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From regular modules to von Neumann regular rings via coordinatization*

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Abstract

In this paper we establish a very close link (in terms of von Neumann's coordinatization) between regular modules introduced by Zelmanowitz, on one hand, and von Neumann regular rings, on the other hand: we prove that the lattice $\mathcal{L}^{fg}(M)$ of all finitely generated submodules of a finitely generated regular module M, over an arbitrary ring, can be coordinatized as the lattice of all principal right ideals of some von Neumann regular ring S.

1 Introduction and Main Results

In 1936, John von Neumann defined a ring R to be regular if for any $r \in R$ there exists $s \in R$ such that $r = rsr$. Motivated by the coordinatization of projective geometry, which was being reworked at that time in

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terms of lattice, von Neumann introduced regular rings as an algebraic tool for studying certain lattice: in fact, regular rings were used to coordinatize complemented modular lattice, a lattice L being coordinatized by a regular ring R if it is isomorphic to the lattice $\mathbf{L}(R_R)$ of all principal right ideals of R.

J. Zelmanowitz in paper [8] followed the original elementwise definition of von Neumann and called a right $R-$module M regular if for any $m \in M$ there exists $g \in Hom_R(M, R)$ such that $mg(m) = m$. Since a morphism $f \in Hom_R(R, M)$ is uniquely given by an element $m \in M$, one can reformulate the regular module defined by Zelmanowitz as follows: for any $f \in Hom_R(R, M)$ there exists $g \in Hom_R(M, R)$ such that $f = f \circ g \circ f$.

In paper [2] it was defined the concept of a regular object with respect to another object (or a relative regular object) in an arbitrary category, which extends the notion of regular module. In particular, when we consider the category $\mathcal{M}_R$ of right $R-$modules we obtain the following definition

Definition 1.1 *Let M and U be two right $R-$modules. We say that M is $U-$regular module if for any $f \in Hom_R(U, M)$ there exists a morphism $g \in Hom_R(M, U)$ such that $f = f \circ g \circ f$.*

Obviously, M is a regular module if and only if M is $R-$regular. The concept of relative regular module has been proved to be an extremely useful tool in the theory of von Neumann regular rings and regular modules (see [2] and [3]).

For a right $R-$module M we denote by $\mathcal{L}^{fg}(M)$ the lattice of all finitely generated submodules of M, partially ordered by inclusion. The aim of this paper is to study the lattice $\mathcal{L}^{fg}(M)$ of all finitely generated submodules of a finitely generated regular module. We first prove that $\mathcal{L}^{fg}(M)$ is a complemented modular lattice.

Definition 1.2 *A lattice $\mathcal{L}$ is coordinatizable if there exists a von Neumann regular ring R such that $\mathcal{L}$ is isomorphic to the lattice $\mathbf{L}(R_R)$ of all principal right ideals of R.*

Using the isomorphism between the lattices $\mathcal{L}^{fg}(M)$ and $\mathcal{L}^{fg}(S_S)$, where S is the endomorphisms ring $End_R(M)$, we obtain our main result:

Theorem 1.3 *If R is an arbitrary ring and M is a finitely generated regular right $R-$module, then $\mathcal{L}^{fg}(M)$ is coordinatizable.*

The key step in proving this theorem is *Proposition 2.3*, which was inspired by a result given by von Neumann, for the case R_R, in his classical paper [6]. In order to adapt his result in the general case of regular modules we use the concept of relative regular module.

A ring R is called strongly regular if for any element $r \in R$ there exists an element $s \in R$ such that $r^2 s = r$. Finally, we will prove:

Theorem 1.4 *Let M be a finitely generated regular right module over an arbitrary ring R. Then $\mathcal{L}^{fg}(M)$ is distributive if and only if $End_R(M)$ is strongly regular ring.*

In this paper, all modules will be right modules. We refer the reder to Birkhoff [1] or Gratzer [5] for elements of lattice theory and to Gooderal [4] for definitions and results about von Neumann regular rings.

2 Proofs of the Main Results

In order to prove $\mathcal{L}^{fg}(M)$ is a complemented modular lattice, we will need the following preliminary result:

Lemma 2.1 *Let M be a regular $R-$module, let U be a finitely generated $R-$module and let $f \in Hom_R(U, M)$. If N is a finitely generated submodule of M, then $f^{-1}(N)$ is finitely generated $R-$module.*

Proof. Since M is a regular module and N is a finitely generated submodule of M, by [8, Theorem 1.6] we obtain that N is a direct summand of M. Therefore there exists a submodule N' of M such that $M = N \oplus N'$. But $M/N = (N + N')/N \simeq N'/(N \cap N') \simeq N'$ and then M/N is isomorphic with a direct summand of M, so M/N is a regular module. Since M/N is a regular module and U is a finitely generated module, by [2, Remark 4.3] it follows that M/N is $U-$regular.

Let $\pi : M \longrightarrow M/N$ be the canonical projection and we consider the morphism of $R-$modules $\pi \circ f : U \longrightarrow M/N$. By [2, Proposition 3.1], we obtain that $Ker(\pi \circ f) = f^{-1}(N)$ is a direct summand of U. Obviously, $f^{-1}(N)$ is a finitely generated $R-$module.

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Proposition 2.2 *Let R be an arbitrary ring and let M be a finitely generated regular right $R-$module. Then $\mathcal{L}^{fg}(M)$ is a complemented modular lattice (where $N_1 \vee N_2 = N_1 + N_2$ and $N_1 \wedge N_2 = N_1 \cap N_2$, for any $N_1, N_2 \in \mathcal{L}^{fg}(M)$).*

Proof. Let $N_1, N_2 \in \mathcal{L}^{fg}(M)$. Obviously, $N_1 + N_2 \in \mathcal{L}^{fg}(M)$ and $N_1 \vee N_2 = N_1 + N_2$. We want to prove that $N_1 \cap N_2 \in \mathcal{L}^{fg}(M)$. We consider the embedding morphism $i : N_1 \hookrightarrow M$. Then $N_1 \cap N_2 = i^{-1}(N_2)$, which is finitely generated by Lemma 2.1. Since $N_1 \wedge N_2 = N_1 \cap N_2$, we may assert that $\mathcal{L}^{fg}(M)$ is a lattice.

In general, the lattice $\mathcal{L}(M)$ of all submodules of an arbitrary module M with the lattice operations $+$ and $\cap$ is modular. Thus, $\mathcal{L}^{fg}(M)$ is a modular lattice, when M is a regular module.

If $N \in \mathcal{L}^{fg}(M)$, by [8, Theorem 1.6] we obtain that N is a direct summand of M. Hence we may find a submodule N' of M such that $M = N \oplus N'$. Since N' is a direct summand of M and M is finitely generated as $R-$module, then N' is finitely generated and therefore N' is a complement of N in $\mathcal{L}^{fg}(M)$. Thus $\mathcal{L}^{fg}(M)$ is a complemented lattice.

Proposition 2.3 *Let M be a finitely generated regular right $R-$module and $S = End_R(M)$. Then the lattices $\mathcal{L}^{fg}(M)$ and $\mathcal{L}^{fg}(S_S)$ are isomorphic.*

Proof. Since M is a finitely generated regular $R-$module, by [8, Corollary 4.2] we obtain that $S = End_R(M)$ is a von Neumann regular ring. We consider an arbitrary $J \in \mathcal{L}^{fg}(S_S)$. Then there exists an idempotent $e \in S$ such that $J = eS$. It follows that $JM = eSM = eM \in \mathcal{L}^{fg}(M)$. Thus we obtain the monotone map

$$\varphi : \mathcal{L}^{fg}(S_S) \longrightarrow \mathcal{L}^{fg}(M), \quad \varphi(J) = JM.$$

We will prove that φ is a lattice isomorphism.

For $N \in \mathcal{L}^{fg}(M)$ we denote $\psi(N) = \{f \in S \,|\, f(M) \leq N\}$. Since N is a direct summand of M, then there exists an idempotent $e \in S$ such that $e(M) = N$. Of course, $e_{|_N} = id_N$. We show that $\psi(N) = eS$.

Consider $f \in S$. It follows that $f(M) \leq M$ and thus $e(f(M)) \leq e(M) = N$. Hence $ef \in \psi(N)$ and therefore $eS \subseteq \psi(N)$. For the converse inclusion we consider $g \in \psi(N)$. Then $g \in S$ and $g(M) \leq N$. Since $e(g(m)) = g(m)$, for all $m \in M$, we obtain that $g \in eS$, so $\psi(N) \subseteq eS$. Hence $\psi(N) = eS \in \mathcal{L}^{fg}(S_S)$ and therefore we obtain the monotone map

$$\psi : \mathcal{L}^{fg}(M) \longrightarrow \mathcal{L}^{fg}(S_S), \quad \psi(N) = \{f \in S \,|\, f(M) \leq N\}$$

On one hand, if $e \in S$ is an arbitrary idempotent, then $\varphi(eS) = eM$ and $\psi(\varphi(eS)) = \psi(eM) = eS$, so $\psi \circ \varphi = id_{\mathcal{L}^{fg}(S_S)}$. On the other hand, $\varphi(\psi(eM)) = \varphi(eS) = eM$, so $\varphi \circ \psi = id_{\mathcal{L}^{fg}(M)}$. Thus, we can conclude that φ and ψ are lattice isomorphisms and therefore the lattices $\mathcal{L}^{fg}(M)$ and $\mathcal{L}^{fg}(S_S)$ are isomorphic.

Remark 2.4 If R is a von Neumann regular ring, by [4, Theorem 1.1] we obtain that the lattice $\mathcal{L}^{fg}(R_R)$ is, in fact, the lattice $\mathbf{L}(R_R)$ of all principal right ideals of R.

We are now in the position to prove our main results:

Proof of Theorem 1.3. When M is a finitely generated regular module, over an arbitrary ring R, by [8, Corollary 4.2] it follows that $S = End_R(M)$ is a von Neumann regular ring. Our Theorem is now a direct consequence of *Definition 1.2*, *Proposition 2.3* and *Remark 2.4*.

Proof of Theorem 1.4. Utumi proved in [7, Theorem 1.1] the following: A regular ring R is strongly regular if and only if the lattice $\mathbf{L}(R_R)$ of all principal right ideals of R is distributive.

Using the lattice isomorphism

$$\mathcal{L}^{fg}(M) \simeq \mathcal{L}^{fg}(S_S) = \mathbf{L}(S_S)$$

our theorem is now obvious.

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