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q-ANALOGUE OF THE ALZER'S INEQUALITY

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ABSTRACT. In this article, we are interested in giving a q-analogue of the Alzer's inequality.

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1. Introduction

In 1964 H. Mink and L. Sathre [15] proved the following inequality

$$\frac{n}{n+1} < \frac{(n!)^{\frac{1}{n}}}{((n+1)!)^{\frac{1}{n+1}}} \; , n \in \mathbf{N}. \tag{1.1}$$

The inequality (1.1) was generalized and refined by H. Alzer in [2]-[4]. He proved in [4] the following inequality:

$$\frac{n}{n+1} \leq \left[\frac{(n+1)\sum_{i=1}^{n} i^r}{n\sum_{i=1}^{n+1} i^r}\right]^{\frac{1}{r}} < \frac{(n!)^{\frac{1}{n}}}{((n+1)!)^{\frac{1}{n+1}}}, n \in \mathbf{N},\ r \in \mathbf{R}_+. \tag{1.2}$$

The lower and upper bounds are the best possible.

Many proofs of the inequality (1.2) and some generalizations were given in ([1],[5]-[7],[9],[10],[12] -[14],[16]-[23]).

The left hand side of the Alzer's inequality (1.2) was generalized by Feng Qi [8] as follows:

$$\frac{n+k}{n+m+k} \leq \left[\frac{\frac{1}{n}\sum_{i=k+1}^{n+k} i^r}{\frac{1}{n+m}\sum_{i=k+1}^{n+m+k} i^r}\right]^{\frac{1}{r}}, \ n, m \in \mathbf{N}, \tag{1.3}$$

where k is a nonnegative integer and $r \in \mathbf{R}_+$. The lower bound is best possible.

The main purpose of this paper is to give a q-analogue of inequalities (1.2) and (1.3).

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2. q-analogue of the Alzer's inequality

Throughout this paper, we consider a positive integer $q \neq 1$ and for $x \in \mathbb{C}$, we write

$$[x]_q = \frac{1-q^x}{1-q}. \tag{2.1}$$

Note that $[x]_q$ tends to x when q tends to 1 (we refer to [11] for more details about q-calculus).
To prove the main result of the paper, we need the following lemma.

Lemma 2.1. *For all $q > 1$, for all nonnegative integers n and k and for all nonnegative real number r, we have*

$$\sum_{i=k+1}^{n+k} [i]_q^r > \frac{[n]_q[n+k]_q^r[n+k+1]_q^r}{[n+1]_q[n+k+1]_q^r - [n]_q[n+k]_q^r}. \tag{2.2}$$

Proof. Let k be a nonnegative integer and r be a nonnegative real number. We prove the result by induction on n.

For $n = 1$, we have

$$\begin{aligned}
[k+1]_q^r - \frac{[k+1]_q^r[k+2]_q^r}{[2]_q[k+2]_q^r - [k+1]_q^r} &= [k+1]_q^r \frac{[2]_q[k+2]_q^r - [k+1]_q^r - [k+2]_q^r}{[2]_q[k+2]_q^r - [k+1]_q^r} \\
&= [k+1]_q^r \frac{q[k+2]_q^r - [k+1]_q^r}{(1+q)[k+2]_q^r - [k+1]_q^r} \\
&= [k+1]_q^r \frac{q(1-q^{k+2})^r - (1-q^{k+1})^r}{(1+q)(1-q^{k+2})^r - (1-q^{k+1})^r} \\
&= [k+1]_q^r \frac{q - (\frac{1-q^{k+1}}{1-q^{k+2}})^r}{1+q-(\frac{1-q^{k+1}}{1-q^{k+2}})^r}.
\end{aligned}$$

Using the fact that $\frac{1-q^{k+1}}{1-q^{k+2}} < 1 < q$, we get

$$[k+1]_q^r > \frac{[k+1]_q^r[k+2]_q^r}{[2]_q[k+2]_q^r - [k+1]_q^r},$$

which achieves the proof of the result for $n = 1$.

Suppose, now, that it is valid for $n > 1$ and let's prove that it's valid for $n+1$. Using the fact that $\sum_{i=k+1}^{n+k+1}[i]_q^r = \sum_{i=k+1}^{n+k}[i]_q^r + [n+k+1]_q^r$, calculating straightforwardly, and simplifying easily, the induction step can be written as

$$\frac{[n+2]_q[n+k+2]_q^r - [n+1]_q[n+k+1]_q^r}{[n+1]_q[n+k+1]_q^r - [n]_q[n+k]_q^r} > \left(\frac{[n+k+2]_q}{[n+k+1]_q}\right)^r.$$

Consider the functions f and g defined on $[n, n+1]$ as follows

$$f(x) = [x+1]_q[x+k+1]_q^r \qquad \text{and} \qquad g(x) = [x]_q[x+k]_q^r.$$

Simple derivation gives

$$f'(x) = \frac{\ln q}{q-1}[q^{x+1}[x+k+1]_q^r + rq^{x+k+1}[x+1]_q[x+k+1]_q^{r-1}]$$

and

$$g'(x) = \frac{\ln q}{q-1}[q^x[x+k]_q^r + rq^{x+k}[x]_q[x+k]_q^{r-1}].$$

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So, for all $x \in [n, n+1]$, we have

$$\begin{aligned}
\frac{f'(x)}{g'(x)} &= \frac{q^{x+1}[x+k+1]_q^r + rq^{x+k+1}[x+1]_q[x+k+1]_q^{r-1}}{q^x[x+k]_q^r + rq^{x+k}[x]_q[x+k]_q^{r-1}} \\
&= \frac{q[x+k+1]_q^r\left(1 + rq^k\frac{[x+1]_q}{[x+k+1]_q}\right)}{[x+k]_q^r\left(1 + rq^k\frac{[x]_q}{[x+k]_q}\right)} \\
&> \left(\frac{[x+k+1]_q}{[x+k]_q}\right)^r.
\end{aligned}$$

From Cauchy's mean-value theorem and the previous inequality, there exists one point $\xi \in (n, n+1)$ such that

$$\frac{[n+2]_q[n+k+2]_q^r - [n+1]_q[n+k+1]_q^r}{[n+1]_q[n+k+1]_q^r - [n]_q[n+k]_q^r} = \frac{f'(\xi)}{g'(\xi)} > \left(\frac{[\xi+k+1]_q}{[\xi+k]_q}\right)^r. \tag{2.3}$$

But,

$$\begin{aligned}
\frac{[\xi+k+1]_q}{[\xi+k]_q} - \frac{[n+k+2]_q}{[n+k+1]_q} &= \frac{(1-q^{\xi+k+1})(1-q^{n+k+1}) - (1-q^{n+k+2})(1-q^{\xi+k})}{(1-q^{\xi+k})(1-q^{n+k+1})} \\
&= \frac{q^{n+k+2} + q^{\xi+k} - q^{\xi+k+1} - q^{n+k+1}}{(1-q^{\xi+k})(1-q^{n+k+1})} \\
&= \frac{q^k(q-1)(q^{n+1}-q^{\xi})}{(1-q^{\xi+k})(1-q^{n+k+1})} > 0.
\end{aligned}$$

Then,

$$\left(\frac{[\xi+k+1]_q}{[\xi+k]_q}\right)^r > \left(\frac{[n+k+2]_q}{[n+k+1]_q}\right)^r.$$

This inequality together with (2.3) gives

$$\frac{[n+2]_q[n+k+2]_q^r - [n+1]_q[n+k+1]_q^r}{[n+1]_q[n+k+1]_q^r - [n]_q[n+k]_q^r} > \left(\frac{[n+k+2]_q}{[n+k+1]_q}\right)^r, \tag{2.4}$$

which proves that the result is valid for $n+1$. □

Now, we are in a situation to prove the main result of this paper.

Theorem 2.2.

$$\frac{[n+k]_q}{[n+m+k]_q} \le \begin{cases} \frac{1}{q^{m(1+\frac{1}{r})}}\left(\frac{[n+m]_q\sum_{i=k+1}^{n+k} q^{-ir}[i]_q^r}{[n]_q\sum_{i=k+1}^{n+k+m} q^{-ir}[i]_q^r}\right)^{\frac{1}{r}}, & \text{if } q \in]0,1[, \\ \left(\frac{[n+m]_q\sum_{i=k+1}^{n+k}[i]_q^r}{[n]_q\sum_{i=k+1}^{n+k+m}[i]_q^r}\right)^{\frac{1}{r}}, & \text{if } q \in]1,+\infty[, \end{cases} \tag{2.5}$$

where $n, m \in \mathbf{N}$, k is a nonnegative integer and $r \in \mathbf{R}_+$. The lower bounds are best possible.

Proof. It is easy to verify that for all positive real $q \neq 1$, we have

$$[n]_q = q^{n-1}[n]_{\frac{1}{q}}$$

and so,

$$\frac{[n+k]_q}{[n+m+k]_q} = \frac{q^{n+k-1}[n+k]_{\frac{1}{q}}}{q^{n+m+k-1}[n+m+k]_{\frac{1}{q}}} = \frac{[n+k]_{\frac{1}{q}}}{q^m[n+m+k]_{\frac{1}{q}}}.$$

Then, to prove the result, it suffices to focus on the case $q > 1$.

Let $q > 1$ and r be a nonnegative real number.
From the previous lemma and the fact that

$$\sum_{i=k+1}^{n+k+1} [i]_q^r = \sum_{i=k+1}^{n+k} [i]_q^r + [n+k+1]_q^r \tag{2.6}$$

we obtain for all $n \in \mathbf{N}$ and k nonnegative integer

$$\frac{1}{[n]_q[n+k]_q^r} \sum_{i=k+1}^{n+k} [i]_q^r > \frac{1}{[n+1]_q[n+k+1]_q^r} \sum_{i=k+1}^{n+k+1} [i]_q^r. \tag{2.7}$$

So, by induction on m, we get for all $n \in \mathbf{N}$ and k, m nonnegative integers

$$\frac{1}{[n]_q[n+k]_q^r} \sum_{i=k+1}^{n+k} [i]_q^r > \frac{1}{[n+m]_q[n+m+k]_q^r} \sum_{i=k+1}^{n+m+k} [i]_q^r.$$

Then,

$$\left(\frac{[n+k]_q}{[n+m+k]_q} \right)^r < \frac{[n+m]_q \sum_{i=k+1}^{n+k} [i]_q^r}{[n]_q \sum_{i=k+1}^{n+k+m} [i]_q^r},$$

which achieves the proof. □

The limit case is given by
$\forall q \in]1, +\infty[$,

$$\lim_{r \to +\infty} \left(\frac{[n+m]_q \sum_{i=k+1}^{n+k} [i]_q^r}{[n]_q \sum_{i=k+1}^{n+k+m} [i]_q^r} \right)^{\frac{1}{r}} = \frac{[n+k]_q}{[n+m+k]_q}. \tag{2.8}$$

$\forall q \in]0, 1[$,

$$\lim_{r \to +\infty} \frac{1}{q^{m(1+\frac{1}{r})}} \left(\frac{[n+m]_q \sum_{i=k+1}^{n+k} q^{-ir} [i]_q^r}{[n]_q \sum_{i=k+1}^{n+k+m} q^{-ir} [i]_q^r} \right)^{\frac{1}{r}} = \frac{[n+k]_q}{[n+m+k]_q}. \tag{2.9}$$

Thus, the lower bound is best possible.
Indeed, using the fact that $0 \leq \frac{[i]_q}{[j]_q} < 1, \forall 1 \leq i < j, \ \forall q \in]1, +\infty[$,

$$\begin{aligned} \lim_{r \to +\infty} \left(\frac{[n+m]_q \sum_{i=k+1}^{n+k} [i]_q^r}{[n]_q \sum_{i=k+1}^{n+k+m} [i]_q^r} \right)^{\frac{1}{r}} &= \lim_{r \to +\infty} \left(\frac{[n+m]_q}{[n]_q} \right)^{\frac{1}{r}} \frac{[n+k]_q}{[n+m+k]_q} \left(\frac{1 + \sum_{i=k+1}^{n+k-1} (\frac{[i]_q}{[n+k]_q})^r}{1 + \sum_{i=k+1}^{n+k+m-1} (\frac{[i]_q}{[n+m+k]_q})^r} \right)^{\frac{1}{r}} \\ &= \frac{[n+k]_q}{[n+m+k]_q}. \end{aligned}$$

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$\forall q \in]0,1[$,

$$\begin{aligned}
\lim_{r\to+\infty} \frac{1}{q^{m(1+\frac{1}{r})}} \left(\frac{[n+m]_q \sum_{i=k+1}^{n+k} q^{-ir}[i]_q^r}{[n]_q \sum_{i=k+1}^{n+k+m} q^{-ir}[i]_q^r} \right)^{\frac{1}{r}} &= \lim_{r\to+\infty} \frac{1}{q^{m(1+\frac{1}{r})}} \left(\frac{[n+m]_q}{[n]_q} \right)^{\frac{1}{r}} \frac{[n+k]_{\frac{1}{q}}}{[n+m+k]_{\frac{1}{q}}} \\
&\times \left(\frac{1+\sum_{i=k+1}^{n+k-1} (\frac{[i]_{\frac{1}{q}}}{[n+k]_{\frac{1}{q}}})^r}{1+\sum_{i=k+1}^{n+k+m-1} (\frac{[i]_{\frac{1}{q}}}{[n+m+k]_{\frac{1}{q}}})^r} \right)^{\frac{1}{r}} \\
&= \frac{1}{q^m} \frac{[n+k]_{\frac{1}{q}}}{[n+m+k]_{\frac{1}{q}}} = \frac{[n+k]_q}{[n+m+k]_q}.
\end{aligned}$$

For $k=0$ and $m=1$, we find the following special case:

Corollary 2.3. *If r, q are positive real numbers and n is a positive integer, then*

$$\frac{[n]_q}{[n+1]_q} \leq \begin{cases} \frac{1}{q^{1+\frac{1}{r}}} \left(\frac{[n+1]_q \sum_{i=1}^{n} q^{-ir}[i]_q^r}{[n]_q \sum_{i=1}^{n+1} q^{-ir}[i]_q^r} \right)^{\frac{1}{r}}, if\ q \in]0,1[, \\ \left(\frac{[n+1]_q \sum_{i=1}^{n} [i]_q^r}{[n]_q \sum_{i=1}^{n+1} [i]_q^r} \right)^{\frac{1}{r}}, if\ q \in]1,+\infty[. \end{cases} \tag{2.10}$$

The lower bounds are best possible.

Remark 2.4. When q tends to 1 ($q \to 1^+$ or $q \to 1^-$), $[n]_q$ tends to n and the inequality (2.10) tends to the Alzer's one.

References

[1] S. Abramovich, J. Barić, M. Matić and J. Peĉarić, *On van de Lune-Alzers inequality*, J. Math. Inequal. 1 (2007), no. 4, 563-587.

[2] Horst Alzer, *On some inequlities involving* $(n!)^{\frac{1}{n}}$, Rocky Mt. J. Math. 24 (1994), no. 3, 867-873.

[3] Horst Alzer, *On some inequlities involving* $(n!)^{\frac{1}{n}}$*,II*, Period. Math. Hung. 28 (1994), no. 3, 229-233.

[4] Horst Alzer, *On an inequality of H. Minc and L. Sathre*, J. Math. Anal. Appl. 179 (1993), 396-402.

[5] G. Bennett, *Meaningful sequences*, Houston J. Math., 33(2007), no.2, 555-580.

[6] S. S. Dragomir and J. Van Hoek, *Some new analytic inequalities and their applications in guessing theory*, J. Math. Anal. Appl. 225 (1998), Issue 2, 542556.

[7] N. Elezovic and J. Pecaric, *On Alzer'z inequality*, J. Math. Anal. Appl. 223(1998), 366-369.

[8] Feng Qi, *Generalization of H. Alzer's Inequality*, J. Math. Anal. Appl. 240 (1999), 294-297.

[9] Feng Qi, *Generalization of H. Alzer's and Kuang's inequality*, Tamkang J. Math. 31 (2000), no 3.

[10] Feng Qi and L. Debnath, *On a new generalization of Alzer's inequality*, Internat J. Math. and Math. Sc 23 (2000), no. 12, 815-818. (1999), no. 6, Article 14.

[11] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its application, Vol 35, Cambridge Univ. Press, Cambridge, UK, 1990.

[12] J. C. Kuang, *Some extensions and refinements of Minc-Sathre inequality*, Math Gaz. 83 (1999), 123-127.

[13] Z. Liu, *New generalizations of H. Alzer's inequality*, Tamkaqng J. Math. 34(2003), no.3, 255-260.

[14] J. S. Martins, *Arithmetic and geometric means, an application to Lorentz sequence spaces*, Math. Nachr. 139 (1988), 281-288.

[15] H. Mink and L. Sathre, *Some inequalities involving* $(r!)^{\frac{1}{r}}$, Proc. Edinburgh Math. Soc. 14 (1964-1965), 41-46.

[16] N. Ozeki, *On some inequalities*, J. College Arts Sci. Chiba Univ.4 (1965), no. 3, 211-214. (Japanese).

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2015, VOLUME 5, ISSUE 2, p.145-150

[17] József Sándor, *On an inequality of Alzer*, J. Math. Anal. Appl. 192 (1995), 1034-1035.
[18] József Sándor, *Comments of an inequality for the sum of powers of positive numbers* , RGMIA Res. Rep. Coll. 2.
[19] József Sándor, *On an inequality of Alzer, II*, Octogon Math. Mag., 11(2003), No.2, 554-555 (1999), no 2, 259-261.
[20] József Sándor, *On an inequality of Alzer for negative powers*, RGMIA, 9(2006), NO.4, Art.4.
[21] J. S. Ume, *An elementary proof of Alzer's inequality*, Math . Japon. 44 (1996), no. 3 , 521-522.
[22] J. S. Ume, *An inequality for positive real numbers*, MIA, 5(2002), no.4, 693-696.
[23] J. S. Ume, *A simple proof of generalized Alzer inequality*, Indian J. Pure Appl. Math, 35 (2004), no. 8, 969-971.

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