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Almost periodic mild solutions for stochastic delay functional differential equations driven by a fractional Brownian motion

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Abstract

In this paper we investigate the existence and stability of quadratic-mean almost periodic mild solutions to stochastic delay functional differential equations driven by fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$, under some suitable assumptions, by means of semigroup of operators and fixed point method.

Key words and phrases: Fractional Brownian motion, Stochastic delay functional differential equations, Quadratic-mean almost periodic solution
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1 Introduction

The fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$ is a zero mean Gaussian process $B^H = \{B_t^H, t \geq 0\}$ with covariance function

$$R_H(s,t) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}). \quad (1)$$

This process was introduced by Kolmogorov and later studied by Mandelbrot and Van Ness, where a stochastic integral representation in terms of a standard Brownian motion was obtained.

From (1) we deduce that $\mathbb{E}(|B_t^H - B_s^H|^2) = |t-s|^{2H}$ and, as a consequence, the trajectories of B^H are almost surely locally α-Hölder continuous for all $\alpha \in (0, H)$. Since B^H is not a semimartingale if $H \neq 1/2$ (see [8]), we cannot use the classical Itô theory to construct a stochastic calculus with respect to the fBm. Over the past years some new techniques have been developed in order to define stochastic integrals with respect to the fBm. In the case $H > \frac{1}{2}$ one can use a pathwise approach to define integrals with respect to the fractional Brownian motion, taking advantage of the results by Young. An alternative approach to define pathwise integrals with respect to an fBm with parameter $H > \frac{1}{2}$ is based on fractional calculus. This approach was introduced by Feyel and De la Pradelle in [5] and it was also developed by Zähle in [10].

We would like to mention that the theory for the stochastic delay functional differential equations driven by a Wiener process have recently been studied intensively (see e.g. [6], [7]).

As, for the same equations driven by a fractional Brownian motion (fBm), even much less has been done, as far as we know, only the works by Boufoussi and Hajji in [2] and Caraballo et al. in [4].

In the present paper, motived by [1], [3], [9] based on the semigroups of operators method and fixed points method, we investigate the existence and stability of quadratic-mean almost periodic mild solutions for stochastic delay functional differential equations

$$\begin{cases} dx(t) &= (Ax(t) + b(t, x(t), x_t))dt + \sigma_H(t)dB_Q^H(t), \quad t \in [0, T], \\ x(t) &= \varphi(t), \quad -r \leq t \leq 0, \quad r \geq 0, \end{cases} \tag{2}$$

where $B_Q^H = \{B_Q^H(t), t \in [0, T]\}$ is a fBm with Hurst index $H \in (\frac{1}{2}, 1)$. Some sufficient conditions on the operator A and the coefficient functions b, σ_H ensuring the existence and stability of quadratic-mean almost periodic mild solutions are presented.

The contents of the paper are as follow. In Section 2 some necessary preliminaries, the relating notations and useful lemmas are introduced. Section 3 contains the main results including some criteria ensuring the existence of quadratic mean almost periodic mild solutions. In Section 4, the stability of the quadratic mean almost periodic mild solution is further discussed. Finally, an example is given to illustrate our results in Section 5.

2 Preliminaries

In this section we introduce some notations, definitions, a technical lemmas and preliminary fact which are used in what follows.

Let $T > 0$ and denote by Υ the linear space of $\mathbb{R}$-valued step functions on $[0, T]$, that is, $\phi \in \Upsilon$ if

$$\phi(t) = \sum_{i=1}^{n-1} z_i \chi_{[t_i, t_{i+1})}(t),$$

where $t \in [0, t]$, $x_i \in \mathbb{R}$ and $0 = t_1 < t_2 < \cdot < t_n = T$. For $\phi \in \Upsilon$ its Wiener integral with respect to B^H as

$$\int_0^T \phi(s) dB^H(s) = \sum_{i=1}^{n-1} z_i \Big(B^H(t_{i+1}) - B^H(t_i) \Big).$$

Let $\mathcal{H}$ be the Hilbert space defined as the closure of Υ with respect to the scalar product $\langle \chi_{[0,t]}, \chi_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s)$. Then the mapping

$$\phi = \sum_{i=1}^{n-1} z_i \chi_{[t_i, t_{i+1})} \mapsto \int_0^T \phi(s) dB^H(s)$$

is an isometry between Υ and the linear space $span\{B^H(t), t \in [0, T]\}$, which can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos of the fBm $\overline{span}^{L^2(\Omega)}\{B^H(t), t \in [0, T]\}$ (see [8]). The image of an element $\phi \in \mathcal{H}$ by this isometry is called the Wiener integral of ϕ with respect to B^H.

Let us now consider the Kernel

$$K_H(t, s) = c_H s^{\frac{1}{2} - H} \int_s^t (u - s)^{H - \frac{3}{2}} u^{H - \frac{1}{2}} du$$

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where $c_H = \left(\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}\right)^{\frac{1}{2}}$, where β denoting the Beta function, and $t > s$. It is not difficult to see that

$$\frac{\partial K_H}{\partial t}(t,s) = c_H \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}}.$$

Let $\mathcal{K}_H : \Upsilon \mapsto L^2([0,T])$ be the linear operator given by

$$\mathcal{K}_H \phi(s)(s) = \int_s^t \phi(t) \frac{\partial K_H}{\partial t}(t,s)dt.$$

Then $(\mathcal{K}_H \chi_{[0,t]})(s) = K_H(t,s)\chi_{[0,t]}(s)$ and $\mathcal{K}_H$ is an isometry between Υ and $L^2([0,T])$ that can be extended to $\mathcal{H}$.
Denoting $L^2_{\mathcal{H}}([0,T]) = \{\phi \in \mathcal{H}, \mathcal{K}_H\phi \in L^2([0,T])\}$, since $H > 1/2$, we have

$$L^{1/H}([0,T]) \subset L^2_{\mathcal{H}}([0,T]). \tag{3}$$

Moreover the following result hold:

Lemma 2.1 ([8]) *For $\phi \in L^{1/H}([0,T])$,*

$$H(2H-1)\int_0^T \int_0^T |\phi(r)||\phi(u)||r-u|^{2H-2}drdu \le c_H \|\phi\|^2_{L^{1/H}([0,T])}.$$

Let us now consider two separable Hilbert spaces $(U, |\cdot|_U, \langle\cdot,\cdot\rangle_U)$ and $(V, |\cdot|_V, \langle\cdot,\cdot\rangle_V)$. Let $L(V,U)$ denote the space of all bounded linear operator from V to U and $Q \in L(V,V)$ de a non-negative self adjoint operator. Denote by $L^0_Q(V,U)$ the space of all $\xi \in L(V,U)$ such that $\xi Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. the norm is given by

$$|\xi|^2_{L^0_Q(V,U)} = \left|\xi Q^{\frac{1}{2}}\right|^2_{HS} = tr(\xi Q \xi^*).$$

Then ξ is called a Q-Hilbert-Schmidt operator from V to U.

Let $\{B^H_n(t)\}_{n\in\mathbb{N}}$ be a sequence of two-side one-dimensional fBm mutually independent on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\{e_n\}_{n\in\mathbb{N}}$ be a complete orthonormal basis in V. Define the V-valued stochastic process $B^H_Q(t)$ by $B^H_Q(t) = \sum_{n=1}^{\infty} B^H_n(t)Q^{\frac{1}{2}e_n}$, $t \ge 0$.
If Q is a non-negative self-adjoint trace class operator, then this series converges in the space V, that is, it holds that $B^H_Q(t) \in L^2(\Omega, V)$. Then, we say that $B^H_Q(t)$ is a V-valued Q-cylindrical fBm with covariance operator Q.

Let $\psi : [0,T] \to L^0_Q(V,U)$ such that

$$\sum_{n=1}^{\infty} \|\mathcal{K}_H(\psi Q^{\frac{1}{2}})e_n\|_{L^2([0,T],U)} < \infty. \tag{4}$$

Definition 2.2 *Let $\psi : [0,T] \to L^0_Q(V,U)$ satisfy (4). Then, its stochastic integral with respect to the fBm B^H_Q is defined for $t \ge 0$ as*

$$\int_0^t \psi(s)dB^H_Q(s) := \sum_{n=1}^{\infty}\int_0^t \psi(s)Q^{\frac{1}{2}}e_n dB^H_n(s) = \sum_{n=1}^{\infty}\int_0^t \Big(\mathcal{K}_H(\psi Q^{\frac{1}{2}}e_n)\Big)(s)dW(s),$$

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where W is a Wiener process.

Notice that if

$$\sum_{n=1}^{\infty} \|\psi Q^{\frac{1}{2}} e_n\|_{L^{1/H}([0,T],U)} < \infty, \tag{5}$$

then in particular (4) holds, which follows immediately from (3).

The following lemma is proved in [8] and obtained as a simple application of Lemma 2.1.

Lemma 2.3 ([8]) *For any $\psi : [0,T] \to L_Q^0(V,U)$ such that (5) holds, and for any $\alpha, \beta \in [0,T]$ with $\alpha > \beta$,*

$$\mathbb{E}\Big|\int_\beta^\alpha \psi(s)dB_Q^H(s)\Big|_U^2 \le cH(2H-1)(\alpha-\beta)^{2H-1}\sum_{n=1}^{\infty}\int_\beta^\alpha |\psi Q^{\frac{1}{2}}e_n|_U^2 ds,$$

where $c = c(H)$. If in addition

$$\sum_{n=1}^{\infty} |\psi Q^{\frac{1}{2}} e_n|_U \ \text{ is uniformly convergent for } t \in [0,T], \tag{6}$$

then

$$\mathbb{E}\Big|\int_\beta^\alpha \psi(s)dB_Q^H(s)\Big|_U^2 \le cH(2H-1)(\alpha-\beta)^{2H-1}\int_\beta^\alpha |\psi(s)|_{L_Q^0(V,U)}^2 ds. \tag{7}$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space introduced above, $\mathcal{F}_t = \mathcal{F}_0$, for $t \ge 0$. The following useful definitions come from [1].

Definition 2.4 *1. A stochastic process $X : [0,T] \to L^2(\Omega, U)$ is said to be continuous, provided that, for any $s \in [0,T]$, $\lim_{t\to s} \mathbb{E}|X(t) - X(s)|_U^2 = 0$.*

2. A stochastic process $X : [0,T] \to L^2(\Omega, U)$ is said to be stochastically bounded, whenever $\lim_{N\to\infty} \sup_{t\in[0,T]} \mathbb{P}[|X(t)|_U > N] = 0$.

Let us consider the Banach space $C([0,T]; L^2(\Omega,U)) = C([0,T]; L^2(\Omega,\mathcal{F},\mathbb{P},U))$ of all continuous and uniformly bounded processes from $[0,T]$ into $L^2(\Omega,U)$ equipped with the sup norm.

Definition 2.5 *A continuous stochastic process $X : [0,T] \to L^2(\Omega,U)$ is said to be quadratic-mean almost periodic, provided that, for each $\epsilon > 0$, the set*

$$J(X,\epsilon) := \Big\{\kappa : \sup_{t\in[0,T]} \mathbb{E}|X(t+\kappa) - X(t)|_U^2 < \epsilon\Big\}$$

is relatively dense in $\mathbb{R}$, i.e., there exists a constant $c = c(\epsilon) > 0$ such that $J(X,\epsilon)\cap[t,t+c] \neq \emptyset$, for any $t \in [0,T]$.

Denote the set of all quadratic-mean almost periodic stochastic processes by $\widehat{C}([0,T], L^2(\Omega,U))$. Notice that this set is a closed subspace of $C([0,T]; L^2(\Omega,U))$. therefore, $\widehat{C}([0,T], L^2(\Omega,U))$ equipped with the sup norm is a Banach space.

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Definition 2.6 *A function $b(t,Y):[0,T]\times L^2(\Omega,U)\to L^2(\Omega,V)$, which is jointly continuous, is said to be quadratic-mean almost periodic in $t\in[0,T]$, uniformly for $Y\in\mathbb{K}$, where $\mathbb{K}\subset L^2(\Omega,U)$ is compact; if for any $\epsilon>0$, there exists a constant $c(\epsilon,\mathbb{K})>0$ such that any interval of length $c(\epsilon,\mathbb{K})$ contains at least a number κ satisfying*

$$\sup_{t\in[0,T]}\left(\mathbb{E}|b(t+\kappa,Y)-b(t,Y)|_V^2\right)<\epsilon,$$

for each stochastic process $Y:[0,T]\to\mathbb{K}$.

The collection of such functions will be denoted by $\widehat{C}([0,T]\times L^2(\Omega,U),L^2(\Omega,V))$.

The following lemma is also proved in [1].

Lemma 2.7 *Let $\widetilde{C}([-r,0];L^2(\Omega,U))$ be the space of all continuous functions from $[-r,0]$ into $L^2(\Omega,U)$ with the sup norm*

$$\|Z\|_{\widetilde{C}([-r,0];L^2(\Omega,U))}=\sup\{|Z(s)|_U;Z\in\widetilde{C},-r\le s\le 0\},$$

$\mathbb{K}\subset L^2(\Omega,U)\times\widetilde{C}([-r,0];L^2(\Omega,U))$ be a compact set. Assume that the function $b(t,x,y):[0,T]\times L^2(\Omega,U)\times\widetilde{C}([-r,0];L^2(\Omega,U))\to L^2(\Omega,V)$ be quadratic-mean almost periodic in $t\in[0,T]$, uniformly for $(x,y)\in\mathbb{K}$; furthermore, there exists a constant $c_1>0$ such that

$$|b(t,x,y)-b(t,\tilde{x},\tilde{y})|_V^2\le c_1\left(|x-\tilde{x}|_U^2+\|y-\tilde{y}\|^2_{\widetilde{C}([-r,0];L^2(\Omega,U))}\right),$$

for $t\in[0,T]$ and $(x,y),(\tilde{x},\tilde{y})\in L^2(\Omega,U)\times\widetilde{C}([-r,0];L^2(\Omega,U))$; then for any quadratic-mean almost periodic stochastic process $\psi:[0,T]\to L^2(\Omega,U)$, the stochastic process $t\to b(t,\psi(t),\psi_t)$ is quadratic-mean almost periodic.

3 Almost periodic mild solutions

In this section we study the existence of quadratic-mean almost periodic mild solutions for stochastic delay functional differential equations

$$\begin{aligned} dx(t) &= (Ax(t)+b(t,x(t),x_t))dt+\sigma_H(t)dB_Q^H(t), \quad t\in[0,T],\\ x(t) &= \varphi(t), \quad -r\le t\le 0, \quad r\ge 0, \end{aligned} \tag{8}$$

where $B_Q^H(t)$ is the fractional Brownian motion which was introduced in the previous section, the initial data $\varphi\in\widetilde{C}([-r,0];L^2(\Omega,U))$ is a function defined by $\varphi_t(s)=\varphi(t+s)$, $s\in[-r,0]$, and $A:Dom(A)\subset U\to U$ is the infinitesimal generator of a strongly continuous semigroup $S(.)$ on U, that is, for $t\ge 0$, it holds $|S(t)|_U\le Me^{\rho t}$, $M\ge 1$, $\rho\in\mathbb{R}$. The coefficients $b:[0,T]\times U\times\widetilde{C}([-r,0];U)\to U$ and $\sigma_H:[0,T]\to L_Q^0(U,V)$ are appropriate functions.

Definition 3.1 *A U-valued process $x(t)$ is called a mild solution of (8) if $x\in\widetilde{C}([-r,T];L^2(\Omega,U))$, $x(t)=\varphi(t)$ for $t\in[-r,0]$, and, for $t\in[0,T]$, satisfies*

$$x(t)=S(t)\varphi(0)+\int_0^t S(t-s)b(s,x(s),x_s)ds+\int_0^t S(t-s)\sigma_H(s)dB_Q^H(s)\quad \mathbb{P}-a.s. \tag{9}$$

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Now, we state our first main result. We will make use of the following assumptions on the coefficients.

($\mathbf{H}b$) The function $b \in \widehat{C}([0,T] \times U \times \widetilde{C}, U)$, and there exists a constant $c_b > 0$ such that

$$|b(t,x,y) - b(t,\tilde{x},\tilde{y})|_U^2 \le c_b\Big(|x-\tilde{x}|_U^2 + \|y-\tilde{y}\|_{\widetilde{C}}^2\Big),$$

where the space $\widetilde{C}$ is defined in Section 2, $(x,y),(\tilde{x},\tilde{y}) \in U \times \widetilde{C}$, $t \in [0,T]$.

($\mathbf{H}\sigma_H$) The function $\sigma_H : [0,T] \to L_Q^0(U,V))$ satisfies the following conditions: for the complete orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ in V, we have

$$\sum_{n=1}^{\infty} \|\sigma_H Q^{1/2} e_n\|_{L^2([0,T];U)} < \infty.$$

$$\sum_{n=1}^{\infty} |\sigma_H(t,x(t))Q^{1/2}e_n|_U \text{ is uniformly convergent for } t \in [0,T].$$

Note that assumption ($\mathbf{H}\sigma_H$) immediately imply that, for every $t \in [0,T]$, $\int_0^t |\sigma_H(s)|^2_{L_Q^0(V,U)} ds < \infty$.

Theorem 3.2 *Under the assumptions on A, the conditions ($\mathbf{H}b$) and ($\mathbf{H}\sigma_H$) , for every $\varphi \in \widetilde{C}([-r,T];L^2(\Omega,U))$, Eq. (8) has a unique quadratic-mean almost periodic mild solution x whenever*

$$\gamma = 2Me^{\rho T}\sqrt{Tc_b} < 1,$$

where c_b is a positive constant.

Proof. We can assume that $\rho > 0$, otherwise we can take $\rho_0 > 0$ such that, for $t \ge 0$, $|S(t)| \le Me^{\rho_0 t}$. Define the operator $\mathcal{L}$ on $\widehat{C}([0,T],U)$ by

$$\begin{aligned}(\mathcal{L}x)(t) &:= S(t)\varphi(0) + \int_0^t S(t-s)b(s,x(s),x_s)ds + \int_0^t S(t-s)\sigma_H(s)dB_Q^H(s) \quad \mathbb{P}-a.s.\\ &:= S(t)\varphi(0) + \Phi x(t) + \Psi(t).\end{aligned} \tag{10}$$

Firstly, it suffices to show that $\Phi x(.)$ is quadratic-mean almost periodic whenever x is quadratic-mean almost periodic.

Indeed, assuming that x is quadratic-mean almost periodic, using condition ($\mathbf{H}b$) and Lemma 2.7, one can see that $s \mapsto b(s,x(s),x_s)$ is quadratic-mean almost periodic. Therefore, for each $\epsilon > 0$, there exists $c(\epsilon) > 0$ such that any interval of length $c(\epsilon)$ contains at least κ satisfying

$$\sup_{0\le t\le T} \mathbb{E}\Big|b(t+\kappa,x(t+\kappa),x_{t+\kappa}) - b(t,x(t),x_t)\Big|_U^2 < \frac{\epsilon}{(TMe^{\rho T})^2}, \tag{11}$$

for $T > 0$ fixed. Furthermore

$$\begin{aligned}
\mathbb{E}|\Phi x(t+\kappa) - \Phi x(t)|_U^2 &= \mathbb{E}\Big|\int_0^t S(t-s)b(s+\kappa, x(s+\kappa), x_{s+\kappa})ds - \int_0^t S(t-s)b(s, x(s), x_s)ds\Big|_U^2 \\
&\leq t\mathbb{E}\int_0^t \Big|S(t-s)\Big(b(s+\kappa, x(s+\kappa), x_{s+\kappa}) - b(s, x(s), x_s)\Big)\Big|_U^2 ds \\
&\leq tM^2e^{2\rho t}\mathbb{E}\int_0^t \Big|b(s+\kappa, x(s+\kappa), x_{s+\kappa}) - b(s, x(s), x_s)\Big|_U^2 ds \\
&\leq TM^2e^{2\rho T}\int_0^t \sup_{0\leq\tau\leq s} \mathbb{E}\Big|b(\tau+\kappa, x(\tau+\kappa), x_{\tau+\kappa}) - b(\tau, x(\tau), x_\tau)\Big|_U^2 ds \\
&< \epsilon.
\end{aligned}$$

Secondly, for the chosen $v > 0$ small enough, we have

$$\begin{aligned}
& \mathbb{E}|\Psi(t+v) - \Psi(t)|^2 \\
= \ & \mathbb{E}\Big|\int_0^{t+v} S(t+v-s)\sigma_H(s)dB_Q^H(s) - \int_0^t S(t-s)\sigma_H(s)dB_Q^H(s)\Big|^2 \\
\leq \ & 2\mathbb{E}\Big|\int_0^t [S(t+v-s) - S(t-s)]\sigma_H(s)dB_Q^H(s)\Big|^2 + 2\mathbb{E}\Big|\int_t^{t+v} S(t-s)\sigma_H(s)dB_Q^H(s)\Big|^2 \\
:= \ & I_1 + I_1.
\end{aligned}$$

Applying inequality (7) to I_1 we get

$$\begin{aligned}
I_1 &\leq 2cH(2H-1)t^{2H-1}\int_0^t \Big|S(t-s)(S(v) - Id)\sigma_H(s)\Big|_{L_Q^0(V,U)}^2 ds \\
&\leq 2cH(2H-1)t^{2H-1}M^2e^{2\rho t}\int_0^t \Big|(S(v) - Id)\sigma_H(s)\Big|_{L_Q^0(V,U)}^2 ds \\
&\leq 2cH(2H-1)t^{2H-1}M^4e^{2\rho t}(1+e^{2\rho v})\int_0^t \Big|\sigma_H(s)\Big|_{L_Q^0(V,U)}^2 ds.
\end{aligned}$$

Applying now inequality (7) to I_2 we obtain

$$I_2 \leq 2cH(2H-1)v^{2H-1}M^2e^{2\rho v}\int_0^{t+v} \Big|\sigma_H(s)\Big|_{L_Q^0(V,U)}^2 ds.$$

We observe that the condition $(\mathbf{H}\sigma_H)$ ensures the existence of a positive constants c_1 and c_2 such that

$$2cH(2H-1)t^{2H-1}M^4e^{2\rho t}(1+e^{2\rho v})\int_0^t \Big|\sigma_H(s)\Big|_{L_Q^0(V,U)}^2 ds \leq c_1,$$

and

$$2cH(2H-1)v^{2H-1}M^2e^{2\rho v}\int_0^{t+v} \Big|\sigma_H(s)\Big|_{L_Q^0(V,U)}^2 ds \leq c_2.$$

Therefore, for the chosen $v > 0$ and all $t \geq 0$ we have

$$\mathbb{E}|\Psi(t+v) - \Psi(t)|^2 \leq c_1 + c_2 = c_3.$$

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From the above discussion, it is clear that the operator $\mathcal{L}$ maps $\widehat{C}([0,T],U)$ into itself.

Finally we claim that $\mathcal{L}$ is a contraction mapping on $\widehat{C}([0,T],U)$. We have

$$\begin{aligned}\mathbb{E}\Big|(\mathcal{L}x)(t)-(\mathcal{L}y)(t)\Big|^2 &= \mathbb{E}\Big|\int_0^t S(t-s)\Big[b(s,x(s),x_s)-b(s,y(s),y_s)\Big]ds\Big|^2\\ &\le 2M^2e^{2\rho t}\mathbb{E}\int_0^t\Big|b(s,x(s),x_s)-b(s,y(s),y_s)\Big|_U^2 ds\\ &\le 2M^2e^{2\rho T}\mathbb{E}\int_0^t\sup_{0\le\tau\le s}\Big|b(\tau,x(\tau),x_\tau)-b(\tau,y(\tau),y_\tau)\Big|_U^2 ds\\ &\le 2TM^2e^{2\rho T}c_b\sup_{0\le\tau\le s}\Big(|x-y|_U^2+\|x-y\|_{\widetilde{C}}^2\Big)\\ &\le 4TM^2e^{2\rho T}c_b\|x-y\|_\infty^2.\end{aligned}$$

Hence,

$$\|(\mathcal{L}x)(t)-(\mathcal{L}y)(t)\|_\infty\le 2Me^{\rho T}\sqrt{Tc_b}\|x-y\|_\infty=\gamma\|x-y\|_\infty. \tag{12}$$

As $\gamma<1$, by (12), we know that $\mathcal{L}$ is a contraction mapping. Hence, by the contraction mapping principle, $\mathcal{L}$ has a unique fixed point x, which obviously is the unique quadratic-mean almost periodic mild solution to Eq. (8).

□

Now, we give another main result. We first need to state the following conditions:

($\mathbf{H}'$) The semigroup $\{S(t)\}_{t\ge0}$ is bounded, i.e., there exists a constant $M_1>0$ such that $|S(t)|_U\le M_1$;

($\mathbf{H}'b$) The function $b\in\widehat{C}([0,T]\times U\times\widetilde{C},U)$, and for each natural number n, there exists a function $\eta_n:\mathbb{R}\to\mathbb{R}^+$ such that

$$\sup_{|x|\le n}\mathbb{E}|b(t,x(t),x_t)|_U^2\le\eta_n(t),\qquad\text{for }(x,x_t)\in U\times\widetilde{C},\ t\in[0,T];$$

($\mathbf{H}'\sigma_H$) The function $\sigma_H:[0,T]\to L_Q^0(U,V))$, and there exists a function $\vartheta:\mathbb{R}\to\mathbb{R}^+$ such that

$$\Big|\sigma_H(t)\Big|^2_{L_Q^0(U,V)}\le\vartheta(t),\qquad\text{for }t\in[0,T];$$

($\mathbf{H}''$) $\displaystyle\liminf_{n\to\infty}\frac{1}{n}\left(\int_0^T\eta_n(s)ds+tr(Q)cH(2H-1)T^{2H-1}\int_0^T\vartheta(s)ds\right)=\Omega<\infty.$

Theorem 3.3 *Let the conditions* ($\mathbf{H}'$), ($\mathbf{H}'b$), ($\mathbf{H}'\sigma_H$) *and* ($\mathbf{H}''$) *be satisfied. Then Eq.* (8) *has a quadratic-mean almost periodic mild solution whenever* $\Omega M_1^2<\frac{1}{3}$.

Proof. Let $\mathcal{L}$ be the operator defined by (10). First, we use the Schauder fixed point theorem to prove that $\mathcal{L}$ has a fixed point. The proof will be given in several steps.

Step 1. Let $\{x_n\}$ be a sequence such that $x_n\to x$. Using the continuity of $b(t,x(t),x_t)$ with respect to $x(t)$ and x_t, we get $b(t,x_n(t),(x_n)_t)\to b(t,x(t),x_t)$ as $n\to\infty$. For each $0\le t\le T$ we have

$$\begin{aligned}\mathbb{E}\Big|(\mathcal{L}x_n)(t)-(\mathcal{L}x)(t)\Big|^2 &= \mathbb{E}\Big|\int_0^t S(t-s)[b(s,x_n(s),(x_n)_s)-b(s,x(s),x_s)]ds\Big|^2\\ &\le 2M_1^2\mathbb{E}\int_0^t\Big|b(s,x_n(s),(x_n)_s)-b(s,x(s),x_s)\Big|_U^2 ds\\ &\le 2TM_1^2\sup_{0\le\tau\le s}\mathbb{E}|b(\tau,x_n(\tau),(x_n)_\tau)-b(\tau,x(\tau),x_\tau)|_U^2\end{aligned}$$

which implies that $\mathcal{L}$ is continuous.

Step 2. Let $D_n = \{x \in \widehat{C}([0,T],U); |x| \leq n\}$, for each natural number n. We want to show that the operator $\mathcal{L}$ maps bounded sets into bounded sets, i.e. there exists a natural number n^* such that $\mathcal{L}D_{n^*} \subset D_{n^*}$. If it is not true, then for each n, there exist $x_n \in D_n$ and $t_n \in [0,T]$ such that $\mathcal{L}x_n(t_n) > n$. This, together with $(\mathbf{H}')$, $(\mathbf{H}'b)$, $(\mathbf{H}'\sigma_H)$ and $(\mathbf{H}'')$ yields

$$\begin{aligned}
n &< |(\mathcal{L}x_n)(t_n)|_U^2 \\
&= \mathbb{E}\left|S(t)\varphi(0) + \int_0^{t_n} S(t_n-s)b(s,x_n(s),(x_n)_s)ds + \int_0^{t_n} S(t_n-s)\sigma_H(s)dB_Q^H(s)\right|^2 \\
&\leq 3\mathbb{E}|S(t)\varphi(0)|^2 + 3\mathbb{E}\left|\int_0^{t_n} S(t_n-s)b(s,x_n(s),(x_n)_s)ds\right|^2 \\
&\quad +3\mathbb{E}\left|\int_0^{t_n} S(t_n-s)\sigma_H(s)dB_Q^H(s)\right|^2 \\
&\leq 3M_1^2\mathbb{E}|\varphi(0)|^2 + 3\int_0^T \mathbb{E}|S(t-s)b(s,x(s),x_s)|^2 ds \\
&+ 3M_1^2 tr(Q)cH(2H-1)T^{2H-1}\int_0^T |\sigma_H(s)|^2_{L_Q^0(U,V)}ds \\
&\leq 3M_1^2\mathbb{E}|\varphi(0)|^2 + 3M_1^2\int_0^T \eta_n(s)ds + 3M_1^2 tr(Q)cH(2H-1)T^{2H-1}\int_0^T \vartheta(s)ds.
\end{aligned} \tag{13}$$

Dividing both sides of (13) by n and taking the lower limit as $n \to \infty$, one obtains

$$1 < \liminf_{n\to\infty} \frac{3M_1^2}{n}\int_0^T \eta_n(s)ds + \frac{3M_1^2 tr(Q)cH(2H-1)T^{2H-1}}{n}\int_0^T \vartheta(s)ds.$$

This is a contradiction to the assumption $\Omega M_1^2 < \frac{1}{3}$. Then $\mathcal{L}D_{n^*} \subset D_{n^*}$.

Step 3. Let D_{n^*} be a bounded set as in Step 2, and $x \in D_{n^*}$. Then for $t_1 < t_2$ we have

$$\begin{aligned}
&\mathbb{E}|(\mathcal{L}x)(t_2) - (\mathcal{L}x)(t_1)|_U^2 \\
\leq\ & 3\mathbb{E}|[S(t_2)-S(t_1)]\varphi(0)|^2 + 3\mathbb{E}\left|\int_0^{t_2} S(t_2-s)b(s,x(s),x_s)ds - \int_0^{t_1} S(t_1-s)b(s,x(s),x_s)ds\right|^2 \\
+\ & 3\mathbb{E}\left|\int_0^{t_2} S(t_2-s)\sigma_H(s)dB_Q^H(s) - \int_0^{t_1} S(t_1-s)\sigma_H(s)dB_Q^H(s)\right|^2 \\
\leq\ & 3\mathbb{E}|[S(t_2)-S(t_1)]\varphi(0)|^2 \\
+\ & 3\mathbb{E}\left|\int_0^{t_2} S(s)b(t_2-s,x(t_2-s),x_{t_2-s})ds - \int_0^{t_1} S(s)b(t_1-s,x(t_1-s),x_{t_1-s})ds\right|^2 \\
+\ & 3\mathbb{E}\left|\int_0^{t_2} S(s)\sigma_H(t_2-s)dB_Q^H(s) - \int_0^{t_1} S(s)\sigma_H(t_1-s)dB_Q^H(s)\right|^2 \\
\leq\ & 3\mathbb{E}|[S(t_2)-S(t_1)]\varphi(0)|^2 + 6\mathbb{E}\left|\int_{t_1}^{t_2} S(s)b(t_2-s,x(t_2-s),x_{t_2-s})ds\right|^2 \\
+\ & 6\mathbb{E}\left|\int_0^{t_1} S(s)[b(t_2-s,x(t_2-s),x_{t_2-s}) - b(t_2-s,x(t_2-s),x_{t_2-s})]ds\right|^2 \\
+\ & 6\mathbb{E}\left|\int_0^{t_1} S(s)[\sigma_H(t_2-s)-\sigma_H(t_1-s)]dB_Q^H(s)\right|^2 + 6\mathbb{E}\left|\int_{t_1}^{t_2} S(s)\sigma_H(t_2-s)dB_Q^H(s)\right|^2.
\end{aligned}$$

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Applying (7) of Lemma 2.3, the assumptions $(\mathbf{H}'b)$ and $(\mathbf{H}'\sigma_H)$, we obtain

$$\begin{aligned}
& \mathbb{E}|(\mathcal{L}x)(t_2)-(\mathcal{L}x)(t_1)|_U^2 \\
\leq\ & 3\mathbb{E}|[S(t_2)-S(t_1)]\varphi(0)|^2 + 6M_1^2\int_{t_1}^{t_2}\mathbb{E}|b(t_2-s,x(t_2-s),x_{t_2-s})|^2ds \\
+\ & 6M_1^2\int_0^{t_1}\mathbb{E}|b(t_2-s,x(t_2-s),x_{t_2-s})-b(t_1-s,x(t_1-s),x_{t_1-s})|^2ds \\
+\ & 6M_1^2tr(Q)cH(2H-1)T^{2H-1}\int_0^{t_1}|\sigma_H(t_2-s)-\sigma_H(t_1-s)|^2_{L_Q^0(U,V)}ds \\
+\ & 6M_1^2tr(Q)cH(2H-1)T^{2H-1}\int_{t_1}^{t_2}|\sigma_H(t_2-s)|^2_{L_Q^0(U,V)}ds \\
\leq\ & 3\mathbb{E}|[S(t_2)-S(t_1)]\varphi(0)|^2 + 6M_1^2\int_{t_1}^{t_2}\eta_{t_2-n}(s)ds \\
& +6M_1^2tr(Q)cH(2H-1)T^{2H-1}\int_{t_1}^{t_2}\vartheta(t_2-s)ds \\
+\ & 6M_1^2\int_0^{t_1}\mathbb{E}|b(t_2-s,x(t_2-s),x_{t_2-s})-b(t_1-s,x(t_1-s),x_{t_1-s})|^2ds \\
+\ & 6M_1^2tr(Q)cH(2H-1)T^{2H-1}\int_0^{t_1}|\sigma_H(t_2-s)-\sigma_H(t_1-s)|^2_{L_Q^0(U,V)}ds.
\end{aligned}$$

Thus $\mathcal{L}$ is equicontinuous.

It remains to prove that $\Theta(t) = \{\mathcal{L}x(t); x \in D_{n^*}\}$ is relatively compact in U. $S(t)$ is compact in U, since it is generated by the dense operator A. Then $\Theta(0) = S(0)x_0$ is relatively compact in U.

Now, for t fixed and for each $\epsilon \in (0,t)$, $x \in D_{n^*}$ we define $\mathcal{L}_\epsilon x(t)$ as follow

$$\mathcal{L}_\epsilon x(t) = S(t)\varphi(0) + \int_0^{t-\epsilon} S(t-s)b(s,x(s),x_s)ds + \int_0^{t-\epsilon} S(t-s)\sigma_H(s)dB_Q^H(s). \tag{14}$$

Then the sets $\Theta_\epsilon(t) = \{\mathcal{L}_\epsilon x(t); x \in D_{n^*}\}$ are relatively compact in U. Moreover, for each $x \in D_{n^*}$, one has

$$|\mathcal{L}x(t)-\mathcal{L}_\epsilon x(t)|_U^2 \leq 2M_1^2\left(\int_{t-\epsilon}^t \eta_n(s)ds + tr(Q)cH(2H-1)T^{2H-1}\int_{t-\epsilon}^t\vartheta(s)ds\right), \tag{15}$$

from which, by combining the condition $(\mathbf{H}'')$, follows that there are relatively compact sets arbitrarily close to $\Theta(t)$ and hence $\Theta(t)$ is also relatively compact in U. Thus, the Arzela-Ascoli theorem implies that $\mathcal{L}D_{n^*}$ is relatively compact, and $\mathcal{L}$ is completely continuous on D_{n^*}.

As a consequence of *Steps* 1-3 together with the Schauder fixed point theorem, we deduce that $\mathcal{L}$ has a fixed point in D_{n^*} which is a quadratic-mean almost periodic mild solution to Eq. (8).

□

Now, we give the third main result. In this sequence, we require the following assumptions.
$(\mathbf{H}''b)$ The function $b \in \widehat{C}([0,T]\times U \times \widetilde{C}, U)$, and there exists a function $\eta : \mathbb{R} \to \mathbb{R}^+$ such that

$$\sup \mathbb{E}|b(t,x(t),x_t)|_U^2 \leq \eta(t), \qquad \text{for } (x,x_t) \in U\times\widetilde{C},\ t\in[0,T];$$

$(\mathbf{H}''')$ The integral $\int_0^t \eta(t-s)ds + tr(Q)cH(2H-1)T^{2H-1}\int_0^t\vartheta(t-s)ds$ exists for all $t \in [0,T]$.

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Theorem 3.4 *Let the conditions* $(\mathbf{H}''b)$, $(\mathbf{H}'\sigma_H)$ *and* $(\mathbf{H}''')$ *be satisfied. Then Eq.* (8) *has a quadratic-mean almost periodic mild solution.*

Proof. We shall also apply the Schauder fixed point theorem to prove this theorem. The proof of *Step* 1 in this theorem is the same as the proof of *Step* 1 in Theorem 3.3 and so is omitted. Now, we start our proof from *Step* 2.

Step 2. Let $D = \{x \in \widehat{C}([0,T],U); |x| \le k\}$, where $k = 3M_1^2(\mathbb{E}|\varphi(0)|^2 + M_2)$ and M_2 is the integral defined in $(\mathbf{H}''')$. We have

$$\begin{aligned}
|(\mathcal{L}x)(t)|_U^2 &= \mathbb{E}\left|S(t)\varphi(0) + \int_0^t S(s)b(t-s,x(t-s),x_{t-s})ds + \int_0^t S(s)\sigma_H(t-s)dB_Q^H(s)\right|^2 \\
&\le 3M_1^2\mathbb{E}|\varphi(0)|^2 + 3\int_0^t \mathbb{E}|S(s)b(t-s,x(t-s),x_{t-s})|^2 ds \\
&+ 3M_1^2 cH(2H-1)T^{2H-1}tr(Q)\int_0^t \sigma_H(t-s)|^2_{L_Q^0(U,V)}ds \\
&\le 3M_1^2\left(\mathbb{E}|\varphi(0)|^2 + \int_0^t \eta(t-s)ds + cH(2H-1)T^{2H-1}tr(Q)\int_0^t \vartheta(t-s)ds\right) = k.
\end{aligned}$$

Therefore, $\mathcal{L}: D \to D$.

Step 3. Let D be a bounded set as in *Step* 2, $t_1 < t_2$ and $x \in D$. We have

$$\begin{aligned}
& \mathbb{E}|(\mathcal{L}x)(t_2) - (\mathcal{L}x)(t_1)|^2 \\
\le\ & 3\mathbb{E}|S(t_2 - S(t_1))\varphi(0)|^2 + 6M_1^2\int_{t_1}^{t_2}\eta(t_2-s)ds + 6M_1^2 cH(2H-1)T^{2H-1}tr(Q)\int_{t_1}^{t_2}\vartheta(t_2-s)ds \\
+\ & 6M_1^2\int_0^{t_1}\mathbb{E}|b(t_2-s,x(t_2-s),x_{t_2-s}) - b(t_1-s,x(t_1-s),x_{t_1-s})|^2 ds \\
+\ & 6M_1^2 cH(2H-1)T^{2H-1}tr(Q)\int_0^{t_1}|\sigma_H(t_2-s)-\sigma_H(t_1-s)|^2_{L_Q^0(U,V)}ds.
\end{aligned}$$

Thus, $\mathcal{L}$ is equicontinuous.

Set $\Theta(t) = \{\mathcal{L}x(t) : x \in D\}$. Fix t, for each $\epsilon \in (0,t)$ and $x \in D$. Let $\mathcal{L}_\epsilon$ be the operator defined by (14); then the sets $\Theta_\epsilon(t) = \{\mathcal{L}_\epsilon x(t) : x \in D\}$ are relatively compact in U. Meanwhile, (15) implies that $\mathcal{L}_\epsilon$ arbitrarily close to $\Theta(t)$ and $\Theta(t)$ is also relatively compact in U. Thus, the ArzelaAscoli theorem implies that $\mathcal{L}D$ is relatively compact, $\mathcal{L}$ is completely continuous on D.

Finally, we can conclude from *Step* 1-2 that $\mathcal{L}D \to D$ is continuous and completely continuous. Thus, $\mathcal{L}$ has a fixed point in D by using the Schauder fixed point theorem. So, it follow that Eq. (8) has at least a quadratic-mean almost periodic mild solution.

□

4 Stability

As in this section we first assume that the operator A is a closed linear operator generating a strongly continuous exponentially stable semigroup $S(.)$ on U, that is, for $t \ge 0$, it holds $|S(t)|_U \le Me^{-\lambda t}$, $M, \lambda > 0$. We also assume in addition to assumption $(\mathbf{H}\sigma_H)$ that $\int_0^\infty e^{\lambda s}|\sigma_H(s)|^2_{L_Q^0(U,V)}ds < \infty$. Our first result on the stability of the quadratic-mean almost periodic mild solution is the following theorem.

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Theorem 4.1 *Under the assumptions on A, the conditions $(\mathbf{H}b)$ and $(\mathbf{H}\sigma_H)$, the quadratic-mean almost periodic mild solution $x(t)$ to Eq. (8) is globally exponentially stable.*

Proof. Using the assumptions, one can choose a positive constant η such that $0 < \eta < \lambda$. One has

$$\begin{aligned} e^{\eta t}\mathbb{E}|x(t)|^2 &\leq 3e^{\eta t}\mathbb{E}|S(t)\varphi(0)|^2 + 3e^{\eta t}\mathbb{E}\bigg|\int_0^t S(t-s)b(s,x(s),x_s)ds\bigg|^2 \\ &+ e^{\eta t}\mathbb{E}\bigg|\int_{-\infty}^t S(t-s)\sigma_H(s)dB_Q^H(s)\bigg|^2 \\ &= 3e^{\eta t}\mathbb{E}|S(t)\varphi(0)|^2 + I_1 + I_2. \end{aligned} \tag{16}$$

Estimating the terms on the right-hand side of (16) yields

$$3e^{\eta t}\mathbb{E}|S(t)\varphi(0)|^2 \leq 3e^{(\eta-\rho)t}M^2\mathbb{E}|\varphi(0)|^2 \to 0 \quad \text{as} \quad t\to\infty, \tag{17}$$

and

$$I_1 \leq 3e^{\eta t}M^2 c_b \int_0^t e^{-\lambda(t-s)}ds \int_0^t e^{-\lambda(t-s)}\Big(|x(s)|_U^2 + \|x_s\|_{\widetilde{C}}^2\Big)ds.$$

For the chosen parameter θ, and any $x(t)\in U$ we have

$$\begin{aligned} I_1 &\leq \tfrac{3}{\lambda}M^2 c_b e^{\eta t}\int_0^t e^{-\lambda(t-s)}\Big(|x(s)|_U^2 + \|x_s\|_{\widetilde{C}}^2\Big)ds \\ &= \tfrac{3}{\lambda}M^2 c_b e^{-\theta t}\int_0^t e^{\theta s}e^{\eta s}\Big(|x(s)|_U^2 + \|x_s\|_{\widetilde{C}}^2\Big)ds. \end{aligned}$$

Now, for any $\epsilon > 0$, there exists a constant $v > 0$ such that $e^{\eta s}|x(s-r)|_U^2 < \epsilon$, for $s \geq v$, and we have

$$\begin{aligned} I_1 &\leq \tfrac{3}{\lambda}M^2 c_b e^{-\theta t}\int_v^t e^{\theta s}e^{\eta s}\Big(|x(s)|_U^2 + \|x_s\|_{\widetilde{C}}^2\Big)ds \\ &+ \tfrac{3}{\lambda}M^2 c_b e^{-\theta t}\int_0^v e^{\theta s}e^{\eta s}\Big(|x(s)|_U^2 + \|x_s\|_{\widetilde{C}}^2\Big)ds \\ &\leq \tfrac{6M^2c_b\epsilon}{\lambda\theta} + \tfrac{3}{\lambda}M^2 c_b e^{-\theta t}\int_0^v e^{\theta s}e^{\eta s}\Big(|x(s)|_U^2 + \|x_s\|_{\widetilde{C}}^2\Big)ds. \end{aligned} \tag{18}$$

Using the fact that $e^{-\theta t} \to 0$ as $t\to\infty$, it follows that there exists a constant $u \geq v$ such that for any $t \geq u$,

$$\frac{3}{\lambda}M^2 c_b e^{-\theta t}\int_{-\infty}^v e^{\theta s}e^{\eta s}\Big(|x(s)|_U^2 + \|x_s\|_{\widetilde{C}}^2\Big)ds < \epsilon - \frac{6M^2c_b\epsilon}{\lambda\theta}. \tag{19}$$

Thus, from (18) and (19), we get for any $t \geq u$,

$$I_1 = 4e^{\eta t}\mathbb{E}\bigg|\int_0^t S(t-s)b(s,x(s),x_s)ds\bigg|^2 < \epsilon,$$

which implies

$$I_1 = 4e^{\eta t}\mathbb{E}\bigg|\int_0^t S(t-s)b(s,x(s),x_s)ds\bigg|^2 \to 0 \quad \text{as} \quad t\to\infty. \tag{20}$$

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Estimating I_2, for any $x(t) \in U$, $t \geq -r$, we have

$$\begin{aligned} I_2 &\leq 3cH(2H-1)M^2T^{2H-1}e^{\eta t}\int_0^t e^{-2\lambda(t-s)}|\sigma_H(s)|^2_{L^0_Q(U,V)}ds \\ &\leq 3cH(2H-1)M^2T^{2H}\int_0^t e^{\lambda s}|\sigma_H(s)|^2_{L^0_Q(U,V)}ds, \end{aligned}$$

and the additional assumption to $(\mathbf{H}\sigma_H)$ ensures the existence of a positive constant ϵ such that

$$3cH(2H-1)M^2T^{2H}\int_0^t e^{\lambda s}|\sigma_H(s)|^2_{L^0_Q(U,V)}ds < \epsilon \quad \text{for all } t \geq -r. \tag{21}$$

Thus, from (17), (20) and (21), we obtain $e^{\eta t}\mathbb{E}|x(t)|^2 \to 0$ as $t \to \infty$. The quadratic-mean almost periodic mild solution of (8) is exponentially stable.

□

Now we study the uniform stability of the quadratic-mean almost periodic mild solution to Eq. (8). We first require the following assumption:
$(\mathbf{H}'''b)$ The function $b \in \widehat{C}([0,T] \times U \times \widetilde{C}, U)$, and there exists a function $c_b : \mathbb{R} \to \mathbb{R}_+$ such that

$$|b(t,x,y)|^2_U \leq c_b(t)\left(|x|^2_U + \|y\|^2_{\widetilde{C}}\right),$$

where $(x,y) \in U \times \widetilde{C}$, $t \in [0,T]$.

Theorem 4.2 *Under the assumptions* $(\mathbf{H}')$, $(\mathbf{H}\sigma_H)$ *and* $(\mathbf{H}'''b)$, *the quadratic-mean almost periodic mild solution to Eq.* (8) *is uniformly stable whenever* $M_1^2 I < \frac{1}{6}$, *where* $I = \int_0^t c_b(s)ds$.

Proof. Let $x(t)$ be a solution of

$$x(t) = S(t)\varphi(0) + \int_0^t S(t-s)b(s,x(s),x_s)ds + \int_0^t S(t-s)\sigma_H(s)dB^H_Q(s) \tag{22}$$

such that $x(0) = x_0$, where $x_0 \in U$. Then

$$\begin{aligned} |x(t)|^2_U &\leq 3\mathbb{E}|S(t)\varphi(0)|^2 + 3\mathbb{E}\left|\int_0^t S(t-s)b(s,x(s),x_s)ds\right|^2 \\ &+ 3\mathbb{E}\left|\int_0^t S(t-s)\sigma_H(s)dB^H_Q(s)\right|^2 \\ &\leq 3M_1^2\mathbb{E}|\varphi(0)|^2 + 3M_1^2\int_0^t c_b(s)\left(|x(s)|^2_U + \|x_s\|^2_{\widetilde{C}}\right)ds \\ &+ 3M_1^2 cH(2H-1)T^{2H-1}tr(Q)\int_0^t |\sigma_H(s)|^2_{L^0_Q(U,V)}ds. \end{aligned}$$

Using the assumption $(\mathbf{H}\sigma_H)$ we obtain

$$\begin{aligned} |x(t)|^2_U &\leq 3M_1^2\|\varphi(0)\|^2_\infty + 6M_1^2\left(\int_0^t c_b(s)ds\right)\|x\|^2_\infty + c_3 \\ &= 3M_1^2\|\varphi(0)\|^2_\infty + 6M_1^2 I\|x\|^2_\infty + c_3, \end{aligned}$$

c_3 is a positive positive constant.
Thus
$$\|x\|_\infty^2 \le 3M_1^2\|\varphi(0)\|_\infty^2 + 6M_1^2 I\|x\|_\infty^2 + c_3,$$
$6M_1^2 I < 1$ yields
$$\|x\|_\infty^2 \le \frac{1}{1-6M_1^2 I}\Big(c_3 + 3M_1^2\|\varphi(0)\|_\infty^2\Big).$$
Therefore, if $\|\varphi(0)\|_\infty^2 < \lambda(\epsilon)$, then $\|x\|_\infty^2 < \epsilon$, which implies that the quadratic-mean almost periodic mild solution to Eq. (8) is uniformly stable.

□

5 Example

Consider the following stochastic evolution equation:

$$\begin{cases} d\xi(t,x) &= \left[\frac{\partial^2}{\partial x^2}\xi(t,x) + \delta[\xi(t,x)(\sin(t)+\sin(\sqrt{2}t))]\right]dt + \sigma_H(t)dB_Q^H(t), \quad t\in[0,1], x\in[0,\pi], \\ \xi(t,0) &= \xi(t,\pi) = 0, \quad t\in[0,1], \\ \xi(t,x) &= \varphi(t,x), \quad t\in[-r,0], \end{cases} \tag{23}$$

where $r \in (0,1)$, $\varphi(.,x) \in \widetilde{C}([-r,0],\mathbb{R})$ and $B_Q^H(t)$ is a Q-cylindrical fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2},1)$ satisfying $tr(Q) = 1$. Denote $U = L^2(\mathbb{P}, L^2[0,\pi])$, and define $A : D(A) \subset U \to U$ given by $A = \frac{\partial^2}{\partial x^2}$ with
$D(A) = \{\xi(.) \in U : \xi'' \in U, \xi' \in U$ is absolutely continuous on $[0,\pi], \xi(0) = \xi(\pi) = 0\}$.

It is well known that a strongly continuous semigroup S, generated by the operator A, satisfies $|S(t)| \le e^{-t}$, for $t \ge 0$. Taking $b(t,\varphi,\varphi_t)(\theta) = \delta[\varphi(\theta)(\sin(t)+\sin(\sqrt{2}t))]$, and σ_H satisfies assumption $(\mathbf{H}\sigma_H)$. Thus one has

$$|b(t,x,x_t) - b(t,y,y_t)|_U^2 \le 4\delta^2|x-y|_U^2.$$

Therefore, Eq. (23) has a quadratic-mean almost periodic mild solution, provided that, $\delta < \frac{\sqrt{3}}{6}$ according to Theorem 3.2.

Let $\eta_n(t) = \delta_n(t) = \delta^2(\sin(t)+\sin(\sqrt{2}t))^2$ for $n \in \mathbb{N}$, Eq. (23) has a quadratic-mean almost periodic mild solution according to Theorem 3.3.

Let $\eta(t) = \delta_n(t) = \delta^2(\sin(t)+\sin(\sqrt{2}t))^2$, Eq. (23) has a quadratic-mean almost periodic mild solution according to Theorem 3.4.

The quadratic-mean almost periodic mild solution to Eq. (23) is exponentially stable according to Theorem 4.1.

The quadratic-mean almost periodic mild solution to Eq. (23) is uniformly stable, provided that, $\delta < \frac{\sqrt{3}}{6}$ according to Theorem 4.2.

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