin \{v, s\}, \tag{4.35}$$

$$h^{\phi_r(t)}_{tt} = 3h^{\phi_r(t)}_{ss} = 3a^s, \;\; s, t \in \Delta_{k+1}, \;\; s \neq t, \tag{4.36}$$

where $a^i, \; a^s$ are some real numbers.

So, we obtain

$$(n_i+2)h^{\phi_r(t)}_{\alpha_i\alpha_i} = (n_j+2)h^{\phi_r(t)}_{\alpha_j\alpha_j}, \;\; \alpha_i \in \Delta_i, \;\; \alpha_j \in \Delta_j, \;\; i,j \in \{1,\ldots,k\}, \;\; i \neq j$$

and

$$(n_i+2)a^i = (n_j+2)a^j, \;\; i,j \in \overline{1,k}, \;\; i \neq j. \tag{4.37}$$

$$h^{\phi_r(t)}_{tt} = (n_i+2)a^i, \;\; i = \overline{1,k}. \tag{4.38}$$

From $3h^{\phi_r(t)}_{ss} = (n_i+2)h^{\phi_r(t)}_{\alpha_i\alpha_i}, \; s \neq t, \; \alpha_i \in \Delta_i, \; i = \overline{1,k}$, we get

$$3a^s = (n_i+2)a^i. \tag{4.39}$$

Let $a^s = m, \; m \in \mathbb{R}$. Thus $a^i = \dfrac{3m}{n_i+2}$, $i = \overline{1,k}$ and this implies

$$h^{\phi_r(t)}_{\alpha_i\alpha_i} = \frac{3m}{n_i+2}, \;\; i = \overline{1,k}, \;\; \alpha_i \in \Delta_i, \tag{4.40}$$

$$h^{\phi_r(t)}_{ss} = m, \;\; s \in \Delta_{k+1}, \;\; s \neq t, \tag{4.41}$$

$$h^{\phi_r(t)}_{tt} = 3m. \tag{4.42}$$

Because $h^{\phi_r(t)}_{11} + h^{\phi_r(t)}_{22} + \ldots + h^{\phi_r(t)}_{nn} = k^r, \; r = \overline{1,3}$, from the above relations we get

$$\sum_{i=1}^{k}\left(n_i \cdot \frac{3m}{n_i+2}\right) + 3m + (n-N-1)m = k^r,$$

$$3m\sum_{i=1}^{k}\frac{n_i}{n_i+2} + 3m + \left(n - \sum_{i=1}^{k} n_i - 1\right) m = k^r,$$

which implies

$$m\left[3\sum_{i=1}^{k}\frac{n_i}{n_i+2} + 3 + n - \sum_{i=1}^{n} n_i - 1\right] = k^r$$

or

$$m\left[3\sum_{i=1}^{k}\left(1 - \frac{2}{n_i+2}\right) + 3 + n - \sum_{i=1}^{k} n_i - 1\right] = k^r$$

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and this is equivalent with

$$m\left[3k-6\sum_{i=1}^{k}\frac{1}{n_i+2}+3+n-N-1\right]=k^r. \tag{4.43}$$

Denoting by $Q=\sum_{i=1}^{k}\frac{1}{n_i+2}$, from (4.43) we find

$$m=\frac{k^r}{n-N+3k+2-6Q}$$

and using (4.40), (4.41) and (4.42) we get

$$h_{\alpha_i\alpha_i}^{\phi_r(t)}=\frac{3k^r}{(n_i+2)(n-N+3k+2-6Q)},\ i=\overline{1,k}\ \alpha_i\in\Delta_i, \tag{4.44}$$

$$h_{ss}^{\phi_r(t)}=\frac{k^r}{n-N+3k+2-6Q},\ s\in\Delta_{k+1},\ s\neq t, \tag{4.45}$$

$$h_{tt}^{\phi_r(t)}=\frac{3k^r}{n-N+3k+2-6Q}. \tag{4.46}$$

Using the relation (4.23) and the relations (4.44), (4.45) and (4.46), we have

$$f_{\phi_r(t)}\leq\frac{3k^r}{n-N+3k+2-6Q}\cdot(n-N-1)\cdot\frac{k^r}{n-N+3k+2-6Q}+$$

$$+C_{n-N-1}^2\cdot\frac{(k^r)^2}{(n-N+3k+2-6Q)^2}+\sum_{1\leq i<j\leq k}\frac{n_i\cdot n_j}{(n_i+2)(n_j+2)}\cdot\frac{(3k^r)^2}{(n-N+3k+2-6Q)^2}+$$

$$+\frac{3k^r}{n-N+3k+2-6Q}\cdot\sum_{i=1}^{k}\frac{n_i\cdot 3k^r}{(n_i+2)(n-N+3k+2-6Q)}+$$

$$+(n-N-1)\cdot\frac{k^r}{n-N+3k+2-6Q}\cdot\sum_{i=1}^{k}\frac{n_i\cdot 3k^r}{(n_i+2)(n-N+3k+2-6Q)}-$$

$$-\sum_{i=1}^{k}n_i\cdot\frac{(3k^r)^2}{(n_i+2)^2(n-N+3k+2-6Q)^2}-(n-N-1)\cdot\frac{(k^r)^2}{(n-N+3k+2-6Q)^2}=$$

$$=\frac{(k^r)^2}{(n-N+3k+2-6Q)^2}\cdot\left[3(n-N-1)+C_{n-N-1}^2+9\sum_{1\leq i<j\leq k}\frac{n_i\cdot n_j}{(n_i+2)(n_j+2)}+\right.$$

$$\left.+9\sum_{i=1}^{k}\frac{n_i}{n_i+2}+3(n-N-1)\sum_{i=1}^{k}\frac{n_i}{n_i+2}-9\sum_{i=1}^{k}\frac{n_i}{(n_i+2)^2}-(n-N-1)\right]=$$

$$=\frac{(k^r)^2}{(n-N+3k+2-6Q)^2}\cdot\left[2(n-N-1)+C_{n-N-1}^2+9\sum_{1\leq i<j\leq k}\frac{n_i\cdot n_j}{(n_i+2)(n_j+2)}+\right.$$

$$\left.+9\sum_{i=1}^{k}\frac{n_i}{n_i+2}+3(n-N-1)\sum_{i=1}^{k}\frac{n_i}{n_i+2}-9\sum_{i=1}^{k}\frac{n_i}{(n_i+2)^2}\right]=$$

$$=\frac{(k^r)^2}{(n-N+3k+2-6Q)^2}\cdot\left[2(n-N-1)+\frac{(n-N-1)(n-N-2)}{2}+\right.$$

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$$+9\sum_{1\le i<j\le k}\frac{(n_i+2)(n_j+2)-2n_i-2n_j-4}{(n_i+2)(n_j+2)}+9\sum_{i=1}^{k}\left(1-\frac{2}{n_i+2}\right)+$$

$$+3(n-N-1)\sum_{i=1}^{k}\left(1-\frac{2}{n_i+2}\right)-9\sum_{i=1}^{k}\left(\frac{1}{(n_i+2)}-\frac{2}{(n_i+2)^2}\right)\Bigg]=$$

$$=\frac{(k^r)^2}{(n-N+3k+2-6Q)^2}\cdot\Bigg\{\frac{(n-N-1)(n-N+2)}{2}+9\sum_{1\le i<j\le k}\left[1-\frac{2(n_i+n_j+2)}{(n_i+2)(n_j+2)}\right]+$$

$$+9k-18\sum_{i=1}^{k}\frac{1}{n_i+2}+3(n-N-1)k-6(n-N-1)\sum_{i=1}^{k}\frac{1}{n_i+2}-$$

$$-9\sum_{i=1}^{k}\frac{1}{n_i+2}+18\sum_{i=1}^{k}\frac{1}{(n_i+2)^2}\Bigg\}=$$

$$=\frac{(k^r)^2}{(n-N+3k+2-6Q)^2}\cdot\Bigg\{\frac{(n-N-1)(n-N+2)}{2}+9C_k^2-$$

$$-18\sum_{1\le i<j\le k}\left[\frac{(n_i+2)+(n_j+2)-2}{(n_i+2)(n_j+2)}\right]+9k-18Q+$$

$$+3(n-N-1)k-6(n-N-1)Q-9Q+18\sum_{i=1}^{k}\frac{1}{(n_i+2)^2}\Bigg\}=$$

$$=\frac{(k^r)^2}{(n-N+3k+2-6Q)^2}\cdot\Bigg\{\frac{(n-N-1)(n-N+2)}{2}+\frac{9k(k-1)}{2}-$$

$$-18\sum_{1\le i<j\le k}\left[\frac{1}{n_i+2}+\frac{1}{n_j+2}\right]+36\sum_{1\le i<j\le k}\frac{1}{(n_i+2)(n_j+2)}+$$

$$+9k-18Q+3(n-N-1)k-6(n-N-1)Q-9Q+18\sum_{i=1}^{k}\frac{1}{(n_i+2)^2}\Bigg\}=$$

$$=\frac{(k^r)^2}{2(n-N+3k+2-6Q)^2}\cdot\Big[(n-N-1)(n-N+2)+9k^2-9k-36(k-1)Q+$$

$$+36\sum_{1\le i<j\le k}\frac{2}{(n_i+2)(n_j+2)}+18k-36Q+6(n-N-1)k-$$

$$-12(n-N-1)Q-18Q+36\frac{1}{(n_i+2)^2}\Bigg]=$$

$$=\frac{(k^r)^2}{2(n-N+3k+2-6Q)^2}\cdot\Big[(n-N-1)(n-N+2)+9k^2-9k-36(k-1)Q+$$

$$+18k-36Q+6(n-N-1)k-12(n-N-1)Q-18Q+36Q^2\Big]=$$

$$=\frac{(k^r)^2}{2(n-N+3k+2-6Q)^2}\cdot\Big[(n-N-1)(n-N+2)+3(n-N-1)k+$$

$$+3(n-N-1)k-6(n-N-1)Q-6(n-N-1)Q+9k^2-9k-36kQ+36Q+$$

$$+18k - 36Q - 18Q + 36Q^2\big] =$$

$$= \frac{(k^r)^2}{2(n-N+3k+2-6Q)^2} \cdot \big[(n-N-1)(n-N+2) + 3(n-N-1)k - 6Q(n-N-1) +$$

$$+9k^2 - 9k + 18k + 3k(n-N-1) - 36(k-1)Q - 36Q - 6Q(n-N-1) - 18Q + 36Q^2\big] =$$

$$= \frac{(k^r)^2}{2(n-N+3k+2-6Q)^2} \cdot \big[(n-N-1)(n-N+2+3k-6Q) + 3k(n-N-1+3k+3) -$$

$$-36kQ + 36Q - 36Q - 6Q(n-N-1) - 18Q + 36Q^2\big] =$$

$$= \frac{(k^r)^2}{2(n-N+3k+2-6Q)^2} \cdot \big[(n-N-1)(n-N+3k+2-6Q) + 3k(n-N+3k+2) -$$

$$-36kQ - 6Qn + 6QN - 12Q + 36Q^2\big] =$$

$$= \frac{(k^r)^2}{2(n-N+3k+2-6Q)^2} \cdot \big[(n-N-1)(n-N+3k+2-6Q) +$$

$$+3k(n-N+3k+2) - 18kQ - 18kQ - 6Qn + 6QN - 12Q + 36Q^2\big] =$$

$$= \frac{(k^r)^2}{2(n-N+3k+2-6Q)^2} \cdot \big[(n-N-1)(n-N+3k+2-6Q) +$$

$$+3k(n-N+3k+2-6Q) - 6Q(n-N+3k+2-6Q)\big] =$$

$$= \frac{(k^r)^2}{2(n-N+3k+2-6Q)^2} \cdot (n-N+3k+2-6Q)(n-N+3k-1-6Q).$$

Thus, we find that

$$f_{\phi_r(t)} \leq \frac{(k^r)^2}{2(n-N+3k+2-6Q)^2} \cdot (n-N+3k+2-6Q)(n-N+3k-1-6Q),$$

so

$$f_{\phi_r(t)} \leq \frac{(k^r)^2}{2} \cdot \frac{n-N+3k-1-6Q}{n-N+3k+2-6Q},$$

from which we have

$$f_{\phi_r(t)} \leq \left(\frac{n^2}{2} \cdot \frac{n-N+3k-1-6Q}{n-N+3k+2-6Q}\right) \cdot \left(H^{\phi_r(t)}\right)^2. \tag{4.47}$$

We have 2 cases

I) if $Q \leq \frac{1}{3}$, then

$$\frac{n-N+3k-3}{n-N+3k} \leq \frac{n-N+3k-1-6Q}{n-N+3k+2-6Q},$$

so, using the relations (4.7), (4.22) and (4.47), we find the relation (4.3).

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II) if $Q > \dfrac{1}{3}$, then

$$\frac{n - N + 3k - 1 - 6Q}{n - N + 3k + 2 - 6Q} < \frac{n - N + 3k - 3}{n - N + 3k},$$

thus, using the relations (4.7), (4.22) and (4.47), we find the relation (4.4).

□

Remark 4.4. In the particular case $n_1 = n_2 = 2$, we have $Q = \frac{1}{2}$ and $N = 4 < n$; it follows that we are in the case b) of the Theorem 4.3 and the inequality (4.4) becomes

$$\delta(2,2) \leq \frac{n^2(n-1)}{2(n+2)}\|H\|^2 + \frac{1}{2}[n(n-1) - 4]\frac{c}{4}. \tag{4.48}$$

In this case, the following improved inequality can be obtained

$$\delta(2,2) \leq \frac{n^2(n-2)}{2(n+1)}\|H\|^2 + \frac{1}{2}[n(n-1) - 4]\frac{c}{4};$$

it improves the inequality (4.48), because $\frac{n-2}{n+1} < \frac{n-1}{n+2}$, for $n > 4$. The complete proof is given in a forthcoming paper (joint work with A. Mihai), submitted for publication.

References

[1] J. Bolton, F. Dillen, J. Fastenakels and L. Vrancken, A best possible inequality for curvature-like tensor fields, *Math. Inequal. Appl.* 12, 663-681, 2009.

[2] B.-Y. Chen, Some pinching and classification theorems for minimal submanifolds, *Arch. Math.* 60, 568-578, 1993.

[3] B.-Y. Chen, δ-Invariants, Inequalities of Submanifolds and Their Applications, in *Topics in Differential Geometry* (Eds. A. Mihai, I. Mihai, R. Miron), Editura Academiei Române, Bucharest, 2008.

[4] B.-Y. Chen and F. Dillen, δ-invariants for Lagrangian submanifolds of complex space forms, *Proceedings RIGA* 2011, Editura Universităţii din Bucureşti, 75-94, 2011.

[5] B.-Y. Chen, F. Dillen, J. Van der Veken and L. Vrancken, Curvature inequalities for Lagrangian submanifolds: the final solution, *Differential Geom. Appl.* 31 (6), 808-819, 2013.

[6] B.-Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, Totally real submanifolds of CP^n satisfying a basic equality, *Arch. Math.* 63, 553-564, 1994.

[7] B.-Y. Chen and C.S. Houh, Totally real submanifolds of a quaternion projective space, *Ann. Mat. Pura Appl.* 120 (1), 185-199, 1979.

[8] Y. Hong and C.S. Houh, Lagrangian submanifolds of quaternion Kaehlerian manifolds satisfying Chen's equality, *Beiträge Algebra Geom.* 39 (2), 413-421, 1998.

[9] S. Ishihara, Quaternion Kaehlerian manifolds, *J. Differ. Geom.* 9, 483-500, 1974.

[10] T. Oprea, Optimizations on Riemannian submanifolds, *Analele Univ. Buc.* LIV 1, 127-136, 2005.

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