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Existence and Uniqueness Results for Nonlinear Implicit Fractional Differential Equations with Boundary Conditions

Mouffak Benchohra[a,b] [1] **and Jamal E. Lazreg**[a]

[a]Laboratory of Mathematics, University of Sidi Bel-Abbès,
P. O. Box 89, Sidi Bel-Abbès 22000, Algeria
e-mail: benchohra@yahoo.com, lazregjamal@yahoo.fr

[b] Department of Mathematics, King Abdulaziz University
P.O. Box 80203, Jeddah 21589, Saudi Arabia.

Abstract

In this paper, we establish the existence and uniqueness of solution for a class of boundary value problems for implicit fractional differential equations with Caputo fractional derivative. The arguments are based upon the Banach contraction principle, Schauder's fixed point theorem and the nonlinear alternative of Leray-Schauder type. As applications, two examples are included to show the applicability of our results.

Key words and phrases: Boundary value problem, Caputo's fractional derivative, implicit fractional differential equations, fractional integral, existence, Green's function, Gronwall's lemma, fixed point.
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1 Introduction

Fractional order differential equations are generalizations of classical integer order differential equations. Fractional differential equations can describe many phenomena in various fields of applied sciences and engineering such as acoustic, control, signal processing, porous media, electrochemistry, viscoelasticity, rheology, polymer physics, proteins, electromagnetics, optics, medicine, economics, astrophysics, chemical engineering, chaotic dynamics, statistical physics, thermodynamics, biosciences, bioengineering, etc. See for example [1, 2, 4, 5, 8, 10, 11, 12, 14, 15], and references therein.

Recently, considerable attention has been given to the existence of solutions of boundary value problem and boundary conditions for implicit fractional differential equations and integral equations with Caputo fractional derivative. See for example [3, 6, 7, 13, 16], and references therein.

[1]Corresponding author

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.60-72

Motivated by the above cited works, the purpose of this paper, is to establish existence and uniqueness results to the following implicit fractional-order differential equation:

$$^cD^\alpha y(t) = f(t, y(t), {}^c D^\alpha y(t)), \text{ for each, } t \in J = [0, T], \ T > 0, \ 1 < \alpha \leq 2, \quad (1)$$

$$y(0) = y_0, \quad y(T) = y_1, \quad (2)$$

where $^cD^\alpha$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and $y_0, \ y_1 \in \mathbb{R}$.

In this paper we present three results for the problem $(1) - (2)$. The first one is based on the Banach contraction principle, the second one on Schauder's fixed point theorem, and the last one on the nonlinear alternative of Leray-Schauder type. Finally, we present two illustrative examples.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}.$$

Definition 2.1 *([11, 14]). The fractional (arbitrary) order integral of the function $h \in L^1([0, T], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} h(s) ds,$$

where Γ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt, \ \alpha > 0$.

Definition 2.2 *([11]). For a function h given on the interval $[0, T]$, the Caputo fractional-order derivative of order α of h, is defined by*

$$(^cD^\alpha h)(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n - \alpha - 1} h^{(n)}(s) ds,$$

where $n = [\alpha] + 1$.

Lemma 2.3 *([11]) Let $\alpha \geq 0$ and $n = [\alpha] + 1$. Then*

$$I^\alpha (^cD^\alpha f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^k(0)}{k!} t^k.$$

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.60-72

We need the following auxiliary lemmas.

Lemma 2.4 *([16]) Let $\alpha > 0$. Then the differential equation*

$$^cD^\alpha k(t) = 0$$

has solutions $k(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\alpha] + 1$.

Lemma 2.5 *([16]) Let $\alpha > 0$. Then*

$$I^{\alpha c} D^\alpha k(t) = k(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\alpha] + 1$.

Theorem 2.6 *([9]) (Banach's fixed point theorem). Let C be a non-empty closed subset of a Banach space X. Then any contraction mapping T of C into itself has a unique fixed point.*

Theorem 2.7 *([9]) (Schauder's fixed point theorem). Let X be a Banach space. C be a closed, convex and nonempty subset of X. Let $N : C \to C$ be a continuous mapping such that $N(C)$ is a relatively compact subset of X. Then N has at least one fixed point in C.*

Theorem 2.8 *([9]) ([Nonlinear Alternative of Leray-Schauder type). Let X be a Banach space with $C \subset X$ closed and convex. Assume U is a relatively open subset of C with $0 \in U$ and $N : \overline{U} \to C$ is a compact map. Then either,*

(i) N has a fixed point in $\overline{U}$; or

(ii) there is a point $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda N(u)$.

3 Existence of Solutions

Let us defining what we mean by a solution of problem (1)-(2).

Definition 3.1 *A function $u \in C^1(J, \mathbb{R})$ is said to be a solution of the problem $(1)-(2)$ is u satisfied equation (1) on J and conditions (2).*

For the existence of solutions for the problem $(1) - (2)$, we need the following auxiliary lemmas:

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.60-72

Lemma 3.2 *Let $1 < \alpha \leq 2$ and $g : J \to \mathbb{R}$ be continuous. A function y is a solution of the fractional boundary value problem*

$$
\begin{gathered}
{}^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t)), \text{ for each, } t \in J, \ 1 < \alpha \leq 2, \\
y(0) = y_0, \quad y(T) = y_1,
\end{gathered}
$$

if and only if, y is a solution of the fractional integral equation

$$
y(t) = l(t) + \int_0^T G(t,s) f\left(s, l(s) + \int_0^T G(t,\tau) g(\tau) d\tau, g(s)\right) ds, \tag{3}
$$

where

$$
l(t) = (1 - \frac{t}{T}) y_0 + \frac{t}{T} y_1 = y_0 + \frac{(y_1 - y_0)}{T} t, \tag{4}
$$

$$
{}^cD^\alpha y(t) = g(t)
$$

and

$$
G(t,s) = \frac{1}{\Gamma(\alpha)} \left\{ \begin{array}{ll} (t-s)^{\alpha-1} - \frac{t}{T}(T-s)^{\alpha-1} & \text{if } \ 0 \leq s \leq t \\ -\frac{t}{T}(T-s)^{\alpha-1} & \text{if } \ t \leq s \leq T. \end{array} \right\} \tag{5}
$$

Proof: By Lemma 2.5 we reduce $(1) - (2)$ to the equation

$$
y(t) = I^\alpha g(t) + c_0 + c_1 t = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds + c_0 + c_1 t
$$

for some constants c_0, and $c_1 \in \mathbb{R}$. Conditions (2) give

$$
c_0 = y_0 \text{ and } c_1 = \frac{1}{T} y_T - \frac{1}{T} y_0 - \frac{1}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g(s) ds.
$$

Then the solution of $(1) - (2)$ is given by

$$
\begin{aligned}
y(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g(s) ds \\
&+ (1 - \frac{t}{T}) y_0 + \frac{t}{T} y_1 \\
&= \frac{1}{\Gamma(\alpha)} \Big[\int_0^t [(t-s)^{\alpha-1} - \frac{t}{T}(T-s)^{\alpha-1}] g(s) ds \\
&- \frac{t}{T} \int_t^T (T-s)^{\alpha-1} g(s) ds \Big] + (1 - \frac{t}{T}) y_0 + \frac{t}{T} y_1.
\end{aligned}
$$

Hence we get (3). Inversely, if y satisfies (3), then equations (1) and (2) hold.

From the expression of $G(t,s)$, it is obvious that $G(t,s)$ is continuous on $[0,T]\times[0,T]$. Denote by

$$G^* := \sup\{|G(t,s)|, \quad (t,s)\in J\times J\}.$$

We are now in a position to state and prove our existence result for the problem (1) − (2) based on Banach's fixed point.

Theorem 3.3 *Assume*

(H1) The function $f : J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *is continuous.*

(H2) There exist constants $K>0$ *and* $0<L<1$ *such that*

$$|f(t,u,v)-f(t,\bar{u},\bar{v})|\le K|u-\bar{u}|+L|v-\bar{v}|$$

for any $u,v,\bar{u},\bar{v}\in\mathbb{R}$ *and* $t\in J$.

If

$$\frac{KTG^*}{1-L}<1, \tag{6}$$

then there exists a unique solution for the boundary value problem (1) − (2).

Proof. The proof will be given in several steps. Transform the problem (1) − (2) into a fixed point problem. Define the operator $N : C(J,\mathbb{R})\to C(J,\mathbb{R})$ by:

$$N(y)(t) = l(t) + \int_0^T G(t,s)k(s)ds, \tag{7}$$

where $k\in C(J)$ satisfies the implicit functional equation

$$k(t) = f(t, y(t), k(t)),$$

l and G are the functions defined by (4) and (5) respectively.

Clearly, the fixed points of operator N are solutions of problem (1) − (2). Let $u,w\in C(J,\mathbb{R})$. Then for $t\in J$, we have

$$(Nu)(t)-(Nw)(t) = \int_0^T G(t,s)\Big(g(s)-h(s)\Big)ds,$$

where $g,h\in C(J,\mathbb{R})$ be such that

$$g(t) = f(t,u(t),g(t)),$$

and

$$h(t) = f(t,w(t),h(t)).$$

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.60-72

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.60-72

Then, for $t \in J$

$$|(Nu)(t) - (Nw)(t)| \leq \int_0^T |G(t,s)||g(s) - h(s)|ds. \tag{8}$$

By $(H2)$ we have

$$\begin{aligned} |g(t) - h(t)| &= |f(t,u(t),g(t)) - f(t,w(t),h(t))| \\ &\leq K|u(t) - w(t)| + L|g(t) - h(t)|. \end{aligned}$$

Thus

$$|g(t) - h(t)| \leq \frac{K}{1-L}|u(t) - w(t)|.$$

By (8) we have

$$\begin{aligned} |(Nu)(t) - (Nw)(t)| &\leq \frac{K}{(1-L)} \int_0^T |G(t,s)||u(s) - w(s)|ds \\ &\leq \frac{KTG^*}{1-L}\|u - w\|_\infty. \end{aligned}$$

Then

$$\|Nu - Nw\|_\infty \leq \frac{KTG^*}{1-L}\|u - w\|_\infty.$$

By (6), the operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point which is a unique solution of the problem $(1) - (2)$.

Our next existence result is based on Schauder's fixed point theorem.

Theorem 3.4 *Assume (H1),(H2) and the following hypothesis holds.*

(H3) There exist $p, q, r \in C(J, \mathbb{R}_+)$ *with* $r^* = \sup_{t \in J} r(t) < 1$ *such that*

$$|f(t,u,w)| \leq p(t) + q(t)|u| + r(t)|w| \text{ for } t \in J \text{ and } u, w \in \mathbb{R}.$$

If

$$\frac{q^* T G^*}{1 - r^*} < 1, \tag{9}$$

where $q^* = \sup_{t \in J} q(t)$, *then the boundary value problem* $(1)-(2)$ *has at least one solution.*

Proof. Let the operator N defined in (7). We shall show that N satisfies the assumption of Schauder's fixed point theorem. The proof will be given in several steps.

Claim 1: *N is continuous.*
Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C(J, \mathbb{R})$. Then for each $t \in J$

$$|N(u_n)(t) - N(u)(t)| \leq \int_0^T |G(t,s)||g_n(s) - g(s)|ds, \tag{10}$$

where $g_n, g \in C(J, \mathbb{R})$ such that

$$g_n(t) = f(t, u_n(t), g_n(t)),$$

and

$$g(t) = f(t, u(t), g(t)).$$

By $(H2)$, we have

$$\begin{aligned} |g_n(t) - g(t)| &= |f(t, u_n(t), g_n(t)) - f(t, u(t), g(t))| \\ &\leq K|u_n(t) - u(t)| + L|g_n(t) - g(t)|. \end{aligned}$$

Then

$$|g_n(t) - g(t)| \leq \frac{K}{1-L}|u_n(t) - u(t)|.$$

Since $u_n \to u$, then we get $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$. And let $\eta > 0$ be such that, for each $t \in J$, we have $|g_n(t)| \leq \eta$ and $|g(t)| \leq \eta$. Then, we have

$$\begin{aligned} |G(t,s)||g_n(s) - g(s)| &\leq |G(t,s)|[|g_n(s)| + |g(s)|] \\ &\leq 2\eta|G(t,s)|. \end{aligned}$$

For each $t \in J$, the function $s \to 2\eta|G(t,s)|$ is integrable on J. Then the Lebesgue Dominated Convergence Theorem and (10) imply that

$$|N(u_n)(t) - N(u)(t)| \to 0 \ as \ n \to \infty,$$

and hence

$$\|N(u_n) - N(u)\|_\infty \to 0 \text{ as } n \to \infty.$$

Consequently, N is continuous.

Let

$$R \geq \frac{(2|y_0| + |y_1|)(1 - r^*) + G^*Tp^*}{M},$$

where $M := 1 - r^* - G^*Tq^*$ and $p^* = \sup_{t \in J} p(t)$. Define

$$D_R = \{u \in C(J, \mathbb{R}) : \|u\|_\infty \leq R\}.$$

It is clear that D_R is a bounded, closed and convex subset of $C(J)$.

Claim 2: $N(D_R) \subset D_R$.

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.60-72

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.60-72

Let $u \in D_R$ we show that $Nu \in D_R$. For each $t \in J$, we have

$$\begin{aligned} |Nu(t)| &\leq |l(t)| + \int_0^T |G(t,s)||g(s)|ds \\ &\leq |y_0| + |y_1 - y_0| + G^* \int_0^T |g(s)|ds \\ &\leq 2|y_0| + |y_1| + G^* \int_0^T |g(s)|ds. \end{aligned} \tag{11}$$

where $g(t) = f(t, u(t), g(t))$.
By (H3), for each $t \in J$, we have

$$\begin{aligned} |g(t)| &= |f(t, u(t), g(t))| \\ &\leq p(t) + q(t)|u(t)| + r(t)|g(t)| \\ &\leq p(t) + q(t)R + r(t)|g(t)| \\ &\leq p^* + q^*R + r^*|g(t)|. \end{aligned}$$

Then

$$|g(t)| \leq \frac{p^* + q^*R}{1 - r^*}.$$

Thus (11) implies that, for each $t \in J$,

$$\begin{aligned} |Nu(t)| &\leq 2|y_0| + |y_1| + \frac{p^* + q^*R}{1 - r^*} G^* T \\ &\leq R. \end{aligned}$$

Then $N(D_R) \subset D_R$.

Claim 3: *$N(D_R)$ is relatively compact.*

Let $t_1, t_2 \in J$, $t_1 < t_2$, and let $u \in D_R$. Then

$$\begin{aligned} |N(u)(t_2) - N(u)(t_1)| &= \left| l(t_2) - l(t_1) + \int_0^T [G(t_2, s) - G(t_1, s)]g(s)ds \right| \\ &= \left| \frac{(y_1 - y_0)}{T}(t_2 - t_1) + \int_0^T [G(t_2, s) - G(t_1, s)]g(s)ds \right| \\ &\leq \left| \frac{(y_1 - y_0)}{T}(t_2 - t_1) \right| + \frac{p^* + q^*R}{1 - r^*} \left| \int_0^T [G(t_2, s) - G(t_1, s)]ds \right|. \end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.

As a consequence of Claims 1 to 3 together with the Arzelá-Ascoli theorem, we conclude that $N : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ is continuous and compact. As a consequence of Schauder's fixed point Theorem, we deduce that N has a fixed point which is a solution of the problem $(1) - (2)$.

Our next existence result is based on nonlinear alternative of Leray-Schauder type.

Theorem 3.5 *Assume (H1)-(H3) and (9) hold. Then the IVP* $(1)-(2)$ *has at least one solution.*

Proof. Consider the operator N defined in (7). We shall show that N satisfies the assumption of Leray-Schauder fixed point theorem. The proof will be given in several claims.

Claim 1: *Clearly N is continuous.*

Claim 2: *N maps bounded sets into bounded sets in $C(J,\mathbb{R})$.*

Indeed, it is enough to show that for any $\rho > 0$, there exist a positive constant ℓ such that for each $u \in B_\rho = \{u \in C(J,\mathbb{R}) : \|u\|_\infty \leq \rho\}$, we have $\|N(u)\|_\infty \leq \ell$.
For $u \in B_\rho$, we have, for each $t \in J$,

$$\begin{aligned} |Nu(t)| &\leq |l(t)| + \int_0^T |G(t,s)||g(s)|ds. \\ &\leq |y_0| + |y_1 - y_0| + G^* \int_0^T |g(s)|ds. \end{aligned}$$

Then

$$|Nu(t)| \leq 2|y_0| + |y_1| + G^* \int_0^T |g(t)|ds. \tag{12}$$

By (H3), for each $t \in J$, we have

$$\begin{aligned} |g(t)| &= |f(t,u(t),g(t))| \\ &\leq p(t) + q(t)|u(t)| + r(t)|g(t)| \\ &\leq p(t) + q(t)\rho + r(t)|g(t)| \\ &\leq p^* + q^*\rho + r^*|g(t)|. \end{aligned}$$

Then

$$|g(t)| \leq \frac{p^* + q^*\rho}{1 - r^*} := M^*.$$

Thus (12) implies that

$$|Nu(t)| \leq 2|y_0| + |y_1| + G^*M^*T.$$

Thus

$$\|Nu\|_\infty \leq 2|y_0| + |y_1| + G^*M^*T := l.$$

Claim 3: *Clearly, N maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.*

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.60-72

We conclude that $N : C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R})$ is continuous and completely continuous.

Claim 4: *A priori bounds.*

We now show there exists an open set $U \subseteq C(J,\mathbb{R})$ with $u \neq \lambda N(u)$, for $\lambda \in (0,1)$ and $u \in \partial U$. Let $u \in C(J,\mathbb{R})$ and $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus for each $t \in J$, we have

$$u(t) = \lambda l(t) + \lambda \int_0^T G(t,s)g(s)ds.$$

This implies by (H2) that, for each $t \in J$, we have

$$|u(t)| \leq 2|y_0| + |y_1| + \int_0^T |G(t,s)||g(s)|ds. \tag{13}$$

And, by (H3), for each $t \in J$, we have

$$\begin{aligned} |g(t)| &= |f(t,u(t),g(t))| \\ &\leq p(t) + q(t)|u(t)| + r(t)|g(t)| \\ &\leq p^* + q^*|u(t)| + r^*|g(t)|. \end{aligned}$$

Thus

$$|g(t)| \leq \frac{1}{1-r^*}(p^* + q^*|u(t)|).$$

Hence

$$\begin{aligned} |u(t)| &\leq \left(2|y_0| + |y_1| + \frac{p^*TG^*}{1-r^*}\right) + \frac{q^*G^*}{1-r^*}\int_0^T |u(s)|ds \\ &\leq \left(2|y_0| + |y_1| + \frac{p^*TG^*}{1-r^*}\right) + \frac{q^*TG^*}{1-r^*}||u||_\infty. \end{aligned}$$

Then

$$||u||_\infty \leq \left(2|y_0| + |y_1| + \frac{p^*TG^*}{1-r^*}\right) + \frac{q^*TG^*}{1-r^*}||u||_\infty.$$

Thus

$$||u||_\infty \leq \frac{M_1}{1 - \dfrac{q^*TG^*}{1-r^*}} := \overline{M},$$

where

$$M_1 = 2|y_0| + |y_1| + \frac{p^*TG^*}{1-r^*}.$$

Let

$$U = \{u \in C(J,\mathbb{R}) : \|u\|_\infty < \overline{M} + 1\}.$$

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.60-72

By our choice of U, there is no $u \in \partial U$ such that $u = \lambda N(u)$, for $\lambda \in (0,1)$. As a consequence of Theorem 2.8, we deduce that N has a fixed point u in $\overline{U}$ which is a solution to $(1)-(2)$.

4 Examples

Example 1. Consider the following boundary value problem

$$ {}^cD^{\frac{3}{2}}y(t) = \frac{1}{3e^{t+2}(1+|y(t)|+|{}^cD^{\frac{3}{2}}y(t)|)}, \text{ for each, } t \in [0,1], \tag{14}$$

$$ y(0)=1, \quad y(1)=2. \tag{15}$$

Set

$$ f(t,u,v) = \frac{1}{3e^{t+2}(1+|u|+|v|)}, \quad t \in [0,1], \ u,v \in \mathbb{R}. $$

Clearly, the function f is jointly continuous.
For any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$

$$ |f(t,u,v) - f(t,\bar{u},\bar{v})| \leq \frac{1}{3e^2}(|u-\bar{u}|+|v-\bar{v}|). $$

Hence condition (H2) is satisfied with $K = \frac{1}{3e^2}$ and $L = \frac{1}{3e^2} < 1$.
From (5) the function G is given by

$$ G(t,s) = \frac{1}{\Gamma(\frac{3}{2})} \left\{ \begin{array}{ll} (t-s)^{\frac{1}{2}} - t(1-s)^{\frac{1}{2}} & \text{if } \ 0 \leq s \leq t \\ -t(1-s)^{\frac{1}{2}} & \text{if } \ t \leq s \leq 1. \end{array} \right\} $$

Clearly $G^* < \frac{2}{\Gamma(\frac{3}{2})}$. Thus condition

$$ \frac{KTG^*}{1-L} < 1, $$

is satisfied with $T = 1$ and $\alpha = \frac{3}{2}$. It follows from Theorem 3.3 that the problem (14)-(15) as a unique solution on J.
Example 2. Consider the following boundary value problem

$$ {}^cD^{\frac{3}{2}}y(t) = \frac{(6+|y(t)|+|{}^cD^{\frac{3}{2}}y(t)|)}{10e^{t+1}(1+|y(t)|+|{}^cD^{\frac{3}{2}}y(t)|)}, \text{ for each, } t \in [0,1], \tag{16}$$

$$ y(0)=1, \quad y(1)=2. \tag{17}$$

Set

$$ f(t,u,v) = \frac{6+|u|+|v|}{10e^{t+1}(1+|u|+|v|)}, \quad t \in [0,1], \ u,v \in \mathbb{R}. $$

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.60-72

ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2014, VOLUME 4, ISSUE 1, p.60-72

Clearly, the function f is jointly continuous.
For each $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \leq \frac{1}{2e}(|u-\bar{u}| + |v-\bar{v}|).$$

Hence condition (H2) is satisfied with $K = L = \frac{1}{2e}$.
Also, we have, for each $u, v, \in \mathbb{R}$ and $t \in [0,1]$

$$|f(t,u,v)| \leq \frac{1}{10e^{t+1}}(6 + |u| + |v|).$$

Thus condition (H3) is satisfied with $p(t) = \frac{3}{5e^{t+1}}$ and $q(t) = r(t) = \frac{1}{10e^{t+1}}$. Clearly $p^* = \frac{3}{5e}$, $q^* = \frac{1}{10e}$ and $r^* = \frac{1}{10e} < 1$.
From (5) the function G is given by

$$G(t,s) = \frac{1}{\Gamma(\frac{3}{2})}\left\{ \begin{array}{ll} (t-s)^{\frac{1}{2}} - t(1-s)^{\frac{1}{2}} & \text{if} \quad 0 \leq s \leq t \\ -t(1-s)^{\frac{1}{2}} & \text{if} \quad t \leq s \leq 1. \end{array} \right\}$$

Clearly $G^* < \frac{2}{\Gamma(\frac{3}{2})}$. Thus condition

$$\frac{q^* T G^*}{1 - r^*} < 1,$$

is satisfied with $T = 1$ and $\alpha = \frac{3}{2}$. It follows from Theorems 3.4 and 3.5 that the problem (16)-(17) at least one solution on J.

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