ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2016, VOLUME 6, ISSUE 1, p.39-46

DOUBLE WEIGHTED LACUNARY ALMOST STATISTICAL CONVERGENCE OF ORDER α

ŞÜKRAN KONCA[1], EKREM SAVAŞ[2] AND SELMAN EKIN[3]

Abstract. In this paper, we define and study the concept of double weighted lacunary almost statistical convergence of order α. Further, some inclusion relations have been examined.

Mathematics Subject Classification (2010): 40B05; 40C05; 40A35; 40G15; 46A45
Key words: Weighted lacunary statistical convergence; double statistical convergence of order α; double almost statistical convergence

Article history:
Received 10 April 2016
Received in revised form 1 June 2016
Accepted 3 June 2016

1. Introduction

The concept of statistical convergence was introduced by Fast [16], Steinhaus [39] and later reintroduced by Schoenberg [38] independently. It turned out to be one of the most active areas of research in summability theory after the works of Fridy [17] and Salat [35]. For further results related with this topic we may refer [3], [18], [28], [31].

A different direction was given to the study of statistical convergence, where the notion of statistical convergence of order α, was introduced by replacing n by n^α in the denominator in the definition of statistical convergence by Çolak [12] and indepently by Bhunia et al. [9]. Later, λ-statistical convergence of order α was introduced by Çolak and Bektaş [13]; λ-statistical convergence of order α of sequences of function by Et, Çınar and Karakaş [15]; lacunary statistical convergence of order α by Şengül and Et [40]; pointwise and uniform statistical convergence of order α by Çınar, Karakaş and Et [11], statistical convergence of order α in probability theory by Das et al. [14], weighted statistical convergence of order α and its applications by Ghosal [19] and many other, different fields of mathematics.

By the convergence of a double sequence we mean the convergence in the Pringsheim's sense. A double sequence $x = (x_{kl})$ is said to converge to the limit L in Pringsheim's sense (shortly, P-convergent to L) if for every $\varepsilon > 0$ there exists an integer N such that $|x_{kl} - L| < \varepsilon$ whenever $k, l > N$ [33]. We shall write this as $\lim_{k,l\to\infty} x_{kl} = L$, where k and l tending to infinity independent of each other and L is called the P-limit of x. A double sequence $x = (x_{kl})$ of real or complex numbers is said to be bounded if $\|x\| = sup_{k,l\geq 0}|x_{kl}| < \infty$. Note that, in contrast to the case for single sequences, a convergent double sequence need not be bounded. We may refer to [1], [7], [10], [21], [22], [29], [30]-[34], for further results related with the concept of double sequence.

The notion of almost convergence and strong almost convergence for single sequences was introduced by Lorentz [23] and by Maddox [24], respectively, and the notion of almost convergence and strong almost convergence for double sequences by Moricz and Rhoades [25] and by Başarır [2], respectively. Some further studies can be seen in ([7], [26]-[27]) and the references therein. Recently, almost statistical convergence of order α and almost lacunary statistical convergence of order α for double sequences have been introduced by Savaş ([36]-[37]).

Başarır and Konca [8] defined a new concept of statistical convergence for single sequences which is called weighted lacunary statistical convergence. In [20] they defined weighted almost lacunary statistical convergence in a real n-normed space. Recently, the notion of weighted lacunary statistical convergence in a locally solid Riesz space for single sequences is studied by Başarır and Konca [5], for double sequences by Konca [22], and in a locally convex topological vector space by Başarır and Konca [6] (see also [4], [21]).

In this paper, we define and study double weighted lacunary almost statistical convergence of order α. Also some inclusion relations have been examined.

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2016, VOLUME 6, ISSUE 1, p.39-46

2. Definitions and Preliminaries

Before beginning of the presentation of the main results, we recall the following basic facts and notations.

Let w_2 be the set of all real or complex double sequences. We denote by c_2 the space of P-convergent sequences. A double sequence $x = (x_{ij})$ is bounded if $\|x\| = sup_{i,j\geq 0}|x_{ij}| < \infty$. Let l_2^∞ and c_2^∞ be the set of all real or complex bounded double sequences and the set of all bounded and convergent double sequences, respectively. Moricz and Rhoades [25] defined the almost convergence of the double sequence as follows:

The double sequence $x = (x_{ij})$ is said to be almost convergent to a number L if

$$P-\lim_{p,q\to\infty}\sup_{m,n}\left|\frac{1}{(p+1)(q+1)}\sum_{i=m}^{m+p}\sum_{j=n}^{n+q}x_{ij}-L\right|=0,$$

that is, the average value of (x_{ij}) taken over any rectangle

$$D=\{(i,j): m\leq i\leq m+p,\ n\leq j\leq n+q\}$$

tends to L as both p and q tend to ∞ and this convergence is uniform in m and n. We denote the space of almost convergent double sequences by $\hat{c}_2$ as

$$\hat{c}_2=\left\{x=(x_{ij}):\lim_{k,l\to\infty}|t_{klpq}(x)-L=0|\text{ uniformly in } p,q\right\},$$

where

$$t_{klpq}(x)=\frac{1}{(k+1)(l+1)}\sum_{i=p}^{k+p}\sum_{j=q}^{l+q}x_{ij}.$$

A double sequence x is called strongly double almost convergent to a number L if

$$P-\lim_{k,l\to\infty}\frac{1}{(k+1)(l+1)}\sum_{i=p}^{k+p}\sum_{j=q}^{l+q}|x_{ij}-L|=0,$$

uniformly in p,q. By $[\hat{c}_2]$ we denote the space of strongly almost convergent double sequences. It is easy to see that the inclusions $c_2^\infty\subset[\hat{c}_2]\subset\hat{c}_2\subset l_2^\infty$ strictly hold.

The double statistical convergence of order α is defined as follows:

Definition 2.1. Let $0<\alpha\leq 1$ be given. The sequence $x=(x_{ij})$ is said to be statistically convergent of order α if there is a real number L such that

$$P-\lim_{m,n\to\infty}\frac{1}{(mn)^\alpha}|\{i\leq m \text{ and } j\leq n: |x_{ij}-L|\geq\varepsilon\}|=0$$

for every $\varepsilon>0$. In this case, we say that x is double statistically convergent of order α to L and we write S_2^α-$\lim_{i,j}x_{ij}=L$. The set of all double statistically convergent sequences of order α will be denoted by

S_2^{α}. If $\alpha = 1$ is taken in this definition, the definition of statistically convergence of a double sequence is obtained (see, [36]).

The double sequence $\theta_{rs} = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing sequences of integers such that $k_0 = 0$, $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$ and $l_0 = 0$, $\bar{h}_s = l_s - l_{s-1} \to \infty$ as $s \to \infty$. Let $k_{rs} = k_r l_s$, $h_{r,s} = h_r \bar{h}_s$ and θ_{rs} is determined by $I_{rs} = \{(kl) : k_{r-1} < k \le k_r \text{ and } l_{s-1} < l \le l_s\}$, $q_r = \frac{k_r}{k_{r-1}}$, $\bar{q}_s = \frac{l_s}{l_{s-1}}$ and $q_{rs} = q_r \bar{q}_s$ [37].

The double lacunary statistical convergence of order α is defined as follows:

Definition 2.2. Let $0 < \alpha \le 1$ be given. The double sequence $x = (x_{ij})$ is said to be lacunary statistically convergent of order α if there is a real number L such that

$$P - \lim_{r,s\to\infty} \frac{1}{{h_{rs}}^{\alpha}} \left|\{(i,j) \in I_{r,s} : |x_{ij} - L| \ge \varepsilon\}\right| = 0$$

for every $\varepsilon > 0$, in this case we say that x is double lacunary statistically convergent of order α to L. In this case, we write $S_{\theta_{rs}}^{\alpha}$-$\lim_{i,j} x_{ij} = L$. The set of all double statistically convergent sequences of order α will be denoted by $S_{\theta_{rs}}^{\alpha}$. If $\alpha = 1$ is taken in this definition, the definition of lacunary statistically convergence of a double sequence is obtained (see, [37]).

Definition 2.3. [1] Let (p_n), $(\bar{p}_m)$ be sequences of positive numbers and $P_n = p_1 + p_2 + ... + p_n$, $\bar{P}_m = \bar{p}_1 + \bar{p}_2 + ... + \bar{p}_m$. Then the transformation given by

$$T_{n,m}(x) = \frac{1}{P_n \bar{P}_m} \sum_{k=1}^{n} \sum_{l=1}^{m} p_k \bar{p}_l x_{kl}$$

is called the Riesz mean of double sequence $x = (x_{kl})$. If $P - lim_{n,m\to\infty} T_{nm}(x) = L$, $L \in \mathbb{R}$, then the sequence $x = (x_{kl})$ is said to be Riesz convergent to L. If $x = (x_{kl})$ is Riesz convergent to L, then we write P_R - $\lim x = L$.

The definition of weighted statistical convergence of order α for double sequences can be defined as follows:

Definition 2.4. Let $0 < \alpha \le 1$ be given. The sequence (x_{ij}) is said to be weighted statistically convergent of order α if there is a real number L such that

$$P - \lim_{m,n\to\infty} \frac{1}{(P_m \bar{P}_n)^{\alpha}} \left|\left\{i \le P_m \text{ and } j \le \bar{P}_n : p_i \bar{p}_j \left|x_{ij} - L\right| \ge \varepsilon\right\}\right| = 0$$

for every $\varepsilon > 0$. In this case, we say that x is double weighted statistically convergent of order α to L and we write $S_{R^2}^{\alpha}$-$\lim_{i,j} x_{ij} = L$. The set of all double weighted statistically convergent sequences of order α will be denoted by $S_{R^2}^{\alpha}$. If $\alpha = 1$ is taken in this definition, the definition of weighted statistical convergence of a double sequence is obtained.

Using the notations of lacunary sequence and Riesz mean for double sequences, Konca and Başarır [21] have given a new definition:

Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence and let (p_k), $(\bar{p}_l)$ be sequences of positive real numbers such that $P_{k_r} := \sum_{k\in(0,k_r]} p_k$ and $\bar{P}_{l_s} := \sum_{l\in(0,l_s]} \bar{p}_l$. If the Riesz transformation of double sequences is RH-regular (it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit), then $\theta'_{r,s} = \left\{\left(P_{k_r}, \bar{P}_{l_s}\right)\right\}$ is a double lacunary sequence, that is; $P_0 = 0$, $0 < P_{k_{r-1}} < P_{k_r}$ and $H_r = P_{k_r} - P_{k_{r-1}} \to \infty$ as $r \to \infty$ and $\bar{P}_0 = 0$, $0 < \bar{P}_{l_{s-1}} < \bar{P}_{l_s}$ and $\bar{H}_s = \bar{P}_{l_s} - \bar{P}_{l_{s-1}} \to \infty$ as $s \to \infty$.

Throughout the paper, we assume that $P_n = p_1 + ... + p_n \to \infty$ $(n \to \infty)$, $\bar{P}_m = \bar{p}_1 + ... + \bar{p}_m \to \infty$ $(m \to \infty)$, such that $H_r = P_{k_r} - P_{k_{r-1}} \to \infty$ as $r \to \infty$ and $\bar{H}_s = \bar{P}_{l_s} - \bar{P}_{l_{s-1}} \to \infty$ as $s \to \infty$.

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2016, VOLUME 6, ISSUE 1, p.39-46

Let $P_{k_{rs}} = P_{k_r}\bar{P}_{l_s}$, $H_{rs} = H_r\bar{H}_s$ and $I'_{rs} = \left\{(kl) : P_{k_{r-1}} < k \leq P_{k_r} \text{ and } \bar{P}_{l_{s-1}} < l \leq \bar{P}_{l_s}\right\}$, $Q_r = \frac{P_{k_r}}{P_{k_{r-1}}}$, $\bar{Q}_s = \frac{\bar{P}_{l_s}}{\bar{P}_{l_{s-1}}}$ and $Q_{rs} = Q_r\bar{Q}_s$. If we take $p_k = 1$, $\bar{p}_l = 1$ for all k and l, then H_{rs}, $P_{k_{rs}}$, Q_{rs} and I'_{rs} reduce to h_{rs}, k_{rs}, q_{rs} and I_{rs}.

3. Main Results

In this section we define double weighted lacunary almost statistically convergent sequences of order α. Also we shall prove some inclusion theorems.

Definition 3.1. Let $0 < \alpha \leq 1$ be given. The double sequence $x = (x_{ij}) \in w_2$ is said to be $\tilde{S}^{\alpha}_{(R^2,\theta)}$-statistical convergent of order α if there is a real number L such that

$$P - \lim_{rs} \frac{1}{H^{\alpha}_{rs}} \left|\{(kl) \in I'_{rs} : p_k\bar{p}_l \left|t_{klpq}(x) - L\right| \geq \varepsilon\}\right| = 0,$$

uniformly in p, q where H^{α}_{rs} denote the α^{th} power of H_{rs}. Incase $x = (x_{ij})$ is $\tilde{S}^{\alpha}_{(R^2,\theta)}$- statistically convergent of order α to L, we write $\tilde{S}^{\alpha}_{(R^2,\theta)} - \lim_{i,j} x_{ij} = L$. We denote the set of all $\tilde{S}^{\alpha}_{(R^2,\theta)}$-statistically convergent sequences of order α by $\tilde{S}^{\alpha}_{(R^2,\theta)}$.

We know that $\tilde{S}^{\alpha}_{(R^2,\theta)}$-statistical convergence of order α is well defined for $0 < \alpha \leq 1$, but it is not well defined for $\alpha > 1$ in general. It is easy to see by taking $x = (x_{ij}) \in w_2$ as fixed.

Definition 3.2. Let $0 < \alpha \leq 1$ be any real number and let t be a positive real number. A sequence x is said to be strongly $\tilde{R}^{\alpha}_{(\theta_{rs})}(t)$-summable of order α, if there is a real number L such that

$$P - \lim_{r,s\to\infty} \frac{1}{H^{\alpha}_{rs}} \sum_{(k,l)\in I_{rs}} p_k p_l \left|t_{klpq}(x) - L\right|^t = 0,$$

uniformly in p, q. We denote the set of all strongly $\tilde{R}^{\alpha}_{(\theta_{rs})}(t)$-summable sequence of order α by $\tilde{R}^{\alpha}_{(\theta_{rs})}(t)$. If we take $p_k = \bar{p}_l = 1$ for all $k, l \in \mathbb{N}$ then $\tilde{R}^{\alpha}_{(\theta_{rs})}(t)$ reduces to the space $\tilde{W}^{\alpha}_{\theta_{rs}}(t)$ (see in [37]).

Theorem 3.3. *If* $0 < \alpha \leq \beta \leq 1$ *then* $\tilde{S}^{\alpha}_{(R^2,\theta_{rs})} \subset \tilde{S}^{\beta}_{(R^2,\theta_{rs})}$.

Proof. Let $0 < \alpha \leq \beta \leq 1$. Then

$$\begin{aligned} &\tfrac{1}{H^{\beta}_{rs}} \sup_{p,q} \left|\{(k,l) \in I'_{rs} : p_k\bar{p}_l \left|t_{klpq}(x) - L\right| \geq \varepsilon\}\right| \\ &\leq \tfrac{1}{H^{\alpha}_{rs}} \sup_{p,q} \left|\{(k,l) \in I'_{rs} : p_k\bar{p}_l \left|t_{klpq}(x) - L\right| \geq \varepsilon\}\right| \end{aligned}$$

for every $\varepsilon > 0$, and finally, we have that $\tilde{S}^{\alpha}_{(R^2,\theta_{rs})} \subset \tilde{S}^{\beta}_{(R^2,\theta_{rs})}$. This proves the result. □

Theorem 3.4. *Let* $0 < \alpha \leq 1$ *and* $\theta_{rs} = \{(k_r, l_s)\}$ *be a double lacunary sequence. If* $\liminf_r Q_r > 1$ *and* $\liminf_s \bar{Q}_s > 1$ *then* $\tilde{S}^{\alpha}_{R^2} \subseteq \tilde{S}^{\alpha}_{(R^2,\theta_{rs})}$.

Proof. Suppose that $\liminf_r Q_r > 1$ and $\liminf_s \bar{Q}_s > 1$, then there exists a $\delta > 0$ such that $Q_r \geq 1 + \delta$ and $\bar{Q}_s \geq 1 + \delta$ for sufficiently large values of r and s, which implies that $\frac{H_r}{P_{kr}} \geq \frac{\delta}{1+\delta}$ and $\frac{\bar{H}_s}{\bar{P}_{l_s}} \geq \frac{\delta}{1+\delta}$. Let

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2016, VOLUME 6, ISSUE 1, p.39-46

$\tilde{S}^{\alpha}_{R^2} - \lim_{(k,l)\to\infty} x_{kl} = L$. Then for sufficiently large values of r and s, we have

$$\begin{aligned}
&\tfrac{1}{(P_{k_r}\bar{P}_{l_s})^{\alpha}} \sup_{p,q} \left|\left\{k \le P_{k_r} \text{ and } l \le \bar{P}_{l_s} : p_k\bar{p}_l \left|t_{klpq}(x) - L\right| \ge \varepsilon\right\}\right| \\
&\ge \tfrac{1}{(P_{k_r}\bar{P}_{l_s})^{\alpha}} \sup_{p,q} \left|\left\{(k,l) \in I'_{rs} : p_k\bar{p}_l \left|t_{klpq}(x) - L\right| \ge \varepsilon\right\}\right| \\
&= \tfrac{H^{\alpha}_{rs}}{(P_{k_r}\bar{P}_{l_s})^{\alpha}} \tfrac{1}{H^{\alpha}_{rs}} \sup_{p,q} \left|\left\{(k,l) \in I'_{rs} : p_k\bar{p}_l \left|t_{klpq}(x) - L\right| \ge \varepsilon\right\}\right| \\
&= \left(\tfrac{H_{rs}}{P_{k_r}\bar{P}_{l_s}}\right)^{\alpha} \tfrac{1}{H^{\alpha}_{rs}} \sup_{p,q} \left|\left\{(k,l) \in I'_{rs} : p_k\bar{p}_l \left|t_{klpq}(x) - L\right| \ge \varepsilon\right\}\right| \\
&\ge \left(\tfrac{\delta}{1+\delta}\right)^{2\alpha} \tfrac{1}{H^{\alpha}_{rs}} \sup_{p,q} \left|\left\{(k,l) \in I'_{rs} : p_k\bar{p}_l \left|t_{klpq}(x) - L\right| \ge \varepsilon\right\}\right|.
\end{aligned}$$

Therefore $\tilde{S}^{\alpha}_{(R^2,\theta_{rs})} - \lim_{(k,l)\to\infty} x_{kl} = L$. □

Theorem 3.5. *Let $0 < \alpha \le 1$ and $\theta_{rs} = \{(k_r, l_s)\}$ be a double lacunary sequence. If $\limsup_r Q^{\alpha}_r < \infty$ and $\limsup_s \bar{Q}^{\alpha}_s < \infty$, then $\tilde{S}^{\alpha}_{(R^2,\theta)} \subseteq \tilde{S}^{\alpha}_{R^2}$.*

Proof. Suppose that $\limsup_r Q^{\alpha}_r < \infty$ and $\limsup_s \bar{Q}^{\alpha}_s < \infty$, then there exists a $K > 0$ such that $Q^{\alpha}_r < K$ and $\bar{Q}^{\alpha}_s < K$ for all $r, s \in \mathbb{N}$. Let $x \in \tilde{S}^{\alpha}_{(R^2,\theta_{rs})}$ with $\tilde{S}^{\alpha}_{(R^2,\theta_{rs})} - \lim_{(k,l)\to\infty} x_{kl} = L$ and

$$N_{rs} := \left|\left\{(k,l) \in I'_{rs} : p_k\bar{p}_l \left|t_{klpq}(x) - L\right| \ge \varepsilon\right\}\right|. \tag{3.1}$$

By (3.1) and the definition of $\tilde{S}^{\alpha}_{(R^2,\theta)}$, given $\varepsilon > 0$, there exists r_0, $s_0 \in \mathbb{N}$ such that $\frac{N_{rs}}{H^{\alpha}_{rs}} < \varepsilon$ for all $r > r_0$ and $s > s_0$. Let $M := \max\{N_{rs} : 1 \le r \le r_0 \text{ and } 1 \le s \le s_0\}$ and let n and m be any integers satisfying $k_{r-1} < n \le k_r$ and $l_{s-1} < m \le l_s$. Hence, for each p and q, we have the following

$$\begin{aligned}
&\tfrac{1}{\left(P_n\bar{P}_m\right)^{\alpha}} \left|\left\{k \le P_n \text{ and } l \le \bar{P}_m : p_k\bar{p}_l \left|t_{klpq}(x) - L\right| \ge \varepsilon\right\}\right| \\
&\le \tfrac{1}{\left(P_{k_{r-1}}\bar{P}_{l_{s-1}}\right)^{\alpha}} \left|\left\{k \le P_{k_r} \text{ and } l \le \bar{P}_{l_s} : p_k\bar{p}_l \left|t_{klpq}(x) - L\right| \ge \varepsilon\right\}\right| \\
&= \tfrac{1}{\left(P_{k_{r-1}}\bar{P}_{l_{s-1}}\right)^{\alpha}} \sum_{i.j=1,1}^{r_0,s_0} N_{ij} + \tfrac{1}{\left(P_{k_{r-1}}\bar{P}_{l_{s-1}}\right)^{\alpha}} \sum_{(r_0<i\le r)\cup(s_0<j\le s)} N_{ij} \\
&\le \tfrac{M r_0 s_0}{\left(P_{k_{r-1}}\bar{P}_{l_{s-1}}\right)^{\alpha}} + \tfrac{1}{\left(P_{k_{r-1}}\bar{P}_{l_{s-1}}\right)^{\alpha}} \sum_{(r_0<i\le r)\cup(s_0<j\le s)} \tfrac{N_{ij}H^{\alpha}_{rs}}{H^{\alpha}_{rs}} \\
&\le \tfrac{M_{r_0 s_0}}{\left(P_{k_{r-1}}\bar{P}_{l_{s-1}}\right)^{\alpha}} + \varepsilon \tfrac{\left(P_{k_r}\bar{P}_{l_s} - P_{k_{r_0}}\bar{P}_{l_{s_0}}\right)^{\alpha}}{\left(P_{k_{r-1}}\bar{P}_{l_{s-1}}\right)^{\alpha}} \\
&\le \tfrac{M_{r_0 s_0}}{\left(P_{k_{r-1}}\bar{P}_{l_{s-1}}\right)^{\alpha}} + \varepsilon \left(\tfrac{P_{k_r}\bar{P}_{l_s}}{P_{k_{r-1}}\bar{P}_{l_{s-1}}}\right)^{\alpha} \\
&= \tfrac{M_{r_0 s_0}}{\left(P_{k_{r-1}}\bar{P}_{l_{s-1}}\right)^{\alpha}} + \varepsilon Q^{\alpha}_r \bar{Q}^{\alpha}_s \le \tfrac{M_{r_0 s_0}}{\left(P_{k_{r-1}}\bar{P}_{l_{s-1}}\right)^{\alpha}} + \varepsilon K^2.
\end{aligned}$$

Since $P_{k_{r-1}} \to \infty$ and $P_{l_{s-1}} \to \infty$ as $r, s \to \infty$, in the sense of Pringsheim limit, it follows that $\frac{1}{P_n\bar{P}_m} \left|\left\{k \le P_n \text{ and } l \le \bar{P}_m : p_k\bar{p}_l \left|t_{klpq}(x) - L\right| \ge \varepsilon\right\}\right| \to 0$ as $m, n \to \infty$. □

Theorem 3.6. *Let $0 < \alpha \le \beta \le 1$ and t be a positive real number, then $\tilde{R}^{\alpha}_{\theta_{rs}}(t) \subseteq \tilde{R}^{\beta}_{\theta_{rs}}(t)$.*

Proof. Let $x = (x_{ij}) \in \tilde{R}^{\alpha}_{\theta_{rs}}(t)$. Then given $\alpha > 0$ and $\beta > 0$ such that $0 < \alpha \le \beta \le 1$ and a positive real number t we write

$$\frac{1}{H^{\beta}_{rs}} \sum_{(k,l)\in I_{rs}} p_k\bar{p}_l \left|t_{klpq}(x) - L\right|^t \le \frac{1}{H^{\alpha}_{rs}} \sum_{(k,l)\in I_{rs}} p_k\bar{p}_l \left|t_{klpq}(x) - L\right|^t$$

and hence we obtain the result. □

We have the following corollary as a consequence of previous theorem.

Corollary 3.7. *Let $0 < \alpha \le \beta \le 1$ and t be a positive real number. Then*

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2016, VOLUME 6, ISSUE 1, p.39-46

(1) If $\alpha = \beta$*, then* $\tilde{R}^{\alpha}_{\theta_{rs}}(t) = \tilde{R}^{\beta}_{\theta_{rs}}(t)$.

(2) $\tilde{R}^{\alpha}_{\theta_{rs}}(t) \subseteq \tilde{R}_{\theta_{rs}}(t)$ *for each* $\alpha \in (0,1]$ *and* $0 < t < \infty$.

Theorem 3.8. *Let* α *and* β *be fixed real numbers such that* $0 < \alpha \leq \beta \leq 1$ *and* $0 < t < \infty$*. If* $I'_{rs} \subseteq I_{rs}$*, then* $\tilde{R}^{\alpha}_{\theta_{rs}} \subset \tilde{S}^{\beta}_{(R^2,\theta_{rs})}$.

Proof. Let $K_{P_{rs}}(\varepsilon) = |\{(k,l) \in I'_{rs} : p_k\bar{p}_l\,|t_{klpq}(x) - L| \geq \varepsilon\}|$. Suppose that $x \in \tilde{R}^{\alpha}_{\theta_{rs}}$. Then for each p and q

$$P - \lim_{rs} \frac{1}{H_{rs}} \sum_{(k,l)\in I_{rs}} p_k\bar{p}_l\,|t_{klpq}(x) - L| = 0.$$

Since

$$\begin{aligned}
&\tfrac{1}{H^{\alpha}_{rs}} \sum_{(k,l)\in I_{rs}} p_k\bar{p}_l\,|t_{klpq}(x) - L| \geq \tfrac{1}{H^{\alpha}_{rs}} \sum_{(k,l)\in I'_{rs}} p_k\bar{p}_l\,|t_{klpq}(x) - L| \\
&= \tfrac{1}{H^{\beta}_{rs}} \sum_{(k,l)\in I'_{rs}} p_k\bar{p}_l\,|t_{klpq}(x) - L| + \tfrac{1}{H^{\alpha}_{rs}} \sum_{(k,l)\in I'_{rs}} p_k\bar{p}_l\,|t_{klpq}(x) - L| \\
&\geq \tfrac{1}{H^{\alpha}_{rs}} \sum_{(k,l)\in I'_{rs}} p_k\bar{p}_l\,|t_{klpq}(x) - L| = \tfrac{1}{H^{\alpha}_{rs}}\,|K_{P_{rs}}(\varepsilon)|\,\varepsilon \\
&\geq \tfrac{1}{H^{\beta}_{rs}}\,|K_{P_{rs}}(\varepsilon)|\,\varepsilon
\end{aligned}$$

for all p and q. This implies that $x \in \tilde{S}^{\alpha}_{(R^2,\theta_{rs})}$. □

Corollary 3.9. *Let* α *be fixed real numbers such* $0 < \alpha \leq 1$

(1) If a sequence is strongly $\tilde{R}^{\alpha}_{\theta_{rs}}$*-summable sequence of order* α *to* L*, then it is* $\tilde{S}^{\alpha}_{(R^2,\theta_{rs})}$*-statistically convergent of order* α *to* L*, i.e.,* $\tilde{R}^{\alpha}_{\theta_{rs}} \subset \tilde{S}^{\alpha}_{(R^2,\theta_{rs})}$.

(2) $\tilde{R}^{\alpha}_{\theta_{rs}} \subset \tilde{S}_{(R^2,\theta_{rs})}$ *for* $0 < \alpha \leq 1$.

Theorem 3.10. *The following statements are true:*

(1) If $p_k < 1$ *and* $\bar{p}_l < 1$ *for all* $k,l \in \mathbb{N}$*, then* $\tilde{W}^{\alpha}_{\theta_{rs}} \subset \tilde{R}^{\alpha}_{\theta_{rs}}$ *with* $\tilde{W}^{\alpha}_{\theta_{rs}}$*-*$P$*-*$\lim x = \tilde{R}^{\alpha}_{\theta_{rs}}$*-*$P$*-*$\lim x = L$.

(2) If $p_k > 1$*,* $\bar{p}_l > 1$ *for all* $k,l \in \mathbb{N}$*, and* $\frac{H_r}{h_r}$ *and* $\frac{\bar{H}_s}{\bar{h}_s}$ *are upper bounded, then* $\tilde{R}^{\alpha}_{\theta_{rs}} \subset \tilde{W}^{\alpha}_{\theta_{rs}}$ *with* $\tilde{R}^{\alpha}_{\theta_{rs}}$*-*$P$*-*$\lim x = \tilde{W}^{\alpha}_{\theta_{rs}}$*-*$P$*-*$\lim x = L$.

Proof. The proof can be done in a similar manner as in [21], Theorem 3.4. □

Theorem 3.11. *The following statements are true:*

(1) If $p_k \leq 1$ *and* $\bar{p}_l \leq 1$ *for all* $k,l \in \mathbb{N}$*, then* $\tilde{S}_{\theta_{rs}} \subseteq \tilde{S}_{(R^2,\theta_{rs})}$ *with* $\tilde{S}_{\theta_{rs}}$*-*P*-*$\lim x = \tilde{S}_{(R^2,\theta_{rs})}$*-*$P$*-*$\lim x = L$.

(2) If $p_k \geq 1$*,* $\bar{p}_l \geq 1$ *for all* $k,l \in \mathbb{N}$*, and* $\frac{H_r}{h_r}$ *and* $\frac{\bar{H}_s}{\bar{h}_s}$ *are upper bounded, then* $\tilde{S}_{(R^2,\theta_{rs})} \subseteq \tilde{S}_{\theta_{rs}}$ *with* $\tilde{S}_{(R^2,\theta_{rs})}$*-*$P$*-*$\lim x = \tilde{S}_{\theta_{rs}}$*-*$P$*-*$\lim x = L$.

Proof. The proof can be done in a similar manner as in the proof of Theorem 3.7 in [21]. □

Theorem 3.12. *If* $\liminf_{rs} \frac{H^{\alpha}_{rs}}{P_{k_r}\bar{P}_{l_s}} > 0$ *then* $\tilde{S}_{R^2} \subseteq \tilde{S}^{\alpha}_{(R^2,\theta)}$.

Proof. For a given $\varepsilon > 0$, we have

$$\{(kl) \in I'_{rs} : p_k\bar{p}_l\,|x_{kl} - L| \geq \varepsilon\} \subset \left\{k \leq P_{k_r} \text{ and } l \leq \bar{P}_{l_s} : p_k\bar{p}_l\,|x_{kl} - L| \geq \varepsilon\right\}.$$

Therefore,

$$\begin{aligned}
&\tfrac{1}{P_{k_r}\bar{P}_{l_s}}\left|\left\{k \leq P_{k_r} \text{ and } l \leq \bar{P}_{l_s} : p_k\bar{p}_l\,|x_{kl} - L| \geq \varepsilon\right\}\right| \\
&\geq \tfrac{1}{P_{k_r}\bar{P}_{l_s}}\,|\{(kl) \in I'_{rs} : p_k\bar{p}_l\,|x_{kl} - L| \geq \varepsilon\}| \\
&= \tfrac{H^{\alpha}_{rs}}{P_{k_r}\bar{P}_{l_s}}\tfrac{1}{H^{\alpha}_{rs}}\,|\{(kl) \in I'_{rs} : p_k\bar{p}_l\,|x_{kl} - L| \geq \varepsilon\}|.
\end{aligned}$$

Since $\liminf_{rs} \frac{H^{\alpha}_{rs}}{P_{k_r}\bar{P}_{l_s}} > 0$, then we have the result by taking the Pringsheim limit as $r \to \infty$. □

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2016, VOLUME 6, ISSUE 1, p.39-46

Theorem 3.13. *Let* $\theta_{rs} = \{(k_r, l_s)\}$ *and* $\theta'_{rs} = \{(u_r, v_s)\}$ *be two double lacunary sequences and let* α, β *be such that* $0 < \alpha \leq \beta \leq 1$ *and* $I'_{rs} \subset J'_{rs}$ *for all* $r,\ s \in N$. *If*

$$\liminf_{rs} \frac{H^{\alpha}_{rs}}{L^{\beta}_{rs}} > 0 \tag{3.2}$$

then $\tilde{S}^{\beta}_{(R^2,\theta')} \subset \tilde{S}^{\alpha}_{(R^2,\theta)}$.

Proof. Suppose that $I'_{rs} \subset J'_{rs}$ for all $r,\ s \in N$ and let (2) be satisfied. For given $\varepsilon > 0$ we have

$$\{(kl) \in I'_{rs} : p_k \bar{p}_l \, |x_{kl} - L| \geq \varepsilon\} \subseteq \{(kl) \in J'_{rs} : p_k \bar{p}_l \, |x_{kl} - L| \geq \varepsilon\}$$

and so

$$\begin{aligned} &\tfrac{1}{L^{\beta}_{rs}} \left|\{(kl) \in J'_{rs} : p_k \bar{p}_l \, |x_{kl} - L| \geq \varepsilon\}\right| \\ &\geq \tfrac{H^{\alpha}_{rs}}{L^{\beta}_{rs}} \tfrac{1}{H^{\alpha}_{rs}} \left|\{(kl) \in I'_{rs} : p_k \bar{p}_l \, |x_{kl} - L| \geq \varepsilon\}\right| \end{aligned}$$

for all $r,\ s \in N$. Now taking the Pringsheim limit as $r,\ s \to \infty$ in the last inequality and using (2) we get $\tilde{S}^{\beta}_{(R^2,\theta')} \subset \tilde{S}^{\alpha}_{(R^2,\theta)}$. □

References

[1] A. M. Alotaibi and C. Çakan, The Riesz convergence and Riesz core of double sequences, *J. Ineq. Appl.* **2012** (56) (2012).

[2] M. Başarır, On the strong almost convergence of double sequences, *Period. Math. Hungar.* **30** (3) (1995), 177-181.

[3] M. Başarır and Ş. Konca, Some sequence spaces derived by Riesz mean in a real 2-normed space, *Iranian J. Sci. Techn.* IJST **38A1** (2014), 25-33. http:/ijsts.shirazu.ac.ir

[4] M. Başarır and Ş. Konca, Some spaces of weighted lacunary B^m_n-convergent sequences defined by an Orlicz function and B^m_n-weighted lacunary statistical convergence, *Contemporary Anal. Appl. Math.* **2** (2) (2014), 254-269. http://dx.doi.org/10.18532/caam.24260

[5] M. Başarır and Ş. Konca, Weighted lacunary statistical convergence in locally solid Riesz spaces, *Filomat.* **28** 10 (2014), 2059-2067. doi. 10.2298/FIL 1410059B

[6] M. Başarır and Ş. Konca, Weighted Lacunary Statistical Convergence in a Locally Convex Topological Vector Space, *Iranian J. Sci. Tech.* (2016), in press.

[7] M. Başarır and Ş. Konca, On some lacunary almost convergent double sequence spaces and Banach limits, Hindawi Publishing Corporation, *Abst. Appl. Anal.* (2012) Article ID 426357, 17 pages. doi:10.1155/2012/426357.

[8] M. Başarır and Ş. Konca, On some spaces of lacunary convergent sequences derived by Norlund-type mean and weighted lacunary statistical convergence, *Arab J. Math. Sci.* **20** (2) (2014) 250-263. doi: 10.1016/j.ajmsc.2013.09.002.

[9] S. Bhunia, P., Das and S. Pal, Restricting statistical convergence, *Acta Math. Hungar.* **134** (1-2) (2012), 153-161.

[10] C. Çakan, B. Altay and H. Coşkun, Double lacunary density and lacunary statistical convergence of double sequences, *Studia Sci. Math. Hungar.* **47** (1) (2010) 35-45.

[11] M. Çınar, M. Karakaş and M. Et, On pointwise and uniform statistical convergence of order α for sequences of functions, *Fixed point Theory Appl.* **33** (2013) 11 pages.

[12] R. Çolak, *Statistical convergence of order* α, In. Modern Methods in Analysis and Its Applications, 121-129 Anamaya Pub., New Delhi 2010.

[13] R. Çolak and C. A. Bektaş, λ-statistical convergence of order α, *Acta Math. Sci. Ser. B.* **31** (3) (2011), 953-959.

[14] P. Das, S. Ghosal and S. Som, Statistical convergence of order α in probability, *Arab J. Math. Sci.* doi:10.1016/j.ajmsc.2014.06.002.

[15] M. Et, M. Çınar and M. Karakaş, On λ-statistical convergence of order α of sequences of function, *J. Ineq. Appl.* **204** (2013), 8 pages.

[16] H. Fast, Sur la convergence statistique, *Colloq. Math.* **2** (3-4) (1951), 241-244.
[17] J. A. Fridy, On statistical convergence, *Analysis.* **5** (1985), 301-313.
[18] J. A. Fridy and C. Orhan, Lacunary statistical convergence, *Pacific J. Math.* **160** (1993), 43–51.
[19] S. Ghosal, Weighted statistical convergence of order α and its applications, *J. Egyptian Math. Soc.* **24** (2016), 60–67.
[20] Ş. Konca and M. Başarır, On some spaces of almost lacunary convergent sequences derived by Riesz mean and weighted almost lacunary statistical convergence in a real n-normed space, *J. Ineq. Appl.* **2014** 81 (2014), 11 pages. doi: 10.1186/1029-242X-2014-81.
[21] Ş. Konca and M. Başarır, Riesz Lacunary Almost Convergent Double Sequence Spaces Defined by Orlicz Functions, *Facta Universitatis Ser. Math. Inform.* **31** (1) (2016), 169-186.
[22] Ş. Konca, Weighted Lacunary Statistical Convergence of Double Sequences in Locally Solid Riesz Spaces, *Filomat.* **30** (3) (2016), 621–629. DOI 10.2298/FIL1603621K
[23] G. G. Lorentz, A contribution to the theory of divergent sequences, *Acta Mathematica.* **80** (1948), 167-190.
[24] I. J. Maddox, On strong almost convergence, *Math. Proc. Camb. Philos. Soc.* **85** (2) (1979), 345-350.
[25] F. Moricz and B. E. Rhoades, Almost convergence of double sequences and strong regularity of summability matrices, *Math. Proc. Cambridge Philos. Soc.* **104** (2) (1988), 283-294.
[26] M. Mursaleen and S. A. Mohiuddine, Banach limit and some new spaces of double sequences, *Turkish J. Math.* **36**, 1 (2012), 121-130.
[27] M. Mursaleen and E. Savaş, Almost regular matrices for double sequences, *Studia Sci. Math. Hungar.* A Quarterly of the Hungarian Academy of Sciences, **40** 1-2 (2003), 205-212.
[28] M. Mursaleen, λ-statistical convergence, *Math. Slovaca.* **50** (1) (2000), 111-115.
[29] M. Mursaleen and C. Belen, On statistical lacunary summability of double sequences, *Filomat.* **28** 2 (2014) 231-239.
[30] M. Mursaleen and O. H. Edely, Statistical convergence of double sequences, *J. Math. Anal. Appl.* **288** (1), 223-231 (2003).
[31] M. Mursaleen, V. Karakaya, M. Ertürk and F. Gürsoy, Weighted statistical convergence and its application to Korovkin type approximation theorem, *Appl. Math. Comput.* **218** (2012), 9132-9137.
[32] R. F. Patterson and E. Savaş, Lacunary statistical convergence of double sequences, *Math. Commun.* **10** (2005) 55-61.
[33] A. Pringsheim, Zur Theorie der zweifach unendlichen Zahlenfolgen, *Math. Ann.* **53** (3) (1900), 289-321.
[34] G. M. Robison, Divergent double sequences and series, *Trans. Am. Math. Soc.* **28** (1926), 50-73.
[35] T. Salat, On statistical convergence of real numbers, *Math. Slovaca.* **30** (1980), 139-150.
[36] E. Savaş, Double almost statistical convergence of order α, *Adv. Difference Equ.* **2013** 62 (2013).
[37] E. Savaş, Double almost lacunary statistical convergence of order α, *Adv. Difference Equ.* **2013** 254 (2013).
[38] I. J. Schoenberg, The integrability of certain functions and related summability methods, *Amer. Math. Monthly.* **66** (1959), 361-375.
[39] H. Steinhaus, Sur la convergence ordinate et la convergence asymptotique, *Colloq. Math.* **2** (1951), 73-84.
[40] H. Şengül and M. Et, On lacunary statistical convergence of order α, *Acta Math. Sci.* **34B** (2) (2014), 473-482.

[1,3]Department of Mathematics, Bitlis Eren University, 13000, Bitlis, Turkey
E-mail address: skonca@beu.edu.tr, selmanekin@hotmail.com

[2]Department of Mathematics, Istanbul Commerce University, 34445, Beyoglu, Istanbul, Turkey
E-mail address: esavas@ticaret.edu.tr

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2016, VOLUME 6, ISSUE 1, p.39-46