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The Laplace transform method for one-dimensional hyperbolic equation with purely integral conditions

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Abstract

The aim of this paper is to prove existence, uniqueness, and continuous dependence upon the data of solution to hyperbolic equation with purely integral conditions. The proofs are based on a priori estimate and Laplace transform method. Finally, we obtain the solution by using a numerical technique for inverting the Laplace transforms.

Keywords and Phrases: Hyperbolic equation, Purely integral conditions, a priori estimates, Laplace transform method.

Mathematical Subject Classification (2010): 35L15, 35L20, 35B45, 44A10.

1 Introduction

In the rectangular domain $D = \{(x,t) : 0 < x < 1,\ 0 < t \leq T\}$, we consider a second order hyperbolic equation

$$\frac{\partial^2 v}{\partial t^2} - \alpha \frac{\partial^2 v}{\partial x^2} = g(x,t), \qquad 0 < x < 1, \qquad 0 < t \leq T, \tag{1}$$

subject to the initial conditions

$$\begin{aligned} v(x,0) &= \Phi(x), \qquad 0 < x < 1, && (2) \\ \frac{\partial v(x,0)}{\partial t} &= \Psi(x), \qquad 0 < x < 1, && (3) \end{aligned}$$

and the purely integral conditions

$$\int_0^1 v\,(x,t)\,dx \quad = \quad E(t), \qquad 0 < t \le T, \tag{4}$$

$$\int_0^1 xv\,(x,t)\,dx \quad = \quad M(t), \qquad 0 < t \le T, \tag{5}$$

where f, φ, ψ, E, and M are known functions, α and T are known positive constants.

In recent years, much attention has been focused on the study of hyperbolic equations with purely integral conditions. Such conditions appear in case, where for instance, direct measurement quantities are impossible and their mean values are known. Such situations take place in studying, for example, in elastodynamic. In this case, conditions $(4)-(5)$ are known as the static and the moment equilibria. The first investigation of this type of problems goes back to [4] in 1996, in which the author proved the existence, uniqueness, and continuous dependence of the solution upon the data of certain hyperbolic problems with only integral boundary conditions. Later, similar problems have been studied in [6, 9, 10, 12, 18] by using the energetic method and the Rothe time-discretization method. We refer the reader to [3, 4, 5, 7, 8, 9, 11, 14, 17, 20, 21, 22] for hyperbolic equations with Neumann and integral condition. For other problems with nonlocal conditions, related to other equations, we apply to [2, 4, 9, 10, 11] and references therein.

The presence of integral terms in the boundary conditions can greatly complicate the application of standard numerical techniques. The main tool used in this paper is the Laplace transform of the problem and then used the numerical technique for the inverse Laplace transform to obtain the solution. We use a numerical method for inverting the Laplace transform to get the solution.

The paper is organized as follows. In Section 2, we begin by introducing certain function spaces which are often used in the next sections, and we reduce the posed problem to one with homogeneous boundary conditions. In Section 3, we first establish a priori estimates, then the uniqueness and continuous dependence are direct consequences. In Section 4, we establish the existence of the solution by the Laplace transform. Finally, numerical examples are provided to demonstrate the validity and applicability of the method.

2 Statement of the problem and notations

Since integral boundary conditions are inhomogeneous, it is convenient to convert problem $(1)-(5)$ to an equivalent problem with homogeneous integral conditions. For this, we introduce a new function $u(x,t)$ representing the deviation of the function $v(x,t)$ from the function

$$u\,(x,t) = v(x,t) - u_1(x,t) \tag{6}$$

where

$$u_1(x,t) = E(t) + 6\left(3x^2 - 2x\right).\left(2M\,(t) - E(t)\right). \tag{7}$$

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Problems $(1)-(5)$ with inhomogeneous integral conditions (4), (5) can be equivalently reduced to the problem of finding a function u satisfying:

$$\frac{\partial^2 u}{\partial t^2} - \alpha \frac{\partial^2 u}{\partial x^2} = f\left(x,t\right), \qquad 0 < x < 1, \qquad 0 < t \leq T, \tag{8}$$

$$u(x,0) = \varphi\left(x\right), \qquad 0 < x < 1, \tag{9}$$

$$\frac{\partial u(x,0)}{\partial t} = \psi\left(x\right), \qquad 0 < x < 1, \tag{10}$$

$$\int_0^1 u\left(x,t\right) dx = 0, \qquad 0 < t \leq T, \tag{11}$$

$$\int_0^1 xu\left(x,t\right) dx = 0, \qquad 0 < t \leq T, \tag{12}$$

where

$$f\left(x,t\right) = g(x,t) - \left(\frac{\partial^2 u_1}{\partial t^2} - \alpha \frac{\partial^2 u_1}{\partial x^2}\right), \tag{13}$$

and

$$\varphi\left(x\right) = \Phi\left(x\right) - u_1\left(x,0\right), \tag{14}$$

$$\psi\left(x\right) = \Psi\left(x\right) - u_1\left(x,0\right). \tag{15}$$

Hence, instead of looking for v, we simply look for u. The solution of problem $(1)-(5)$ will be obtained by the relations $(6)-(7)$.

We introduce the appropriate function spaces that will be used in the rest of the note. Let H be a Hilbert space with a norm $\left\|\cdot\right\|_H$.

Let $L^2(0,1)$ be the standard function space.

Definition 1 *(i) Denote by $L^2(0,T;H)$ the set of all measurable abstract functions $u\left(.,t\right)$ from $(0,T)$ into H such that*

$$\left\|u\right\|_{L^2(0,T;H)} = \left(\int_0^T \left\|u\left(.,t\right)\right\|_H^2 dt\right)^{1/2} < \infty. \tag{16}$$

ii) Let $C(0,T;H)$ be the set of all continuous functions $u(\cdot,t) : (0,T) \to H$ with

$$\left\|u\right\|_{C(0,T;H)} = \max_{0\leq t\leq T} \left\|u\left(.,t\right)\right\|_H < \infty. \tag{17}$$

We denote by $C_0\left(0,1\right)$ the vector space of continuous functions with compact support in $(0,1)$. Since such functions are Lebesgue integrable with respect to dx, we can define on $C_0(0,1)$ the bilinear form given by

$$((u,w)) = \int_0^1 \Im_x^m u \cdot \Im_x^m w dx, \qquad m \geq 1, \tag{18}$$

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where

$$\Im_x^m u = \int_0^x \frac{(x-\xi)^{m-1}}{(m-1)!} u(\xi,t)\, d\xi, \qquad m \geq 1. \tag{19}$$

The bilinear form (18) is considered as a scalar product on $C_0(0,1)$ for which $C_0(0,1)$ is not complete.

Definition 2 *Denote by $B_2^m(0,1)$, a completion $C_0(0,1)$ for the scalar product (18), which is denoted $(.,.)_{B_2^m(0,1)}$, called Bouziani space. By the norm of function u from $B_2^m(0,1)$, $m \geq 1$, we understand the nonnegative number :*

$$\|u\|_{B_2^m(0,1)} = \left(\int_0^1 (\Im_x^m u)^2\, dx \right)^{1/2} = \|\Im_x^m u\| \qquad m \geq 1. \tag{20}$$

Lemma 3 *For all $m \in \mathbb{N}^*$, the following inequality holds:*

$$\|u\|^2_{B_2^m(0,1)} \leq \frac{1}{2} \|u\|^2_{B_2^{m-1}(0,1)}. \tag{21}$$

Corollary 4 *For all $m \in \mathbb{N}^*$, we have the elementary inequality*

$$\|u\|^2_{B_2^m(0,1)} \leq \left(\frac{1}{2}\right)^m \|u\|^2_{L^2(0,1)}. \tag{22}$$

Definition 5 *We denote by $L^2(0,T;B_2^m(0,1))$ the space of functions which are square integrable in the Bochner sense, with the scalar product*

$$(u,w)_{L^2(0,T;B_2^m(0,1))} = \int_0^T (u(.,t), w(.,t))_{B_2^m(0,1)}\, dt. \tag{23}$$

Since the space $B_2^m(0,1)$ is a Hilbert space, it can be shown that $L^2(0,T;B_2^m(0,1))$ is a Hilbert space as well. The set of all continuous abstract functions in $[0,T]$ equipped with the norm

$$\sup_{0\leq t\leq T} \|u(.,t)\|_{B_2^m(0,1)}$$

is denoted $C(0,T;B_2^m(0,1))$.

Corollary 6 *For every $u \in L^2(0,1)$, from which we deduce the continuity of the imbedding $L^2(0,1) \longrightarrow B_2^m(0,1)$, for $m \geq 1$.*

Lemma 7 (***Gronwall Lemma***) *Let $f_1(t)$, $f_2(t) \geq 0$ be two integrable functions on $[0,T]$, $f_2(t)$ is nondecreasing. If*

$$f_1(\tau) \leq f_2(\tau) + c\int_0^\tau f_1(t)\, dt, \qquad \forall \tau \in [0,T], \tag{24}$$

where $c \in \mathbb{R}^+$, then

$$f_1(t) \leq f_2(t) \exp(ct), \qquad \forall t \in [0,T]. \tag{25}$$

Proof. The proof is the same as that of Lemma 1.3.19 in [16]. ■

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3 Uniqueness and Continuous dependence of the Solution

We first establish an a priori estimate, the uniqueness and continuous dependence of the solution with respect to the data are immediate consequences.

Theorem 8 *If $u(x,t)$ is a solution of problem $(8)-(12)$ and $f \in C\left(\overline{D}\right)$, then we have*

$$\begin{aligned} &\|u\|^2_{C(0,T;L^2(0,1))} \\ \leq\ & c_1\left(\int_0^\tau \|f(.,t)\|^2_{B_2^1(0,1)}\,dt + \|\varphi\|^2_{L^2(0,1)} + \|\psi\|^2_{B_2^1(0,1)}\right), \end{aligned} \tag{26}$$

$$\begin{aligned} &\left\|\frac{\partial u}{\partial t}\right\|^2_{C\left(0,T;B_2^1(0,1)\right)} \\ \leq\ & c_2\left(\int_0^\tau \|f(.,t)\|^2_{B_2^1(0,1)}\,dt + \|\varphi\|^2_{L^2(0,1)} + \|\psi\|^2_{B_2^1(0,1)}\right), \end{aligned} \tag{27}$$

where

$$c_1 = \frac{\max(1,2\alpha)\exp T}{\alpha},$$
$$c_2 = \max(1,2\alpha)\exp T$$

and $0 \leq \tau \leq T$.

Proof. Taking the scalar product in $B_2^1(0,1)$ of equation (8) and $\frac{\partial u}{\partial t}$, and integrating over $(0,\tau)$, we have

$$\begin{aligned} &\int_0^\tau \left(\frac{\partial^2 u(.,t)}{\partial t^2}, \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(0,1)} dt - \alpha\int_0^\tau \left(\frac{\partial^2 u(.,t)}{\partial x^2}, \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(0,1)} dt \\ =\ & \int_0^\tau \left(f(.,t), \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(0,1)} dt. \end{aligned} \tag{28}$$

The integration by parts on the left-hand side of (28), we obtain

$$\begin{aligned} &\frac{1}{2}\left\|\frac{\partial u(.,\tau)}{\partial t}\right\|^2_{B_2^1(0,1)} - \frac{1}{2}\|\psi\|^2_{B_2^1(0,1)} + \frac{\alpha}{2}\|u(.,\tau)\|^2_{L^2(0,1)} - \alpha\|\varphi\|^2_{L^2(0,1)} \\ =\ & \int_0^\tau \left(f(.,t), \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(0,1)} dt. \end{aligned} \tag{29}$$

By the Cauchy inequality, the right-hand side of (28) is bounded by

$$\frac{1}{2}\int_0^\tau \|f(.,t)\|^2_{B_2^1(0,1)}\,dt + \frac{1}{2}\int_0^\tau \left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{B_2^1(0,1)} dt. \tag{30}$$

With substitution of (30) into (29), yields

$$\begin{aligned} & \left\| \frac{\partial u\left(.,\tau\right)}{\partial t} \right\|^2_{B_2^1(0,1)} + \alpha \left\| u\left(.,\tau\right) \right\|^2_{L^2(0,1)} \qquad (31) \\ \leq \; & \max\left(1, 2\alpha\right) \left(\int_0^\tau \left\| f\left(.,t\right) \right\|^2_{B_2^1(0,1)} dt + \left\| \varphi \right\|^2_{L^2(0,1)} + \left\| \psi \right\|^2_{B_2^1(0,1)} \right) \\ & + \int_0^\tau \left\| \frac{\partial u\left(.,t\right)}{\partial t} \right\|^2_{B_2^1(0,1)} dt \\ \leq \; & \max\left(1, 2\alpha\right) \left(\int_0^\tau \left\| f\left(.,t\right) \right\|^2_{B_2^1(0,1)} dt + \left\| \varphi \right\|^2_{L^2(0,1)} + \left\| \psi \right\|^2_{B_2^1(0,1)} \right) \\ & + \int_0^\tau \left(\left\| \frac{\partial u\left(.,t\right)}{\partial t} \right\|^2_{B_2^1(0,1)} + \alpha \left\| u\left(.,t\right) \right\|^2_{L^2(0,1)} \right) dt, \end{aligned}$$

and by Gronwall Lemma**,** we have

$$\begin{aligned} & \left\| \frac{\partial u\left(.,\tau\right)}{\partial t} \right\|^2_{B_2^1(0,1)} + \alpha \left\| u\left(.,\tau\right) \right\|^2_{L^2(0,1)} \qquad (32) \\ \leq \; & \max\left(1, 2\alpha\right) \left(\int_0^T \left\| f\left(.,t\right) \right\|^2_{B_2^1(0,1)} dt + \left\| \varphi \right\|^2_{L^2(0,1)} + \left\| \psi \right\|^2_{B_2^1(0,1)} \right). \end{aligned}$$

Since the right-hand side of (32) is independent of τ, we take the supremum with respect to τ from 0 to T in the left-hand side, thus obtaining (26) and (27). ■

Corollary 9 *If problem $(8)-(12)$ has a solution, then this solution is unique and depends continuously on (f, φ, ψ).*

4 Existence of the Solution

The Laplace transform is an efficient method for solving many differential equations and partial differential equations. The main difficulty with Laplace transform method is in inverting the Laplace domain solution into the real domain. In this section we shall apply the Laplace transform technique to find solutions of partial differential equations, we have the Laplace transform

$$V\left(x,s\right) = \mathcal{L}\left\{ v\left(x,t\right); t \longrightarrow s \right\} = \int_0^\infty v\left(x,t\right) \exp\left(-st\right) dt, \qquad (33)$$

where s is positive reel parameter. Taking the Laplace transforms on both sides of (1), we have

$$s^2 V\left(x,s\right) - \alpha \frac{d^2}{dx^2}\left[V\left(x,s\right)\right] = G\left(x,s\right) + s\Phi\left(x\right) + \Psi\left(x\right) \;, \qquad (34)$$

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where $G(x,s) = \mathcal{L}\{g(x,t); t \longrightarrow s\}$. Similarly, we have

$$\int_0^1 V(x,s)\,dx = A(s), \tag{35}$$

$$\int_0^1 xV(x,s)\,dx = B(s), \tag{36}$$

where

$$A(s) = \mathcal{L}\{E(t); t \longrightarrow s\}$$

and

$$B(s) = \mathcal{L}\{M(t); t \longrightarrow s\}.$$

Thus, considered equation is reduced in boundary value problem governed by second order inhomogeneous ordinary differential equation. We obtain a general solution of (31) as

$$\begin{aligned} V(x,s) &= -\frac{\sqrt{\alpha}}{s}\int_0^x [G(\tau,s) + s\Phi(\tau) + \Psi(\tau)\ ]\sinh\left(\frac{s}{\sqrt{\alpha}}[x-\tau]\right)d\tau \\ &\quad + C_1(s)\exp\left(-\frac{s}{\sqrt{\alpha}}x\right) + C_2(s)\exp\left(\frac{s}{\sqrt{\alpha}}x\right), \end{aligned} \tag{37}$$

where $C_1$and C_2 are arbitrary functions of s. With substitution of (34) into $(32)-(33)$, we have

$$\begin{aligned} & C_1(s)\int_0^1 \exp\left(-\frac{s}{\sqrt{\alpha}}x\right)dx + C_2(s)\int_0^1 \exp\left(\frac{s}{\sqrt{\alpha}}x\right)dx \\ = & \frac{\sqrt{\alpha}}{s}\int_0^1\left[[F(\tau,s) + s\varphi(\tau) + \psi(\tau)]\int_\tau^1 \sinh\left(\frac{s}{\sqrt{\alpha}}[x-\tau]\right)dx\right]d\tau \\ & + A(s), \end{aligned} \tag{38}$$

$$\begin{aligned} & C_1(s)\int_0^1 x\exp\left(-\frac{s}{\sqrt{\alpha}}x\right)dx + C_2(s)\int_0^1 x\exp\left(\frac{s}{\sqrt{\alpha}}x\right)dx \\ = & \frac{\sqrt{\alpha}}{s}\int_0^1\left[[G(\tau,s) + s\Phi(\tau) + \Psi(\tau)\ ]\int_\tau^1 x\sinh\left(\frac{s}{\sqrt{\alpha}}[x-\tau]\right)dx\right]d\tau \\ & + B(s), \end{aligned} \tag{39}$$

where

$$\begin{pmatrix} C_1(s) \\ C_2(s) \end{pmatrix} = \begin{pmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{pmatrix}^{-1} \times \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix}, \tag{40}$$

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and

$$\begin{aligned}
a_{11}(s) &= \int_0^1 \exp\left(-\frac{s}{\sqrt{\alpha}}x\right)dx, \\
a_{12}(s) &= \int_0^1 \exp\left(\frac{s}{\sqrt{\alpha}}x\right)dx, \\
a_{21}(s) &= \int_0^1 x\exp\left(-\frac{s}{\sqrt{\alpha}}x\right)dx, \qquad (41)\\
a_{22}(s) &= \int_0^1 x\exp\left(\frac{s}{\sqrt{\alpha}}x\right)dx, \\
b_1(s) &= \frac{\sqrt{\alpha}}{s}\int_0^1 \left[\left[G(\tau,s)+s\Phi(\tau)+\Psi(\tau)\ \right]\int_\tau^1 \sinh\left(\frac{s}{\sqrt{\alpha}}[x-\tau]\right)dx\right]d\tau \\
&\quad +A(s),
\end{aligned}$$

$$\begin{aligned}
b_2(s) &= \frac{\sqrt{\alpha}}{s}\int_0^1 \left[\left[G(\tau,s)+s\Phi(\tau)+\Psi(\tau)\ \right]\int_\tau^1 x\sinh\left(\frac{s}{\sqrt{\alpha}}[x-\tau]\right)dx\right]d\tau \\
&\quad +B(s).
\end{aligned}$$

If it is not possible to calculate the integrals directly, then we calculate it numerically. We approximate similarly as given in [2]. If the Laplace inversion is possible directly for (37), in this case we shall get our solution. Gauss's formula **(25.4.30)** given in Abramowitz and Stegun [1] may be employed to calculate these integrals numerically, we have the following approximations for the integrals:

$$\begin{aligned}
&\int_0^1 \exp\left(\pm\frac{s}{\sqrt{\alpha}}x\right)dx \\
&\simeq \frac{1}{2}\sum_{i=1}^{N} w_i \exp\left(\pm\frac{s}{\sqrt{\alpha}}[x_i+1]\right),
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 x\exp\left(\pm\frac{s}{\sqrt{\alpha}}x\right)dx \\
&\simeq \frac{1}{2}\sum_{i=1}^{N} w_i \left(\frac{1}{2}[x_i+1]\right)\exp\left(\pm\frac{s}{\sqrt{\alpha}}[x_i+1]\right),
\end{aligned}$$

$$\begin{aligned}
&\int_0^x \left[G(\tau,s)+s\Phi(\tau)+\Psi(\tau)\ \right]\sinh\left(\frac{s}{\sqrt{\alpha}}[x-\tau]\right)d\tau \\
&\simeq \frac{x}{2}\sum_{i=1}^{N} w_i\left[G\left(\frac{x}{2}[x_i+1];s\right)+s\Phi\left(\frac{x}{2}[x_i+1]\right)+\Psi\left(\frac{x}{2}[x_i+1]\right)\right]\times \\
&\quad\times\sinh\left(\frac{s}{\sqrt{\alpha}}\left[x-\frac{x}{2}[x_i+1]\right]\right),
\end{aligned}$$

$$
\begin{aligned}
&\int_0^1 \left[\left[G(\tau, s) + s\Phi(\tau) + \Psi(\tau) \right] \int_\tau^1 \sinh\left(\frac{s}{\sqrt{\alpha}} [x - \tau] \right) dx \right] d\tau \\
\simeq\ & \frac{1}{4} \sum_{i=1}^{N} w_i \left[G\left(\frac{1}{2} [x_i + 1]; s \right) + s\Phi\left(\frac{1}{2} [x_i + 1] \right) + \Psi\left(\frac{1}{2} [x_i + 1] \right) \right] \left(1 - \frac{1}{2} [x_i + 1] \right) \times \\
&\times \sum_{i=1}^{N} w_j \sinh\left(\frac{s}{\sqrt{\alpha}} \left[\frac{1}{2} \left[\left(1 - \frac{1}{2} [x_i + 1] \right) x_j + \left(1 + \frac{1}{2} [x_i + 1] \right) \right] - \frac{1}{2} (x_i + 1) \right] \right),
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 \left[\left[F(\tau, s) + s\varphi(\tau) + \psi(\tau) \right] \int_\tau^1 x \sinh\left(\frac{s}{\sqrt{\alpha}} [x - \tau] \right) dx \right] d\tau \\
\simeq\ & \frac{1}{4} \sum_{i=1}^{N} w_i \left[G\left(\frac{1}{2} [x_i + 1]; s \right) + s\Phi\left(\frac{1}{2} [x_i + 1] \right) + \Psi\left(\frac{1}{2} [x_i + 1] \right) \right] \left(1 - \frac{1}{2} [x_i + 1] \right) \times \\
&\times \left(\frac{1}{2} \left[\left(1 - \frac{1}{2} [x_i + 1] \right) x_j + \left(1 + \frac{1}{2} [x_i + 1] \right) \right] \right) \times \\
&\times \sum_{i=1}^{N} w_j \sinh\left(\frac{s}{\sqrt{\alpha}} \left[\frac{1}{2} \left[\left(1 - \frac{1}{2} [x_i + 1] \right) x_j + \left(1 + \frac{1}{2} [x_i + 1] \right) \right] - \frac{1}{2} (x_i + 1) \right] \right), \quad (42)
\end{aligned}
$$

where x_i and w_i are the abscissa and weights, defined as

$$
x_i : i^{th} \text{ zero of } P_n(x), \qquad \omega_i \ = 2/\left(1 - x_i^2\right) \left[P_n'(x) \right]^2.
$$

Their tabulated values can be found in [1] for different values of N.

Numerical inversion of Laplace transform Sometimes, an analytical inversion of a Laplace domain solution is difficult to obtain; therefore a numerical inversion method must be used. A nice comparison of four frequently used numerical Laplace inversion algorithms is given by Hassan Hassanzadeh, Mehran Pooladi-Darvish [15]. In this work we use the Stehfest's algorithm [19] which it is easy to implement. This numerical technique was first introduced by Graver [13] and its algorithm then offered by [19]. Stehfest's algorithm approximates the time domain solution as

$$
v(x, t) \approx \frac{\ln 2}{t} \sum_{n=1}^{2m} \beta_n V\left(x; \frac{n \ln 2}{t} \right), \tag{43}
$$

where, m is the positive integer,

$$
\beta_n = (-1)^{n+m} \sum_{k=\left[\frac{n+1}{2}\right]}^{\min(n,m)} \frac{k^m (2k)!}{(m-k)!k!\,(k-1)!\,(n-k)!\,(2k-n)!}, \tag{44}
$$

and $[q]$ denotes the integer part of the real number q. Then by relations $(6)-(7)$ we obtain the unknown function u.

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5 Numerical Examples

In this section, we report some results of numerical computations using Laplace transform method proposed in the previous section. These technique are applied to solve the problem defined by $(1)-(5)$ for particular functions g, Φ, Ψ, E, M, and positive constant α. The method of solution is easily implemented on the computer, used Matlab 7.9.3 program. The numerical results in triple figures (figure 1, 2, 3) are in excellent agreement with the exact solution.

Example 10 *We take*

$$\begin{aligned} g(x,t) &= 0, \quad 0<x<1, \quad 0<t\le T, \quad \alpha=1, \\ \Phi(x) &= \exp(-x), \quad 0<x<1, \\ \Psi(x) &= 0, \quad 0<x<1, \\ E(t) &= \left(1-e^{-1}\right)\cosh(t), \quad 0<t\le T\ , \\ M(t) &= \left(1-e^{-1}\right)\cosh(t), \quad 0<t\le T\ , \end{aligned}$$

in this case exact solution given by

$$v(x,t) = e^{-x}\cosh(t), \quad 0<x<1, \quad 0<t\le T.$$

The method of solution is easily implemented on the computer, numerical results obtained by $N = 8$ in (42) and $m = 5$ in (43), then we compared the exact solution with numerical solution. For $t = 0.10$, $x \in [0.10, 0.90]$, we calculate v numerically using the proposed method of solution and compare it with the exact solution in Table 1.

x	0.10	0.30	0.50	0.70	0.90
v *exact*	0.9093654	0.7445254	0.6095658	0.490703	0.4086042
v *numerical*	0.9093851	0.7443921	0.6097452	0.500183	0.4080919
error	0.000217	-0.0001790	0.0002943	0.0022295	-0.0012538

Table 1

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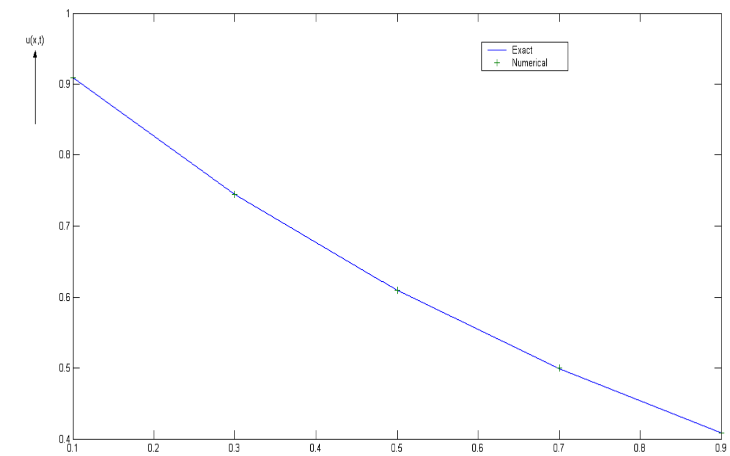

FIGURE 1. Comparision between numerical and exact solution for t=0.1, 0<x<1.

Example 11 *We take*

$$
\begin{aligned}
g(x,t) &= \pi^2\left(1+\pi^2\right)\exp\left(-\pi^2 t\right)\sin(\pi x), \qquad 0<x<1, \qquad 0<t\le T, \qquad \alpha=1,\\
\Phi(x) &= \sin(\pi x), \qquad 0<x<1,\\
\Psi(x) &= -\pi^2\sin(\pi x), \qquad 0<x<1,\\
E(t) &= \frac{2}{\pi}\exp\left(-\pi^2 t\right), \qquad 0<t\le T\ ,\\
M(t) &= -\frac{1}{\pi}\exp\left(-\pi^2 t\right), \qquad 0<t\le T,
\end{aligned}
$$

in this case exact solution given by

$$v(x,t)=\exp\left(-\pi^2 t\right)\sin(\pi x), \qquad 0<x<1, \qquad 0<t\le T.$$

For $t=0.10$, $x\in[0.10,0.90]$, we calculate v numerically using the proposed method of solution and compare it with the exact solution in Table 2:

x	0.10	0.30	0.50	0.70	0.90
v *exact*	0.1151730	0.3015269	0.3727078	0.3015270	0.1151730
v *numerical*	0.1150014	0.3015291	0.3728361	0.3012199	0.1152185
error	-0.0014899	0.0000073	0.0003442	-0.0010185	0.0003951

Table 2

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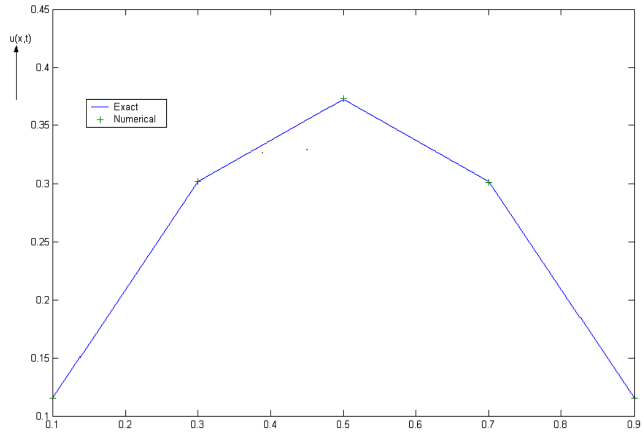

FIGURE 2. Comparision between numerical and exact solution for $t=0.1$, $0<x<1$

Example 12 *We take*

$$
\begin{aligned}
g(x,t) &= \left(1+4\pi^2\right)e^{-t}\cos(2\pi x), \qquad 0<x<1, \qquad 0<t\le T \qquad \alpha=1,\\
\Phi(x) &= \cos(2\pi x), \qquad 0<x<1,\\
\Psi(x) &= -\cos(2\pi x), \qquad 0<x<1,\\
E(t) &= 0, \qquad 0<t\le T\ ,\\
M(t) &= 0, \qquad 0<t\le T,
\end{aligned}
$$

in this case exact solution given by

$$v(x,t)=e^{-t}\cos(2\pi x), \qquad 0<x<1, \qquad 0<t\le T.$$

For $t=0.1$, $x\in[0.1,0.9]$, we calculate v numerically using the proposed method of solution and compare it with the exact solution in Table 3:

x	0.10	0.30	0.50	0.70	0.90
v *exact*	0.7320288	-0.2796101	-0.9048374	-0.2796101	0.7320288
v *numerical*	0.7324162	-0.2795921	-0.9047562	-0.2795421	0.7321329
error	0.0005292	-0.0000644	-0.0000897	-0.0002432	0.0001422

Table 3

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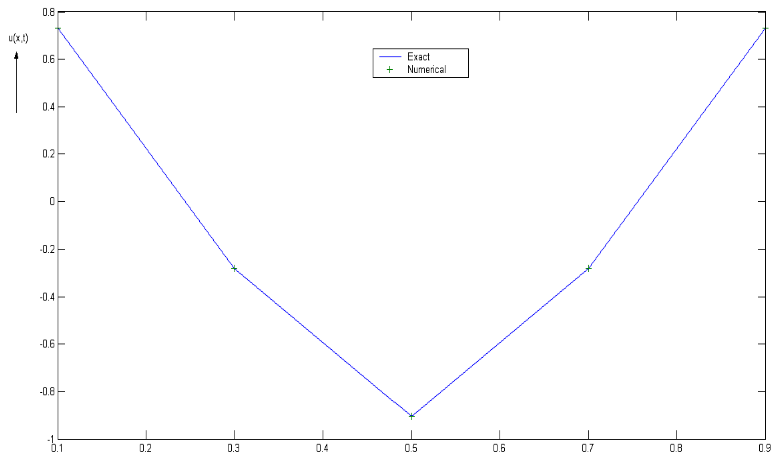

FIGURE 3. Comparision between numerical and exact solutions for $t=0.1$, $0<x<1$.

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