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POSITIVE SOLUTIONS OF A NONLINEAR THREE-POINT EIGENVALUE PROBLEM WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this paper, we study the existence of positive solutions of a three-point integral boundary value problem (BVP) for the following second-order differential equation

$$u''(t) + \lambda a(t) f(u(t)) = 0, \quad 0 < t < 1,$$

$$u'(0) = 0, \; u(1) = \alpha \int_0^\eta u(s)ds,$$

where $\lambda > 0$ is a parameter, $0 < \eta < 1$, $0 < \alpha < \frac{1}{\eta}$. By using the properties of the Green's function and Krasnoselskii's fixed point theorem on cones, the eigenvalue intervals of the nonlinear boundary value problem are considered, some sufficient conditions for the existence of at least one positive solutions are established.

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1. Introduction

In this work, we study the existence of positive solutions of a three-point integral boundary value problem (BVP) for the following second-order differential equation:

$$u''(t) + \lambda a(t) f(u(t)) = 0, \; t \in (0, 1), \tag{1.1}$$

$$u'(0) = 0, \; u(1) = \alpha \int_0^\eta u(s)ds, \tag{1.2}$$

where $0 < \eta < 1$ and $0 < \alpha < \frac{1}{\eta}$, λ is a positive parameter, and

(H1) $f \in C([0, \infty), [0, \infty))$;
(H2) $a \in C([0, 1], [0, \infty))$ and there exists $t_0 \in [0, \eta]$ such that $a(t_0) > 0$.

The study of the existence of solutions of multi-point boundary value problems for linear second-order ordinary differential equations was initiated by II'in and Moiseev [12]. Then Gupta [8] studied three-point boundary value problems for nonlinear second-order ordinary differential equations. Since then, the existence of positive solutions for nonlinear second order three-point boundary-value problems has been studied by many authors by using the fixed point theorem, nonlinear alternative of the Leray-Schauder approach, or coincidence degree theory. We refer the reader to [1], [2], [4], [6], [7], [10], [11], [17]-[27], [29], [30], [32]-[36] and the references therein.

Liu [20] proved the existence of single and multiple positive solutions for the three-point boundary value problem (BVP)

$$u''(t) + a(t)f(u(t)) = 0,\ t \in (0,1), \tag{1.3}$$

$$u'(0) = 0,\ u(1) = \beta u(\eta), \tag{1.4}$$

where $0 < \eta < 1$ and $0 < \beta < 1$.

Recently, Ma [28] studied the second-order three-point boundary value problem (BVP)

$$u''(t) + \lambda a(t)f(u(t)) = 0,\ t \in (0,1), \tag{1.5}$$

$$u(0) = \beta u(\eta),\ u(1) = \alpha u(\eta), \tag{1.6}$$

where $\lambda > 0$ is a parameter, $\alpha > 0$, $\beta > 0$, $0 < \eta < 1$, $a \in C([0,1],[0,\infty))$, $f \in C([0,\infty),[0,\infty))$ and there exists $x_0 \in (0,1)$ such that $a(x_0) > 0$. She obtained the existence of single and multiple positive solutions by using Krasnoselskii's fixed point theorem in cones [16].

Boundary value problems with integral boundary conditions for ordinary differential equations represent a very interesting and important class of problems, and arise in the study of various physical, biological and chemical processes, such as heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics. They include two, three, multi-point and nonlocal BVPs as special cases. The existence of positive solutions for such class of problems has attracted much attention (see [3], [5], [13]-[15], [31], [34], [37] and the references therein).

In [34], Tariboon and Sitthiwirattham investigated the existence of positive solutions of the following three-point integral boundary value problem (BVP)

$$u''(t) + a(t)f(u(t)) = 0,\ t \in (0,1), \tag{1.7}$$

$$u(0) = 0,\ u(1) = \alpha \int_0^\eta u(s)ds, \tag{1.8}$$

where $0 < \eta < 1$ and $0 < \alpha < \frac{2}{\eta^2}$, $f \in C([0,\infty),[0,\infty))$, $a \in C([0,1],[0,\infty))$ and there exists $t_0 \in [\eta,1]$ such that $a(t_0) > 0$. They showed the existence of at least one positive solution if f is either superlinear or sublinear by applying Krasnoselskii's fixed point theorem in cones [16].

In [9], by using Leggett-Williams fixed-point theorem, the authors considered the multiplicity of positive solutions of the following three-point integral boundary value problem (BVP)

$$u''(t) + f(t,u(t)) = 0,\ t \in (0,T), \tag{1.9}$$

$$u(0) = \beta u(\eta),\ u(T) = \alpha \int_0^\eta u(s)ds, \tag{1.10}$$

where $0 < \eta < T$, $0 < \alpha < \frac{2T}{\eta^2}$, $0 \le \beta < \frac{2T-\alpha\eta^2}{\alpha\eta^2-2\eta+2T}$, and $f \in C([0,T]\times[0,\infty),[0,\infty))$.

Motivated greatly by the above-mentioned excellent works, the aim of this paper is to establish some sufficient conditions for the existence of at least one positive solutions of the BVP (1.1) and (1.2). Our ideas are similar those used in [28], but a little different.

We firstly give the corresponding Green's function for the associated linear BVP and some of its properties. Moreover, by applying Krasnoselskii's fixed point theorem, we derive an interval of λ on which there exists a positive solution for the three-point integral boundary value problem (1.1) and (1.2).

As applications, some interesting examples are presented to illustrate the main results. The key tool in our approach is the following Krasnoselskii's fixed point theorem in a cone [16].

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Theorem 1.1. [16]. *Let E be a Banach space, and let $K \subset E$ be a cone. Assume that Ω_1, Ω_2 are open bounded subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \longrightarrow K$$

be a completely continuous operator such that either
(i) $\|Au\| \leq \|u\|, \;\; u \in K \cap \partial\Omega_1,$ *and* $\|Au\| \geq \|u\|, \;\; u \in K \cap \partial\Omega_2$; *or*
(ii) $\|Au\| \geq \|u\|, \;\; u \in K \cap \partial\Omega_1,$ *and* $\|Au\| \leq \|u\|, \;\; u \in K \cap \partial\Omega_2$
hold. Then A has a fixed point in $K \cap (\overline{\Omega}_2 \backslash\ \Omega_1)$.

2. Preliminaries

Lemma 2.1. *Let $\alpha\eta \neq 1$. Then for $y \in C([0,1], \mathbb{R})$, the problem*

$$u''(t) + y(t) = 0, \;\; t \in (0,1), \tag{2.1}$$

$$u'(0) = 0, \;\; u(1) = \alpha \int_0^\eta u(s)ds, \tag{2.2}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)y(s)ds, \tag{2.3}$$

where $G(t,s) : [0,1] \times [0,1] \to \mathbb{R}$ is the Green's function defined by

$$G(t,s) = \frac{1}{2(1-\alpha\eta)} \begin{cases} 2(1-s) - \alpha(\eta - s)^2 - 2(1-\alpha\eta)(t-s), & s \leq \min\{\eta, t\}; \\ 2(1-s) - \alpha(\eta - s)^2, & t \leq s \leq \eta; \\ 2(1-s) - 2(1-\alpha\eta)(t-s), & \eta \leq s \leq t; \\ 2(1-s), & \max\{\eta, t\} \leq s. \end{cases} \tag{2.4}$$

Proof. From (2.1), we have

$$u(t) = u(0) - \int_0^t (t-s)y(s)ds. \tag{2.5}$$

Integrating (2.5) from 0 to η, where $\eta \in (0,1)$, we have

$$\int_0^\eta u(s)ds = u(0)\eta - \frac{1}{2}\int_0^\eta (\eta - s)^2 y(s)ds.$$

Since

$$u(1) = u(0) - \int_0^1 (1-s)y(s)ds,$$

from $u(1) = \alpha \int_0^\eta u(s)ds$, we have

$$(1-\alpha\eta)u(0) = \int_0^1 (1-s)y(s)ds - \frac{\alpha}{2}\int_0^\eta (\eta - s)^2 y(s)ds.$$

Therefore,

$$u(0) = \frac{1}{1-\alpha\eta}\int_0^1 (1-s)y(s)ds - \frac{\alpha}{2(1-\alpha\eta)}\int_0^\eta (\eta-s)^2 y(s)ds,$$

from which it follows that

$$\begin{aligned} u(t) \;\; = \;\; & \frac{1}{1-\alpha\eta}\int_0^1 (1-s)y(s)ds - \frac{\alpha}{2(1-\alpha\eta)}\int_0^\eta (\eta-s)^2 y(s)ds \\ & - \int_0^t (t-s)y(s)ds. \end{aligned}$$

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If $t \leq \eta$, then

$$\begin{aligned}
u(t) &= \int_0^\eta \frac{1-s}{1-\alpha\eta} y(s)ds - \int_0^\eta \frac{\alpha(\eta-s)^2}{2(1-\alpha\eta)} y(s)ds + \int_\eta^1 \frac{1-s}{1-\alpha\eta} y(s)ds \\
&\quad - \int_0^t (t-s)y(s)ds \\
&= \int_0^t \frac{2(1-s)-\alpha(\eta-s)^2-2(1-\alpha\eta)(t-s)}{2(1-\alpha\eta)} y(s)ds + \int_\eta^1 \frac{1-s}{1-\alpha\eta} y(s)ds \\
&\quad + \int_t^\eta \frac{2(1-s)-\alpha(\eta-s)^2}{2(1-\alpha\eta)} y(s)ds \\
&= \int_0^1 G(t,s)y(s)ds.
\end{aligned}$$

If $t \geq \eta$, then

$$\begin{aligned}
u(t) &= \int_0^t \frac{1-s}{1-\alpha\eta} y(s)ds - \int_0^\eta \frac{\alpha(\eta-s)^2}{2(1-\alpha\eta)} y(s)ds + \int_t^1 \frac{1-s}{1-\alpha\eta} y(s)ds \\
&\quad - \int_0^t (t-s)y(s)ds \\
&= \int_0^\eta \frac{2(1-s)-\alpha(\eta-s)^2-2(1-\alpha\eta)(t-s)}{2(1-\alpha\eta)} y(s)ds + \int_t^1 \frac{1-s}{1-\alpha\eta} y(s)ds \\
&\quad + \int_\eta^t \frac{(1-s)-(1-\alpha\eta)(t-s)}{1-\alpha\eta} y(s)ds \\
&= \int_0^1 G(t,s)y(s)ds.
\end{aligned}$$

This completes the proof. □

For convenience, we define

$$g(s) = \frac{1}{1-\alpha\eta}(1-s), \quad s \in [0,1].$$

For the Green's function $G(t,s)$, we have the following two lemmas.

Lemma 2.2. *Let $0 < \eta < 1$ and $0 < \alpha < \frac{1}{\eta}$. Then the Green's function in (2.4) satisfies*

$$0 \leq G(t,s) \leq g(s), \tag{2.6}$$

for each $s, t \in [0,1]$.

Proof. First of all, note that by (2.4) it follows that $G(t,s) \leq g(s)$ for any $(t,s) \in [0,1] \times [0,1]$.
Next, we will prove that $G(t,s) \geq 0$ for any $(t,s) \in [0,1] \times [0,1]$.
If $\eta \leq s \leq t$, then

$$\begin{aligned}
G(t,s) &= \frac{1}{1-\alpha\eta}\Big((1-s)-(1-\alpha\eta)(t-s)\Big) \\
&= \frac{1}{1-\alpha\eta}\Big((1-t)+\alpha\eta(t-s)\Big) \geq 0.
\end{aligned}$$

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If $t \le s \le \eta$, then

$$\begin{aligned} G(t,s) &= \frac{1}{2(1-\alpha\eta)}\Big(2(1-s)-\alpha(\eta-s)^2\Big) \\ &\ge \frac{1}{2(1-\alpha\eta)}\Big(2(1-s)-\frac{(\eta-s)^2}{\eta}\Big) \\ &= \frac{2\eta-\eta^2-s^2}{2\eta(1-\alpha\eta)} \\ &\ge \frac{2\eta-2\eta^2}{2\eta(1-\alpha\eta)} \\ &= \frac{1-\eta}{1-\alpha\eta} \\ &\ge 0. \end{aligned}$$

If $s \le \min\{\eta, t\}$, then

$$\begin{aligned} G(t,s) &= \frac{1}{2(1-\alpha\eta)}\Big(2(1-s)-\alpha(\eta-s)^2-2(1-\alpha\eta)(t-s)\Big) \\ &= \frac{1}{2(1-\alpha\eta)}\Big(2(1-t)-\alpha\eta^2-\alpha s^2+2\alpha\eta t\Big) \\ &\ge \frac{1}{2(1-\alpha\eta)}\Big(2(1-t)+\alpha\eta(t-\eta)+\alpha s(\eta-s)\Big). \end{aligned}$$

We distinguish the following two cases:
If $s \le \eta \le t$, then $G(t,s) \ge 0$.
If $s \le t \le \eta$, then

$$\begin{aligned} G(t,s) &= \frac{1}{2(1-\alpha\eta)}\Big(2(1-t)-\alpha\eta^2-\alpha s^2+2\alpha\eta t\Big) \\ &\ge \frac{1}{2(1-\alpha\eta)}\Big(2(1-t)-\alpha(\eta-t)^2\Big) \\ &\ge \frac{1}{2(1-\alpha\eta)}\Big(2(1-t)-\frac{(\eta-t)^2}{\eta}\Big) \\ &= \frac{2\eta-\eta^2-t^2}{2\eta(1-\alpha\eta)} \\ &\ge \frac{2\eta-2\eta^2}{2\eta(1-\alpha\eta)} \\ &= \frac{1-\eta}{1-\alpha\eta} \\ &\ge 0. \end{aligned}$$

The proof is completed. □

Lemma 2.3. *Let $0 < \eta < 1$ and $0 < \alpha < \frac{1}{\eta}$. Then for any $(t,s) \in [0,\eta] \times [0,1]$, $G(t,s) \ge \gamma g(s)$, where*

$$0 < \gamma = 1 - \eta < 1. \tag{2.7}$$

Proof. If $s = 1$, then by Lemma 2.2, the result follows. Now we suppose that $(t,s) \in [0,\eta] \times [0,1)$.

If $t \le s \le \eta$, then

$$\begin{aligned}
\frac{G(t,s)}{g(s)} &= \frac{2(1-s)-\alpha(\eta-s)^2}{2(1-s)} \\
&\ge \frac{2(1-s)-\frac{(\eta-s)^2}{\eta}}{2(1-s)} \\
&= \frac{2\eta-\eta^2-s^2}{2\eta(1-s)} \\
&\ge \frac{1-\eta}{1-s} \\
&\ge \gamma.
\end{aligned}$$

If $s \le t \le \eta$, then

$$\begin{aligned}
\frac{G(t,s)}{g(s)} &= \frac{2(1-s)-\alpha(\eta-s)^2-2(1-\alpha\eta)(t-s)}{2(1-s)} \\
&= \frac{2(1-t)-\alpha\eta^2-\alpha s^2+2\alpha\eta t}{2(1-s)} \\
&\ge \frac{2(1-t)-\alpha(\eta-t)^2}{2\eta(1-s)} \\
&\ge \frac{1-\eta}{1-s} \\
&\ge \gamma.
\end{aligned}$$

If $t \le \eta \le s$, then

$$\frac{G(t,s)}{g(s)} = 1 \ge \gamma.$$

Therefore,

$$G(t,s) \ge \gamma g(s), \quad (t,s) \in [0,\eta]\times[0,1].$$

The proof is completed. □

Let $E = C([0,1],\mathbb{R})$, and only the sup norm is used. It is easy to see that the BVP (1.1) and (1.2) has a solution $u = u(t)$ if and only if u is a fixed point of operator A_λ, where A_λ is defined by

$$\begin{aligned}
A_\lambda u(t) &= \frac{\lambda}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(u(s))ds - \frac{\lambda\alpha}{2(1-\alpha\eta)}\int_0^\eta (\eta-s)^2 a(s)f(u(s))ds \\
&\quad -\lambda\int_0^t (t-s)a(s)f(u(s))ds \\
&= \lambda\int_0^1 G(t,s)a(s)f(u(s))ds.
\end{aligned}$$

Denote

$$K = \left\{u \in E : u \ge 0, \min_{t\in[0,\eta]} u(t) \ge \gamma\|u\|\right\}, \tag{2.8}$$

where γ is defined in (2.7). It is obvious that K is a cone in E.

Lemma 2.4. *Assume that* (H1) *and* (H2) *hold,* $0 < \eta < 1$ *and* $0 < \alpha < \frac{1}{\eta}$*. Then the operator* $A_\lambda : K \to K$ *is completely continuous.*

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Proof. For $u \in K$, according to the definition of A_λ, Lemma 2.2 and Lemma 2.3, it is easy to prove that $A_\lambda K \subset K$. By the Ascoli-Arzela theorem, it is easy to show that $A_\lambda : K \to K$ is completely continuous. □

In what follows, for the sake of convenience, set

$$\Lambda_1 = \frac{1}{1-\alpha\eta}\int_0^1 (1-s)a(s)ds, \quad \Lambda_2 = \frac{\gamma}{2(1-\alpha\eta)}\int_0^\eta \Big(2(1-\eta)+\alpha(\eta^2-s^2)\Big)a(s)ds,$$
$$f_0 = \lim_{u\to 0+}\frac{f(u)}{u}, \quad f_\infty = \lim_{u\to\infty}\frac{f(u)}{u}.$$

3. Main results

In this section, we will state and prove our main results.

Theorem 3.1. *Suppose that* (H1) *and* (H2) *hold,* $0<\eta<1$ *and* $0<\alpha<\frac{1}{\eta}$. *If* $\Lambda_1 f_0 < \Lambda_2 f_\infty$, *then for each* $\lambda \in (\frac{1}{\Lambda_2 f_\infty}, \frac{1}{\Lambda_1 f_0})$, *the BVP* (1.1) *and* (1.2) *has at least one positive solution.*

Proof. Let $\lambda \in (\frac{1}{\Lambda_2 f_\infty}, \frac{1}{\Lambda_1 f_0})$, and choose $\epsilon > 0$ such that

$$\frac{1}{\Lambda_2(f_\infty-\epsilon)} \le \lambda \le \frac{1}{\Lambda_1(f_0+\epsilon)}. \tag{3.1}$$

By the definition of f_0, there exists $\rho_1 > 0$ such that

$$f(u) \le (f_0+\epsilon)u, \text{ for } u \in (0,\rho_1]. \tag{3.2}$$

Let $\Omega_{\rho_1} = \{u \in E : \|u\| < \rho_1\}$, then from (3.1), (3.2) and Lemma 2.2, for any $u \in K \cap \partial\Omega_{\rho_1}$, we have

$$\begin{aligned} A_\lambda u(t) &= \lambda \int_0^1 G(t,s)a(s)f(u(s))ds \\ &\le \lambda \int_0^1 g(s)a(s)f(u(s))ds \\ &= \frac{\lambda}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(u(s))ds \\ &\le \frac{\lambda}{1-\alpha\eta}\int_0^1 (1-s)a(s)(f_0+\epsilon)u(s)ds \\ &\le \lambda\Lambda_1(f_0+\epsilon)\|u\| \le \|u\|, \end{aligned}$$

which yields

$$\|A_\lambda u\| \le \|u\|, \quad \text{for } u \in K \cap \partial\Omega_{\rho_1}. \tag{3.3}$$

Further, by the definition of f_∞, there exists $\widehat{\rho}_2 > 0$ such that

$$f(u) \ge (f_\infty-\epsilon)u, \text{ for } u \in [\widehat{\rho}_2, \infty). \tag{3.4}$$

Now, set $\rho_2 = \max\left\{2\rho_1, \frac{\widehat{\rho}_2}{\gamma}\right\}$ and $\Omega_{\rho_2} = \{u \in E : \|u\| < \rho_2\}$. Then $u \in K \cap \partial\Omega_{\rho_2}$ implies that

$$u(t) \ge \gamma\|u\| \ge \widehat{\rho}_2, \quad t \in [0,\eta],$$

and so,

$$
\begin{aligned}
A_\lambda u(\eta) &= \lambda\int_0^1 G(\eta,s)a(s)f(u(s))ds\\
&\geq \lambda\int_0^\eta G(\eta,s)a(s)f(u(s))ds\\
&\geq \lambda\int_0^\eta G(\eta,s)a(s)(f_\infty-\epsilon)u(s)ds\\
&\geq \lambda\gamma(f_\infty-\epsilon)\|u\|\int_0^\eta G(\eta,s)a(s)ds\\
&= \lambda\gamma(f_\infty-\epsilon)\|u\|\int_0^\eta \frac{2(1-s)-\alpha(\eta-s)^2-2(1-\alpha\eta)(\eta-s)}{2(1-\alpha\eta)}a(s)ds\\
&= \lambda(f_\infty-\epsilon)\|u\|\frac{\gamma}{2(1-\alpha\eta)}\int_0^\eta \Big(2(1-\eta)+\alpha(\eta^2-s^2)\Big)a(s)ds\\
&= \lambda\Lambda_2(f_\infty-\epsilon)\|u\|\geq\|u\|.
\end{aligned}
$$

This implies that

$$\|A_\lambda u\|\geq\|u\|,\quad \text{for } u\in K\cap\partial\Omega_{\rho_2}. \tag{3.5}$$

Therefore, from (3.3), (3.5) and Theorem 1.1, it follows that A_λ has a fixed point u with $\rho_1\leq\|u\|\leq\rho_2$ in $K\cap(\overline{\Omega}_{\rho_2}\backslash\Omega_{\rho_1})$, which is a desired positive solution of the BVP (1.1) and (1.2). □

By Theorem 3.1 we can easily obtain the following corollary.

Corollary 3.2. *Assume that* (H1) *and* (H2) *hold,* $0<\eta<1$ *and* $0<\alpha<\frac{1}{\eta}$. *Then we have*

(1) If $f_0=0,\ f_\infty=\infty$, *then for each* $\lambda\in(0,\infty)$, *the BVP* (1.1) *and* (1.2) *has at least one positive solution.*

(2) If $f_\infty=\infty,\ 0<f_0<\infty$, *then for each* $\lambda\in(0,\frac{1}{\Lambda_1 f_0})$, *the BVP* (1.1) *and* (1.2) *has at least one positive solution.*

(3) If $f_0=0,\ 0<f_\infty<\infty$, *then for each* $\lambda\in(\frac{1}{\Lambda_2 f_\infty},\infty)$, *the BVP* (1.1) *and* (1.2) *has at least one positive solution.*

Theorem 3.3. *Suppose that* (H1) *and* (H2) *hold,* $0<\eta<1$ *and* $0<\alpha<\frac{1}{\eta}$. *If* $\Lambda_1 f_\infty<\Lambda_2 f_0$, *then for each* $\lambda\in(\frac{1}{\Lambda_2 f_0},\frac{1}{\Lambda_1 f_\infty})$, *the BVP* (1.1) *and* (1.2) *has at least one positive solution.*

Proof. Let $\lambda\in(\frac{1}{\Lambda_2 f_0},\frac{1}{\Lambda_1 f_\infty})$, and choose $\epsilon>0$ such that

$$\frac{1}{\Lambda_2(f_0-\epsilon)}\leq\lambda\leq\frac{1}{\Lambda_1(f_\infty+\epsilon)}. \tag{3.6}$$

By the definition of f_0, there exists $\rho_1>0$ such that

$$f(u)\geq(f_0-\epsilon)u,\ \text{for } u\in(0,\rho_1]. \tag{3.7}$$

Let $\Omega_{\rho_1}=\{u\in E:\|u\|<\rho_1\}$. Hence, for any $u\in K\cap\partial\Omega_{\rho_1}$, from (3.6), (3.7), we get

$$
\begin{aligned}
A_\lambda u(\eta) &\geq \lambda\int_0^\eta G(\eta,s)a(s)f(u(s))ds\\
&\geq \lambda\int_0^\eta G(\eta,s)a(s)(f_0-\epsilon)u(s)ds\\
&\geq \lambda\Lambda_2(f_0-\epsilon)\|u\|\geq\|u\|.
\end{aligned}
$$

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Therefore

$$\|A_\lambda u\| \geq \|u\|, \quad \text{for } u \in K \cap \partial\Omega_{\rho_1}. \tag{3.8}$$

By the definition of f_∞, there exists $\rho_0 > 0$ such that

$$f(u) \leq (f_\infty + \epsilon)u, \ \text{for } u \in [\rho_0, \infty). \tag{3.9}$$

Next, we consider two cases:

If f is bounded. Let $f(u) \leq L$ for all $u \in [0, \infty)$. Set $\rho_2 = \max\{2\rho_1, \lambda\Lambda_1 L\}$ and $\Omega_{\rho_2} = \{u \in E : \|u\| < \rho_2\}$, then for $u \in K \cap \partial\Omega_{\rho_2}$, we have

$$\begin{aligned} A_\lambda u(t) &\leq \frac{\lambda}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(u(s))ds \\ &\leq \lambda L\Lambda_1 \leq \rho_2 \leq \|u\|. \end{aligned}$$

Therefore

$$\|A_\lambda u\| \leq \|u\|, \quad \text{for } u \in K \cap \partial\Omega_{\rho_2}. \tag{3.10}$$

If f is unbounded, then from $f \in C([0,\infty),[0,\infty))$, we know that there is ρ_2: $\rho_2 \geq \max\left\{2\rho_1, \gamma^{-1}\rho_0\right\}$ such that

$$f(u) \leq f(\rho_2), \ \text{for } u \in [0, \rho_2]\,. \tag{3.11}$$

Let $\Omega_{\rho_2} = \{u \in E : \|u\| < \rho_2\}$, then for $u \in K \cap \partial\Omega_{\rho_2}$, we have

$$\begin{aligned} A_\lambda u(t) &\leq \frac{\lambda}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(u(s))ds \\ &\leq \frac{\lambda}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(\rho_2)ds \\ &\leq \lambda\Lambda_1\rho_2(f_\infty+\epsilon) \leq \rho_2 = \|u\|. \end{aligned}$$

Thus

$$\|A_\lambda u\| \leq \|u\|, \quad \text{for } u \in K \cap \partial\Omega_{\rho_2}. \tag{3.12}$$

It follows from Theorem 1.1, that A_λ has a fixed point in $K \cap (\overline{\Omega}_{\rho_2} \backslash \Omega_{\rho_1})$, such that $\rho_1 \leq \|u\| \leq \rho_2$. □

From Theorem 3.3, we have

Corollary 3.4. *Assume that* (H1) *and* (H2) *hold,* $0 < \eta < 1$ *and* $0 < \alpha < \frac{1}{\eta}$. *Then we have*

1. *If* $f_0 = \infty,\ f_\infty = 0$, *then for each* $\lambda \in (0, \infty)$, *the BVP* (1.1) *and* (1.2) *has at least one positive solution.*
2. *If* $f_\infty = 0,\ 0 < f_0 < \infty$, *then for each* $\lambda \in (\frac{1}{\Lambda_2 f_0}, \infty)$, *the BVP* (1.1) *and* (1.2) *has at least one positive solution.*
3. *If* $f_0 = \infty,\ 0 < f_\infty < \infty$, *then for each* $\lambda \in (0, \frac{1}{\Lambda_1 f_\infty})$, *the BVP* (1.1) *and* (1.2) *has at least one positive solution.*

4. Examples

In this section we present some examples to illustrate our main results.

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Example 4.1. Consider the boundary value problem

$$u''(t) + tu^p = 0, \quad 0 < t < 1, \tag{4.1}$$

$$u'(0) = 0, \quad u(1) = 2\int_0^{\frac{1}{4}} u(s)ds. \tag{4.2}$$

Set $\alpha = 2$, $\eta = 1/4$, $a(t) = t$, $f(u) = u^p$ ($p \in (0,1) \cup (1,\infty)$). We can show that $0 < \alpha = 2 < 4 = 1/\eta$. Now we consider the existence of positive solutions of the problem (4.1) and (4.2) in two cases.
Case 1: $p > 1$. In this case, $f_0 = 0$, $f_\infty = \infty$. Then, by Corollary 3.2, the BVP (4.1) and (4.2) has at least one positive solution.
Case 2: $p \in (0,1)$. In this case, $f_0 = \infty$, $f_\infty = 0$. Then, by Corollary 3.4, the BVP (4.1) and (4.2) has at least one positive solution.

Example 4.2. Consider the boundary value problem

$$u''(t) + \frac{aue^{2u}}{b + e^u + e^{2u}} = 0, \quad 0 < t < 1, \tag{4.3}$$

$$u'(0) = 0, \quad u(1) = 2\int_0^{\frac{1}{3}} u(s)ds. \tag{4.4}$$

Set $\alpha = 2$, $\eta = 1/3$, $a(t) \equiv 1$, $f(u) = (aue^{2u})/(b+e^u+e^{2u})$. we consider the existence of positive solutions of the problem (4.3) and (4.4) into the following two cases.

Case 1: If $a = 5$, $b = 8$, then $f_0 = \frac{1}{2}$, $f_\infty = 5$. By calculating, it is easy to obtain that $0 < \alpha = 2 < 3 = 1/\eta$, $\gamma = \frac{2}{3}$. Again

$$\begin{aligned}
\Lambda_1 &= \frac{1}{1-\alpha\eta}\int_0^1 (1-s)a(s)ds = \frac{3}{2}, \\
\Lambda_2 &= \frac{\gamma}{2(1-\alpha\eta)}\int_0^\eta \Big(2(1-\eta) + \alpha(\eta^2 - s^2)\Big)a(s)ds = \frac{40}{81}, \\
\Lambda_1 f_0 &= \frac{3}{4}, \quad \Lambda_2 f_\infty = \frac{200}{81}.
\end{aligned}$$

By Theorem 3.1, we know that for any $\lambda \in (\frac{81}{200}, \frac{4}{3})$, the BVP (4.3) and (4.4) has at least one positive solution $u \in C[0,1]$.

Case 2: If $a = 5$, $b = -2$, then $f_0 = \infty$, $f_\infty = 5$ and $\Lambda_1 f_\infty = \frac{15}{2}$.

Therefore, by Corollary 3.4, we know that for any $\lambda \in (0, \frac{2}{15})$, the BVP (4.3) and (4.4) has at least one positive solution $u \in C[0,1]$.

Example 4.3. Consider the boundary value problem

$$u''(t) + \frac{1}{5}u(1 - \frac{1}{1+u^2}) = 0, \quad 0 < t < 1, \tag{4.5}$$

$$u'(0) = 0, \quad u(1) = \int_0^{\frac{1}{2}} u(s)ds, \tag{4.6}$$

where $\alpha = 1$, $\eta = 1/2$, $a(t) \equiv \frac{1}{5}$, $f(u) = u(1 - \frac{1}{1+u^2})$. By calculating, we have $0 < \alpha = 1 < 2 = 1/\eta$, $\gamma = \frac{1}{2}$, $f_0 = 0$, $f_\infty = 1$. Again

$$\Lambda_2 = \frac{\gamma}{2(1-\alpha\eta)}\int_0^\eta \Big(2(1-\eta) + \alpha(\eta^2 - s^2)\Big)a(s)ds = \frac{7}{120}.$$

Hence, by Corollary 3.2, for any $\lambda \in (\frac{120}{7}, \infty)$, the BVP (4.5) and (4.6) has at least one positive solution $u \in C[0,1]$.

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