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COMPLEMENT GRAPHS AND TOTAL INFLUENCE NUMBER

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Abstract. The total influence number is a natural extension of the graph parameter known as the influence number. The total influence number can be viewed as vertex labeling problem are concerned with the maximizing the sum of the labels. For a vertex subset $S \subseteq V$ of a graph $G = (V, E)$, the total influence number of a vertex $v \in S$ is, denoted by $\eta_T(v) = \sum_{u \in \overline{S}} \frac{1}{2^{d(u,v)}}$. The total influence number of a vertex subset S is $\eta_T(S) = \sum_{v \in S} \eta_T(v) = \sum_{v \in S} \sum_{u \in \overline{S}} \frac{1}{2^{d(u,v)}}$. The total influence number of a graph G is $\eta_T(G) = \max_{S \subseteq V} \eta_T(S)$. In this paper, we show how to find a maximum total influence set on various basic complement graphs.

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1. Introduction

Graph labeling is the most important problem in the field of graph theory. If the vertices of the graph are assigned values subject to certain conditions then it is known as graph labeling. Most of the graph labeling problems have the following three common characteristics: a set of numbers for assignment of vertex labels, a rule that assigns a label to edge and some condition(s) that these labels must satisfy.

The graph labeling problem that appears in graph theory has a fast development recently. This problem was first introduced by Alex Rosa [1] around 1967 as means of attacking the problem of cyclically decomposing the complete graph into the tees. Numerous variations of labeling have been investigated in the literature such as graceful, harmonious, magic, antimagic, bimagic, cordial and prime etc. [3]. Many graph labeling problems seek to find the smallest integer label required to satisfy certain constraints. Other problems seek to minimize the sum of all of the labels. One of the most famous of these problems is the chromatic sum [2]. A useful survey to know about the numerous labeling methods is by J.A.Gallian [4].

In this paper, we study the total influence number as a graph parameter. The total influence number can be viewed as vertex labeling problems are concerned with the sum of the labels. It is another type of the graph parameter known as the influence number. The concept of the influence number is introduced as a graph parameter in the social networks. This problem can also be considered in transmitters and receivers. The applications of the influence number can be extended for the total influence number. For psycology, we consider the situation when a person is influenced by multiple other people. Using

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transmitters and receivers, we allow the receiver to obtain a boosted signal by using the service of each transmitter instead of only connecting to the closest transmitter.

The total influence number is a new approach to the concept of graph labeling, introduced Daugherty and et al. [5]. Although many vertex labeling problems are concerned with the sum of all of the labels study to minimize the sum, the influence and total influence numbers have the aim of maximizing the sum. This means that these parameters attempt to maximize the profit associated with each vertex.

Throughout this paper, the following notation will be used. Let $G = (V, E)$ be a simple, connected graph. The vertex set and edge set of a graph is denoted by $V(G)$ and $E(G)$, respectively. It is assumed that $V(G)$ will be abbreviated V. For a vertex subset $S \subseteq V$, $\overline{S} = V - S$ denotes the complement of S with respect to V.

The shortest distance in G between two vertices u and v will be denoted $d(u, v)$. For any vertex u, let $d(u, S) = \min_{v\in S} d(u, v)$. Then $d(u, S) = 0$ if and only if $u \in S$

The total influence number of a vertex $v \in S$ is

$$\eta_T(v) = \sum_{u\in\overline{S}} \frac{1}{2^{d(u,v)}}.$$

The total influence number of a vertex subset S is

$$\eta_T(S) = \sum_{v\in S} \eta_T(v) = \sum_{v\in S}\sum_{u\in\overline{S}} \frac{1}{2^{d(u,v)}}.$$

The total influence number of a graph G is $\eta_T(G) = \max_{S\subseteq V} \eta_T(S)$. A set S is called η_T-set if $\eta_T(S) = \eta_T(G)$ [5, 6].

The paper proceeds as follows. In section 2, for the total influence number, known results are given. In section 3, the total influence number of cycle and wheel graph are studied. In section 4, exact values for the total influence number of some complement graphs are determined.

Definition 1.1. [5] *A vertex subset S is called an alternating set if and only if S is either (1) the empty set or (2) a maximal independent set such that $\exists u \in S \ni \forall v \in S$, $d(u, v) = 2k$ for some $k \in \mathbb{Z}$.*

Theorem 1.2. [7] *If f is continious on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.*

To find the absolute maximum and minimum values of a continious function f on a closed, bounded set D: 1. Find the values of f at the critical points of f in D. 2. Find the extreme values of f on the boundary of D. 3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

2. Basic Results On The Total Influence Number

Theorem 2.1. [5] *The total influence number of*

(a) *the complete graph K_n is*

$$\eta_T(K_n) = \begin{cases} \frac{n^2}{8} & \text{if } n \text{ is even,} \\ \frac{n^2-1}{8} & \text{if } n \text{ is odd.} \end{cases}$$

(b) *the star $K_{1,n}$ is*

$$\eta_T(K_{1,n}) = \begin{cases} \frac{(n+2)^2}{16} & \text{if } n \text{ is even,} \\ \frac{(n+1)(n+3)}{16} & \text{if } n \text{ is odd.} \end{cases}$$

(c) *the double star $DS_{n,m}$ is*

$$\eta_T(DS_{n,m}) = \begin{cases} \frac{1}{16}n^2 + \frac{3}{8}n + \frac{1}{16}m^2 + \frac{3}{8}m + \frac{1}{16}nm + \frac{3}{4} & \text{if } n, m \text{ are even,} \\ \frac{1}{16}n^2 + \frac{3}{8}n + \frac{1}{16}m^2 + \frac{3}{8}m + \frac{1}{16}nm + \frac{11}{16} & \text{otherwise.} \end{cases}$$

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(d) *the complete bipartite grapf $K_{n,m}$ is*

$$\eta_T(K_{n,m}) = \begin{cases} \frac{mn}{2} & \text{if } n \geq \frac{m}{2}, \\ \frac{(2n+m)^2}{16} & \text{if } n < \frac{m}{2},\ m \text{ is even}, \\ \frac{(2n+m+1)(2n+m-1)}{16} & \text{if } n < \frac{m}{2},\ m \text{ is odd}. \end{cases}$$

Theorem 2.2. [5] *For a path P_n $(n > 1)$, a vertex subset S has maximum total influence if and only if it is a non-empty alternating set.*

Corollary 2.3. *The total influence number of path, P_n is*

$$\eta_T(P_n) = \begin{cases} \frac{(10)2^{-n}+6n-10}{9} & \text{if } n \text{ is even}, \\ \frac{(8)2^{-n}+6n-10}{9} & \text{if } n \text{ is odd}. \end{cases}$$

Theorem 2.4. [5] *For any graph G=(V,E), with vertex partitions V_1 and V_2 and a set $S \subseteq V$ let $S_1 = V_1 \cap S$, $S_2 = V_2 \cap S$, $\overline{S} = V - S$, $\overline{S}_1 = V_1 - S_1$ and $\overline{S}_2 = V_2 - S_2$. Then,*

$$\eta_T(S) = \eta_T(S_1, \overline{S}_1) + \eta_T(S_2, \overline{S}_1) + \eta_T(S_2, \overline{S}_2) + \eta_T(S_1, \overline{S}_2)$$

3. Total Influence Number of Cycle and Wheel Graphs

Theorem 3.1. *For a graph C_n with $n \geq 6$, total influence number is*

$$\eta_T(C_n) = \begin{cases} \frac{2n}{3} + 2^{-\frac{n+1}{2}}(\frac{1}{9} - n) - \frac{2}{9} & \text{if } n \text{ is odd}, \\ \frac{2n}{3} - \frac{2n}{3}2^{-\frac{n}{2}} & \text{if } n \text{ and } \frac{n}{2} \text{ are even}, \\ \frac{2n}{3} - \frac{5n}{6}2^{-\frac{n}{2}} & \text{if } n \text{ is even and } \frac{n}{2} \text{ is odd}. \end{cases}$$

Proof. Consider a cycle C_n with $V(C_n) = \{v_1, v_2, ..., v_n\}$, a total influence set S and vertex partitions V_1 and V_2. The vertex set of C_n can be partitioned into two pieces $V_1 = \{v_1, v_2, ...v_{\lceil \frac{n}{2} \rceil}\}$ and $V_2 = \{v_{\lceil \frac{n}{2} \rceil+1}, v_{\lceil \frac{n}{2} \rceil+2}, ..., v_n\}$. $V_1 \cup V_2 = V(C_n)$. For the vertex subset S, let $S_1 = S \cap V_1$, $S_2 = S \cap V_2$, $\overline{S}_1 = V_1 - S_1$ and $\overline{S}_2 = V_2 - S_2$. By Theorem 2.4, we can write the following expression for $\eta_T(S)$.

$$\eta_T(S) = \eta_T(S_1, \overline{S}_1) + \eta_T(S_1, \overline{S}_2) + \eta_T(S_2, \overline{S}_1) + \eta_T(S_2, \overline{S}_2).$$

By Corollary 2.3, we know $\eta_T(S_1, \overline{S}_1) = \eta_T(P_{\lceil \frac{n}{2} \rceil})$, $\eta_T(S_2, \overline{S}_2) = \eta_T(P_{\lfloor \frac{n}{2} \rfloor})$ and by Theorem 2.2, we also know that S_1 and S_2 are alternating sets. Let v_1 be vertex in S_1. There are exactly two alternating sets for S_2. For summing unknow terms in the above expression, we have four cases depending on n.

Case 1. Let n is odd and $\lceil \frac{n}{2} \rceil$ is odd.

Case 1.1. Let $v_{\lceil \frac{n}{2} \rceil+1} \in S_2$. Then $S_1 = \{v_1, v_3, ..., v_{\lceil \frac{n}{2} \rceil}\}$, $S_2 = \{v_{\lceil \frac{n}{2} \rceil+1}, v_{\lceil \frac{n}{2} \rceil+3}, ..., v_{n-1}\}$. Thus, we get $\eta_T(S_1, \overline{S}_2) + \eta_T(S_2, \overline{S}_1) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{i}{2^i}$.

Case 1.2. Let $v_{\lceil \frac{n}{2} \rceil+1} \notin S_2$. Then $S_1 = \{v_1, v_3, ..., v_{\lceil \frac{n}{2} \rceil}\}$, $S_2 = \{v_{\lceil \frac{n}{2} \rceil+2}, v_{\lceil \frac{n}{2} \rceil+4}, ..., v_n\}$. Thus, we get $\eta_T(S_1, \overline{S}_2) + \eta_T(S_2, \overline{S}_1) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{i}{2^i}$.

It is easy to see that these results are equal.

Case 2. Let n is odd and $\lceil \frac{n}{2} \rceil$ is even.

Case 2.1. Let $v_{\lceil \frac{n}{2} \rceil+1} \in S_2$. Then $S_1 = \{v_1, v_3, ..., v_{\lceil \frac{n}{2} \rceil-1}\}$, $S_2 = \{v_{\lceil \frac{n}{2} \rceil+1}, v_{\lceil \frac{n}{2} \rceil+3}, ..., v_n\}$. Thus, we get $\eta_T(S_1, \overline{S}_2) + \eta_T(S_2, \overline{S}_1) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{i}{2^i}$.

Case 2.2. Let $v_{\lceil \frac{n}{2} \rceil+1} \notin S_2$. Then $S_1 = \{v_1, v_3, ..., v_{\lceil \frac{n}{2} \rceil-1}\}$, $S_2 = \{v_{\lceil \frac{n}{2} \rceil+2}, v_{\lceil \frac{n}{2} \rceil+4}, ..., v_{n-1}\}$. Thus, we get $\eta_T(S_1, \overline{S}_2) + \eta_T(S_2, \overline{S}_1) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{i}{2^i}$.

It is easy to see that these results are equal.

Case 3. Let n is even and $\frac{n}{2}$ is odd.
Case 3.1. Let $v_{\frac{n}{2}+1} \in S_2$. Then $S_1 = \{v_1, v_3, ..., v_{\frac{n}{2}}\}$, $S_2 = \{v_{\frac{n}{2}+1}, v_{\frac{n}{2}+3}, ..., v_n\}$. Thus, we get $\eta_T(S_1, \overline{S}_2) + \eta_T(S_2, \overline{S}_1) = \sum_{i=1}^{\lfloor \frac{n}{4} \rfloor} \frac{4i}{2^{2i}}$.
Case 3.2. Let $v_{\frac{n}{2}+1} \notin S_2$. Then $S_1 = \{v_1, v_3, ..., v_{\frac{n}{2}}\}$, $S_2 = \{v_{\frac{n}{2}+2}, v_{\frac{n}{2}+4}, ..., v_{n-1}\}$. Thus, we get $\eta_T(S_1, \overline{S}_2) + \eta_T(S_2, \overline{S}_1) = \sum_{i=1}^{\lfloor \frac{n}{4} \rfloor} \frac{4i-2}{2^{2i-1}} + \frac{n}{2^{\frac{n+2}{2}}}$.
It is easy to see that second result is smaller than first result.
Case 4. Let n and $\frac{n}{2}$ are even.
Case 4.1. Let $v_{\frac{n}{2}+1} \in S_2$. Then $S_1 = \{v_1, v_3, ..., v_{\frac{n}{2}-1}\}$, $S_2 = \{v_{\frac{n}{2}+1}, v_{\frac{n}{2}+3}, ..., v_{n-1}\}$. Thus, we get $\eta_T(S_1, \overline{S}_2) + \eta_T(S_2, \overline{S}_1) = \sum_{i=1}^{\lfloor \frac{n}{4} \rfloor} \frac{4i-2}{2^{2i-1}}$.
Case 4.2. Let $v_{\frac{n}{2}+1} \notin S_2$. Then $S_1 = \{v_1, v_3, ..., v_{\frac{n}{2}-1}\}$, $S_2 = \{v_{\frac{n}{2}+2}, v_{\frac{n}{2}+4}, ..., v_n\}$. Thus, we get $\eta_T(S_1, \overline{S}_2) + \eta_T(S_2, \overline{S}_1) = \sum_{i=1}^{\lfloor \frac{n}{4} \rfloor} \frac{4i}{2^{2i}} - \frac{n}{2^{\frac{n+2}{2}}}$.
It is easy to see that second result is smaller than first result.
By Case 1, 2, 3.1, 4.1, we have

$$\begin{cases} \frac{2n}{3} + 2^{-\frac{n+1}{2}}(\frac{1}{9} - n) - \frac{2}{9} & \text{if } n \text{ is odd,} \\ \frac{2n}{3} - \frac{2n}{3}2^{-\frac{n}{2}} & \text{if } n \text{ and } \frac{n}{2} \text{ are even,} \\ \frac{2n}{3} - \frac{5n}{6}2^{-\frac{n}{2}} & \text{if } n \text{ is even and } \frac{n}{2} \text{ is odd.} \end{cases}$$

□

Lemma 3.2. *Consider a wheel graph $W_{1,n}$ with $n \geq 6$. Let S_1 and S_2 be total influence sets of $W_{1,n}$. Assume without loss of generality that the center vertex, labeled c, is in S_1 and S_2. Let $S_1 = X_1 \cup X_2 \cup ... \cup X_t \cup \{c\}$ such that any element of X_i can not be consecutive to any element of X_j, where $i \neq j$. Set $|X_1| = x_1$, $|X_2| = x_2$,..., $|X_t| = x_t$, where $t \geq 2$ and $x_1, x_2, ...x_t \geq 2$. Additionally, let each X_i consist of vertices having consecutive indices, where $i \in \{1, 2, ..., t\}$. Let $S_2 = X \cup \{c\}$ such that $|X| = x_1 + x_2 + ... + x_t$ and X consist of vertices having consecutive indices. Then $\eta_T(S_1) > \eta_T(S_2)$.*

Proof. We can write the following equalities for S_1 and S_2.

$$\begin{aligned} \eta_T(S_1) =& 2\frac{1}{2} + 2\frac{1}{4}(n - (x_1 + x_2 + ... + x_t) - 1) + \frac{1}{4}(x_1 - 2)(n - (x_1 + x_2 + ... + x_t)) \\ &+ ... + 2\frac{1}{2} + 2\frac{1}{4}(n - (x_1 + x_2 + ... + x_t) - 1) \\ &+ \frac{1}{4}(x_t - 2)(n - (x_1 + x_2 + ... + x_t)) + \frac{1}{2}(n - (x_1 + x_2 + ...x_t)). \\ =& \frac{t}{2} + \frac{1}{4}(x_1 + x_2 + ... + x_t)(n - (x_1 + x_2 + ... + x_t)) \\ &+ \frac{1}{2}(n - (x_1 + x_2 + ... + x_t)). \end{aligned} \tag{3.1}$$

$$\begin{aligned} \eta_T(S_2) =& 2\frac{1}{2} + 2\frac{1}{4}(n - (x_1 + x_2 + ... + x_t) - 1) \\ &+ \frac{1}{4}(x_1 + x_2 + ... + x_t - 2)(n - (x_1 + x_2 + .. + x_t)) \\ &+ \frac{1}{2}(n - (x_1 + x_2 + ... + x_t)). \end{aligned} \tag{3.2}$$

By (3.1), (3.2) and $t \geq 2$, $\eta_T(S_1) > \eta_T(S_2)$. □

Lemma 3.3. *Consider a wheel graph $W_{1,n}$ with $n \geq 6$. Let S_1 and S_2 be total influence sets of $W_{1,n}$. Assume without loss of generality that the center vertex, labeled c, is in S_1 and S_2. Let $X_1 \cup X_2 \cup ... \cup X_t \cup \{c\}$*

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be a partition of S_1 such that any element of X_i can not be consecutive to any element of X_j, where $i \neq j$. Let $|X_1| = x_1$, $|X_2| = x_2$,..., $|X_t| = x_t$, where $t \geq 2$ and $x_1, x_2, ...x_t \geq 2$. Additionally, let each X_i consist of vertices having consecutive indices, where $i \in \{1, ..., t\}$. Let $S_2 = Y \cup \{c\}$, where $Y = \{v_i : for \quad \forall v_i, v_j \in V(W_{1,n}) - \{c\}, (v_i, v_j) \notin E(W_{1,n})\}$. Then $\eta_T(S_2) > \eta_T(S_1)$.

Proof. For S_1 by (3.1), and for S_2, we have following equalities

$$\begin{aligned}\eta_T(S_1) =& \frac{t}{2} + \frac{1}{4}(x_1 + x_2 + ... + x_t)(n - (x_1 + x_2 + ... + x_t)) \\ &+ \frac{1}{2}(n - (x_1 + x_2 + ...x_t)). \\ \eta_T(S_2) =& \frac{1}{4}(x_1 + x_2 + ... + x_t)(n - (x_1 + x_2 + ... + x_t) + 2) \\ &+ \frac{1}{2}(n - (x_1 + x_2 + ...x_t)).\end{aligned}$$

We prove by induction on the number of sets which have vertices with consecutive indices, t. ($t \geq 2$.)
i. for $t = 2$,

$$\begin{aligned}\eta_T(S_1) =& 1 + \frac{1}{4}(x_1 + x_2)(n - (x_1 + x_2)) + \frac{1}{2}(n - (x_1 + x_2)). \\ \eta_T(S_2) =& \frac{1}{4}(x_1 + x_2)(n - (x_1 + x_2)) + 2\frac{1}{4}(x_1 + x_2) + \frac{1}{2}(n - (x_1 + x_2))\end{aligned}$$

Since $x_1, x_2 \geq 2$, it is obvious that $\eta_T(S_2) > \eta_T(S_1)$.
ii. We suppose that the claim holds for $t = k$.

$$\begin{aligned}\eta_T(S_1) =& \frac{1}{2}k + \frac{1}{4}(x_1 + x_2 + ... + x_k)(n - (x_1 + x_2 + ... + x_k)) \\ &+ \frac{1}{2}(n - (x_1 + x_2 + ...x_k)). \\ \eta_T(S_2) =& \frac{1}{4}(x_1 + x_2 + ... + x_k)(n - (x_1 + x_2 + ... + x_k)) + 2\frac{1}{4}(x_1 + x_2 + ... + x_k) \\ &+ \frac{1}{2}(n - (x_1 + x_2 + ...x_k)).\end{aligned}$$

Since assuming $\eta_T(S_2) > \eta_T(S_1)$, we obtain

$$x_1 + x_2 + ... + x_k > k. \tag{3.3}$$

iii. for $t = k + 1$,

$$\begin{aligned}\eta_T(S_1) =& \frac{1}{2}(k + 1) + \frac{1}{4}(x_1 + x_2 + ... + x_k + x_{k+1})(n - (x_1 + x_2 + ... + x_k + x_{k+1})) \\ &+ \frac{1}{2}(n - (x_1 + x_2 + ...x_(k + 1))). \\ \eta_T(S_2) =& \frac{1}{4}(x_1 + x_2 + ... + x_k + x_{k+1})(n - (x_1 + x_2 + ... + x_k + x_{k+1})) \\ &+ 2\frac{1}{4}(x_1 + x_2 + ... + x_k + x_{k+1}) + \frac{1}{2}(n - (x_1 + x_2 + ...x_{k+1})).\end{aligned}$$

$$\eta_T(S_2) - \eta_T(S_1) = \frac{1}{2}(x_1 + x_2 + ...x_k + x_{k+1} - k - 1). \tag{3.4}$$

By (3.3) and (3.4), it is obvious that $\eta_T(S_2) > \eta_T(S_1)$. Thus, we point out that a result of Lemma is that choosing S_2 instead of S_1 . □

Theorem 3.4. *The total influence number of $W_{1,n}$ with $n \geq 6$ is*

$$\eta_T(W_{1,n}) = \begin{cases} \frac{n^2+8n-1}{16} & \textit{if } n \textit{ is odd,} \\ \frac{n^2+8n}{16} & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. Let S is a total influence set of $W_{1,n}$. By Lemma 3.2 and Lemma 3.3, we must choose S_2 in Lemma 3.3 as S. Let $|S| = x+1$ such that x is the number of vertices on the cycle of $W_{1,n}$ in S. Let $f(x) := \eta_T(S)$, thus we have

$$f(x) = 2\frac{1}{2}x + \frac{1}{4}x(n-x-2) + \frac{1}{2}(n-x).$$

Bound is $0 \leq x \leq \lfloor \frac{n}{2} \rfloor$. Solving $f_x(x) = 0$ gives $x = \frac{n}{2}$. When n is even, we have $f(\frac{n}{2}) = \frac{n^2+8n}{16}$. When n is odd, x can be $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$. Since $0 \leq x \leq \lfloor \frac{n}{2} \rfloor$, we ignore $x = \lceil \frac{n}{2} \rceil$. For $x = \lfloor \frac{n}{2} \rfloor$, we have $f(\frac{n-1}{2}) = \frac{n^2+8n-1}{16}$. After examining at the boundaries of x, we get $f(0) = \frac{n}{2}$ at $x = 0$ and also get $f(\frac{n}{2}) = \frac{n^2+8n}{16}$ when n is even, $f(\frac{n-1}{2}) = \frac{n^2+8n-1}{16}$ when n is odd at $x = \lfloor \frac{n}{2} \rfloor$. Thus, the total influence number of $W_{1,n}$ is

$$\eta_T(W_{1,n}) = \begin{cases} \frac{n^2+8n-1}{16} & \text{if } n \text{ is odd,} \\ \frac{n^2+8n}{16} & \text{if } n \text{ is even.} \end{cases}$$

□

4. Total Influence Number of Some Complement Graphs

Theorem 4.1. *For a complement of complete graph $\overline{K}_n$, the total influence number is $\eta_T(\overline{K}_n) = 0$.*

Proof. A complement of complete graph $\overline{K}_n$ contains isolated n vertices. Since these vertices don't influence each other, total influence number of $\overline{K}_n$ is 0. □

Theorem 4.2. *For a graph $\overline{K}_{n,m}$, the total influence number is*

$$\eta_T(\overline{K}_{n,m}) = \begin{cases} \frac{n^2+m^2-2}{8} & \textit{if } n, m \textit{ are odd,} \\ \frac{n^2+m^2}{8} & \textit{if } n, m \textit{ are even,} \\ \frac{n^2+m^2-1}{8} & \textit{if } n \textit{ is even, } m \textit{ is odd or } m \textit{ is even, } n \textit{ is odd.} \end{cases}$$

Proof. A complement of complete bipartite graph $\overline{K}_{n,m}$ with $n \leq m$ contains two complete graphs K_n and K_m which are not connected to each other. By Theorem 2.1, we know total influence number of K_n and K_m. A set S is an η_T-set if and only if it contains exactly $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$ vertices from K_n and exactly $\lfloor \frac{m}{2} \rfloor$ or $\lceil \frac{m}{2} \rceil$ vertices from K_m. Thus we get $\eta_T(\overline{K}_{n,m}) = \eta_T(K_n) + \eta_T(K_m)$. □

Theorem 4.3. *For a graph $\overline{K}_{1,n-1}$, the total influence number is*

$$\eta_T(\overline{K}_{1,n-1}) = \begin{cases} \frac{(n-1)^2}{8} & \textit{if } n \textit{ is odd,} \\ \frac{(n-1)^2-1}{8} & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. A complement of star graph $\overline{K}_{1,n-1}$ contains a complete graph K_{n-1} and an isolated vertex. Let S be a total influence set of $\overline{K}_{1,n-1}$. We know total influence number of K_{n-1} by Theorem 2.1 and the isolated vertex doesn't influence any vertices of K_{n-1}, so it can be in S or $\overline{S}$. Thus we get $\eta_T(\overline{K}_{1,n-1}) = \eta_T(K_{n-1})$. □

Theorem 4.4. *For the graph $\overline{tK_2}$ with $V(\overline{tK_2}) = \{v_1, v_2, ..., v_{2t}\}$, let X and Y be vertex sets such that $X = \{v_1, v_3, ..., v_{2t-1}\}$, $Y = \{v_2, v_4, ..., v_{2t}\}$ and $|X| = |Y| = t$. Consider a total influence sets S, let $x = |X \cap S|$ and $y_1 + y_2 = |Y \cap S|$ such that y_1 and y_2 are the number of vertices non-adjacent to one of x vertices and adjacent to all of x vertices, respectively. Then S is an η_T-set if and only if following condition or its complement is satisfied:*

$$(x = \lceil \tfrac{t}{2} \rceil,\ y_1 = \lfloor \tfrac{t}{2} \rfloor,\ y_2 = 0).$$

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Furthermore,

$$\eta_T(\overline{tK_2}) = \begin{cases} \frac{t^2}{2} & \textit{if } t \textit{ is even,} \\ \frac{2t^2-1}{4} & \textit{if } t \textit{ is odd.} \end{cases}$$

Proof. Using the definitions of x, y_1, y_2 and $f(x, y_1, y_2) := \eta_T(S)$, we have

$$\begin{aligned} f(x, y_1, y_2) &= \frac{1}{2}x(t-x) + \frac{1}{4}(x-y_1) + \frac{1}{2}(x-1)(x-y_1) \\ &+ \frac{1}{2}x(t-x-y_2) + \frac{1}{2}(y_1+y_2)(t-(y_1+y_2)) \\ &+ \frac{1}{2}y_1(t-x) + \frac{1}{4}y_2 + \frac{1}{2}y_2(y_2-1) + \frac{1}{2}y_2(t-x-y_2). \end{aligned}$$

Bounds are $0 \le x \le t$, $0 \le y_1 \le x$ and $0 \le y_2 \le t-x$. Solving the system $f_x(x, y_1, y_2) = 0$, $f_{y_1}(x, y_1, y_2) = 0$, $f_{y_2}(x, y_1, y_2) = 0$ doesn't give a solution. So we search to the maximum of $f(x, y_1, y_2)$ by looking at the boundaries for x, y_1, y_2 and we do this search from Theorem 1.2.

Case 1. For $x = 0$, we maximize $f(0, y_1, y_2) = \frac{1}{4}y_1 - \frac{1}{4}y_2 - y_1y_2 + y_1t + y_2t - \frac{1}{2}y_1^2 - \frac{1}{2}y_2^2$. Solving the system $f_{y_1}(0, y_1, y_2) = 0$ and $f_{y_2}(0, y_1, y_2) = 0$ doesn't give a solution. So we must seek the maximum of the function at the boundaries of y_1 and y_2.

Case 1.1. For $y_1 = 0$ ($y_1 = x = 0$), we have $f(0, 0, y_2) = \frac{1}{4}y_2 - y_2(y_2-t) + \frac{1}{2}y_2(y_2-1)$. Solving $f_{y_2}(0, 0, y_2) = 0$ gives $y_2 = \frac{4t-1}{4}$ and we get $f(0, 0, \lfloor\frac{4t-1}{4}\rfloor) = \frac{2t^2-t-1}{4}$, $f(0, 0, \lceil\frac{4t-1}{4}\rceil) = \frac{2t^2-t}{4}$. From the boundaries of y_2, we have $f(0, 0, t) = \frac{2t^2-t}{4}$ and $f(0, 0, 0) = 0$. Since $0 < |S| < 2t$, we can ignore $f(0, 0, 0) = 0$.

Case 1.2. For $y_2 = 0$ and $y_2 = t - x = t$, since $0 \le y_1 \le x$ and $x = 0$, y_1 just takes the value 0. From examining the boundaries of y_2, we have same results as Case 1.1.

By Case 1, the function is maximized at $y_1 = 0$ and $y_2 = t$.

Case 2. For $x = t$, we maximize $f(t, y_1, y_2) = -\frac{1}{2}y_1^2 - y_1y_2 + \frac{1}{4}y_1 - \frac{1}{2}y_2^2 - \frac{1}{4}y_2 + \frac{1}{2}t^2 - \frac{1}{4}t$. Solving the system $f_{y_1}(t, y_1, y_2) = 0$ and $f_{y_2}(t, y_1, y_2) = 0$ doesn't give a solution. So must seek the maximum of the function at the boundaries of y_1 and y_2.

Case 2.1. For $y_1 = 0$ and $y_1 = x = t$, since $0 \le y_2 \le t - x$ and $x = t$, y_2 just takes the value 0. So we have $f(t, 0, 0) = \frac{2t^2-t}{4}$ and $f(t, t, 0) = 0$, from the boundaries of y_1. Since $0 < |S| < 2t$, we can ignore $f(t, t, 0) = 0$.

Case 2.2. For $y_2 = 0$ ($y_2 = t - x = 0$), we have $f(t, y_1, 0) = \frac{1}{4}t - \frac{1}{4}y_1 - \frac{1}{2}(t-1)(y_1-t) - \frac{1}{2}y_1(y_1-t)$. Solving $f_{y_1}(t, y_1, 0) = 0$, we have $y_1 = \frac{1}{4}$. But this value isn't integer. From examining the boundaries of y_1, we have the same results as Case 2.1.

By Case 2, the function is maximized at $y_1 = 0$ and $y_2 = 0$.

Case 3. For $y_1 = 0$, we maximize $f(x, 0, y_2) = -\frac{1}{4}(y_2 + x)(2y_2 - 4t + 2x + 1)$. Solving the system $f_x(x, 0, y_2) = 0$ and $f_{y_1}(x, 0, y_2) = 0$ doen't give a solution. So we must seek the maximum of the function at the boundaries of x and y_2.

Case 3.1. For $x = 0$, $x = t$, these cases are equivalent to Case 1.1, 2.1, respectively.

Case 3.2. For $y_2 = 0$, we maximize $f(x, 0, 0) = \frac{1}{4}x + \frac{1}{2}x(x-1) + x(t-x)$. Solving $f_x(x, 0, 0) = 0$ gives $x = \frac{4t-1}{4}$ and we get $f(\lfloor\frac{4t-1}{4}\rfloor, 0, 0) = \frac{2t^2-t-1}{4}$, $f(\lceil\frac{4t-1}{4}\rceil, 0, 0) = \frac{2t^2-t}{4}$. From the boundaries of x, we have $f(t, 0, 0) = \frac{2t^2-t}{4}$ and $f(0, 0, 0) = 0$. Since $0 < |S| < 2t$, we can ignore $f(0, 0, 0) = 0$.

Case 3.3. For $y_2 = t - x$, we maximize $f(x, 0, t) = \frac{1}{4}t(2t-1)$. From solving $f_x(x, 0, t) = 0$, we don't find suitable x value. From the boundaries of x, we get $f(0, 0, t) = \frac{2t^2-t}{4}$ and $f(t, 0, t) = 0$. Since $0 < |S| < 2t$, we can ignore $f(t, 0, t) = 0$

By Case 3, the function is maximized at $x = t$ and $y_2 = 0$ or $x = 0$ and $y_2 = t$.

Case 4. For $a = x$, we maximize $f(x, x, y_2) = y_2t - \frac{1}{4}y_2 - 2y_2x + 2tx - \frac{1}{2}y_2^2 - 2x^2$. Solving the system $f_x(x, x, y_2) = 0$ and $f_{y_2}(x, x, y_2) = 0$ doesn't give a sloution. So we must seek the maximum of the function at the boundaries of x and y_2.

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Case 4.1. For $x = 0$, $x = t$, these cases are same as Case 1.1, 2.1, respectively.

Case 4.2. For $y_2 = 0$, we maximize $f(x, x, 0) = 2x(t - x)$. Solving $f_x(x, 0, 0) = 0$ gives $x = \frac{t}{2}$. $x = \frac{t}{2}$ is only valid when t is even, we need to try both $x = \lfloor \frac{t}{2} \rfloor = \frac{t-1}{2}$ and $x = \lceil \frac{t}{2} \rceil = \frac{t+1}{2}$ when t is odd.

Consequently, we find the maximum value of this case,

$$\begin{cases} f(\frac{t}{2}, \frac{t}{2}, 0) = \frac{t^2}{2} & \text{if t is even,} \\ f(\frac{t+1}{2}, \frac{t-1}{2}, 0) = \frac{2t^2-1}{4} & \text{if t is odd.} \end{cases}$$

After examining at the boundaries of x, we don't obtain a result, since $0 < |S| < 2t$.

Case 4.3. For $y_2 = t - x$, we maximize $f(x, x, t - x) = \frac{1}{4}(t - x)(2t + 2x - 1)$. From solving $f_x(x, x, t - x) = 0$, we have $x = \frac{1}{4}$. But this value isn't integer. From the boundaries of x, we get $f(0, 0, t) = \frac{2t^2-t}{4}$ and $f(t, t, 0) = 0$ but since $0 < |S| < 2t$ we can ignore $f(t, t, 0) = 0$.

By Case 4, the function is maximized at $x = \lceil \frac{t}{2} \rceil$, $y_1 = \lfloor \frac{t}{2} \rfloor$ and $y_2 = 0$.

Case 5. For $y_2 = 0$, we maximize $f(x, y_1, 0) = \frac{1}{4}y_1 - \frac{1}{4}x + y_1 t - y_1 x + tx - \frac{1}{2}y_1^2 - \frac{1}{2}x^2$. Solving the system $f_x(x, y_1, 0) = 0$ and $f_{y_1}(x, y_1, 0) = 0$ doesn't give a solution. So we must seek the maximum of the function at the boundaries of x and y_1. Examinations at $x = 0$, $x = t$, $y_1 = 0$, $y_1 = x$ are equivalent to Case 1.2, 2.2, 3.2, 4.2, respectively.

Case 6. For $y_2 = t - x$, we maximize $f(x, y_1, t - x) = -\frac{1}{4}(y_1 - t)(2y_1 + 2t - 1)$. Solving the system of $f_x(x, y_1, t - x) = 0$ and $f_{y_1}(x, y_1, t - x) = 0$ doesn't give a solution and so we must seek the maximum of the function at the boundaries of x and y_1. Examinations at $x = 0$, $x = t$, $y_1 = 0$, $y_1 = x$ are equivalent to Case 1.2, 2.2, 3.3, 4.3, respectively.

From all the cases, total influence number of $\overline{tK_2}$ is $\frac{t^2}{2}$ when t is even and $\frac{2t^2-1}{4}$ when t is odd. $\square$

Definition 4.5. [5] *A double star $DS_{n,m}$ is a tree with exactly 2 non-leaf vertices u and v such that $deg(u) = n + 1$ and $deg(v) = m + 1$.*

Theorem 4.6. *Consider a complement of double star $\overline{DS}_{n,m}$, it contains K_{n+m} graph and exactly two vertices u and v such that u is adjacent to m vertices, v is adjacent to n vertices of K_{n+m}. Let X be the set of n vertices of K_{n+m} and Y be the set of m vertices of K_{n+m}. $|X| = n$, $|Y| = m$. Consider a total influence set S. Let $x = |X \cap S|$ and $y = |Y \cap S|$. Then S is an η_T-set if and only if the following conditions or their complements are satisfied:*

n	m	u	v	x	y	
odd	*odd*	$\in S$	$\notin S$	n	$\frac{m-n}{2}$	$n \leq m - \frac{1}{2}$
even	*even*					
odd	*even*	$\in S$	$\notin S$	n	$\frac{m-n-1}{2}$	
even	*odd*					
odd	*odd*	$\in S$	$\notin S$	$\frac{m+n}{2}$	0	$n \geq m + \frac{1}{2}$
even	*even*					
odd	*even*	$\in S$	$\notin S$	$\frac{m+n+1}{2}$	0	
even	*odd*					
odd	*odd*	$\in S$	$\in S$	0	$\frac{m+n-2}{2}$	$n = m$
even	*even*			$\frac{m+n-2}{2}$	0	
odd	*even*	$\in S$	$\in S$	0	$\frac{m+n-1}{2}$	
even	*odd*			$\frac{m+n-1}{2}$	0	

Furthermore,

$$\eta_T(\overline{DS}_{n,m}) = \begin{cases} \frac{m^2+2mn+3m+n^2+5n+1}{8} & \text{if } n \leq m - \frac{1}{2}, \\ \frac{m^2+2mn+5m+n^2+3n+1}{8} & \text{if } n \geq m + \frac{1}{2}, \\ \frac{m^2+2mn+3m+n^2+3n+2}{8} & \text{if } n = m. \end{cases}$$

Proof. We examine two cases depending on u and v's membership in S. Two cases are comprehensive because other cases are complements of these cases.

Case 1. Let $u \in S$, $v \notin S$, $x = |X \cap S|$, $y = |Y \cap S|$ and $f(x, y) := \eta_T(S)$. In this case, we have

$$f(x,y) = \frac{1}{2}(m-y) + \frac{1}{4}(n-x) + \frac{1}{8} + \frac{1}{2}x(n-x) + \frac{1}{2}x + \frac{1}{2}x(m-y) + \frac{1}{2}y(m-y) + \frac{1}{2}y(n-x) + \frac{1}{4}y.$$

Bounds are $0 \le x \le n$ and $0 \le y \le m$. Solving the system $f_x(x, y) = 0$ and $f_y(x, y) = 0$ doesn't give a solution and we must seek the maximum of the function at the boundaries of x and y.

Case 1.1. For $x = 0$, we have $f(0, y) = \frac{1}{2}m + \frac{1}{4}n - \frac{1}{4}y + \frac{1}{2}ny + \frac{1}{2}y(m-y) + \frac{1}{8}$. Solving $f_y(0, y) = 0$ gives $y = \frac{2m+2n-1}{4}$. If $n \le m + \frac{1}{2}$, $\frac{2m+2n-1}{4} \in [0, m]$. We need to try both $y = \lfloor \frac{2m+2n-1}{4} \rfloor$ and $y = \lceil \frac{2m+2n-1}{4} \rceil$ depending on n and m.

i: if n is odd, m is odd or n is even, m is even, we have $f(0, \frac{m+n-2}{2}) = \frac{m^2+2mn+3m+n^2+n-1}{8}$ and $f(0, \frac{m+n}{2}) = \frac{m^2+2mn+3m+n^2+n+1}{8}$.

ii: if n is odd, m is even or n is even, m is odd, we have $f(0, \frac{m+n-1}{2}) = \frac{m^2+2mn+3m+n^2+n+1}{8}$ and $f(0, \frac{m+n+1}{2}) = \frac{m^2+2mn+3m+n^2+n-1}{8}$.

From the boundaries of y, we get $f(0, 0) = \frac{4m+2n+1}{8}$ and $f(0, m) = \frac{2m+2n+4mn+1}{8}$.

Case 1.2. For $x = n$, we have $f(n, y) = \frac{1}{2}m + \frac{1}{2}n - \frac{1}{4}y + \frac{1}{2}n(m-y) + \frac{1}{2}y(m-y) + \frac{1}{8}$. Solving $f_y(n, y) = 0$ gives $y = \frac{2m-2n-1}{4}$. If $n \le m - \frac{1}{2}$, $\frac{2m-2n-1}{4} \in [0, m]$. We need to try both $y = \lfloor \frac{2m-2n-1}{4} \rfloor$ and $y = \lceil \frac{2m-2n-1}{4} \rceil$ depending on n and m.

i: if n is odd, m is odd or n is even, m is even, we have $f(n, \frac{m-n-2}{2}) = \frac{m^2+2mn+3m+n^2+5n-1}{8}$ and $f(n, \frac{m-n}{2}) = \frac{m^2+2mn+3m+n^2+5n+1}{8}$.

ii: if n is odd, m is even or n is even, m is odd, we have $f(n, \frac{m-n-1}{2}) = \frac{m^2+2mn+3m+n^2+5n+1}{8}$ and $f(n, \frac{m-n+1}{2}) = \frac{m^2+2mn+3m+n^2+5n-1}{8}$.

From the boundaries of y, we get $f(n, 0) = \frac{4m+4n+4mn+1}{8}$ and $f(n, m) = \frac{2m+4n+1}{8}$.

Case 1.3. For $y = 0$, we have $f(x, 0) = \frac{1}{2}m + \frac{1}{4}n + \frac{1}{4}x + \frac{1}{2}mx + \frac{1}{2}x(n-x) + \frac{1}{8}$. Solving $f_x(x, 0) = 0$ gives $x = \frac{2m+2n+1}{4}$. If $n \ge m + \frac{1}{2}$, $\frac{2m+2n+1}{4} \in [0, n]$. We need to try both $x = \lfloor \frac{2m+2n+1}{4} \rfloor$, $x = \lceil \frac{2m+2n+1}{4} \rceil$ depending on n and m.

i: if n is odd, m is odd or n is even, m is even, we have $f(\frac{m+n}{2}, 0) = \frac{m^2+2mn+5m+n^2+3n+1}{8}$ and $f(\frac{m+n+2}{2}, 0) = \frac{m^2+2mn+5m+n^2+3n-1}{8}$.

ii: if n is odd, m is even or n is even, m is odd, we have $f(\frac{m+n-1}{2}, 0) = \frac{m^2+2mn+5m+n^2+3n-1}{8}$ and $f(\frac{m+n+1}{2}, 0) = \frac{m^2+2mn+5m+n^2+3n+1}{8}$.

From the boundaries of x, we get $f(0, 0) = \frac{4m+2n+1}{8}$ and $f(n, 0) = \frac{4m+4n+4mn+1}{8}$.

Case 1.4. For $y = m$, we have $f(x, m) = \frac{1}{4}m + \frac{1}{4}n + \frac{1}{4}x + \frac{1}{2}m(n-x) + \frac{1}{2}x(n-x) + \frac{1}{8}$. Solving $f_x(x, m) = 0$ gives $x = \frac{2n-2m+1}{4}$. If $n \ge m - \frac{1}{2}$, $\frac{2n-2m+1}{4} \in [0, n]$. We need to try both $x = \lfloor \frac{2n-2m+1}{4} \rfloor$, $x = \lceil \frac{2n-2m+1}{4} \rceil$ depending on n and m.

i: if n is odd, m is odd or n is even, m is even, we have $f(\frac{n-m}{2}, m) = \frac{m^2+2mn+m+n^2+3n+1}{8}$ and $f(\frac{n-m+2}{2}, m) = \frac{m^2+2mn+m+n^2+3n-1}{8}$.

ii: if n is odd, m is even or n is even, m is odd, we have $f(\frac{n-m-1}{2}, m) = \frac{m^2+2mn+m+n^2+3n-1}{8}$ and $f(\frac{n-m+1}{2}, m) = \frac{m^2+2mn+m+n^2+3n+1}{8}$.

From the boundaries of x, we get $f(0, m) = \frac{2m+2n+4mn+1}{8}$ and $f(n, m) = \frac{2m+4n+1}{8}$.

Case 2. Let $u, v \in S$, $x = |X \cap S|$, $y = |Y \cap S|$ and $f(x, y) := \eta_T(S)$. In this case, we have

$$f(x,y) = \frac{1}{2}(n-x) + \frac{1}{4}(m-y) + \frac{1}{2}(m-y) + \frac{1}{4}(n-x) + \frac{1}{2}x(n-x) + \frac{1}{2}x(m-y) + \frac{1}{2}y(n-x) + \frac{1}{2}y(m-y).$$

Bounds are $0 \le x \le n$ and $0 \le y \le m$. Solving the system $f_x(x, y) = 0$ and $f_y(x, y) = 0$ doesn't give a solution and we must seek the maximum of the function at the boundaries of x and y.

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Case 2.1. For $x = 0$, we have $f(0, y) = \frac{3}{4}m + \frac{3}{4}n - \frac{3}{4}y + \frac{1}{2}ny + \frac{1}{2}y(m - y)$. Solving $f_y(0, y) = 0$ gives $y = \frac{2m+2n-3}{4}$. If $n \le m + \frac{3}{2}$, $\frac{2m+2n-3}{4} \in [0, m]$. We need to try both $y = \lfloor \frac{2m+2n-3}{4} \rfloor$ and $y = \lceil \frac{2m+2n-3}{4} \rceil$ depending on n, m

i: if n is odd, m is odd or n is even, m is even, we have
$f(0, \frac{m+n-2}{2}) = \frac{m^2+2mn+3m+n^2+3n+2}{8}$ and $f(0, \frac{m+n}{2}) = \frac{m^2+2mn+3m+n^2+3n}{8}$.

ii: if n is odd, m is even or n is even, m is odd, we have
$f(0, \frac{m+n-3}{2}) = \frac{m^2+2mn+3m+n^2+3n}{8}$ and $f(0, \frac{m+n-1}{2}) = \frac{m^2+2mn+3m+n^2+3n+2}{8}$.
From the boundaries of y, we get $f(0, 0) = \frac{3m+3n}{4}$ and $f(0, m) = \frac{3n+2mn}{4}$.

Case 2.2. For $x = n$, we have $f(n, y) = \dfrac{3}{4}m - \dfrac{3}{4}y + \dfrac{1}{2}n(m - y) + \dfrac{1}{2}y(m - y)$. Solving $f_y(n, y) = 0$ gives $y = \frac{2m-2n-3}{4}$. If $n \le m - \frac{3}{2}$, $\frac{2m-2n-3}{4} \in [0, m]$. We need to try both $y = \lfloor \frac{2m-2n-3}{4} \rfloor$ and $y = \lceil \frac{2m-2n-3}{4} \rceil$ depending on n and m.

i: if n is odd, m is odd or n is even, m is even, we have
$f(n, \frac{m-n-2}{2}) = \frac{m^2+2mn+3m+n^2+3n+2}{8}$ and $f(n, \frac{m-n}{2}) = \frac{m^2+2mn+3m+n^2+3n}{8}$.

ii: if n is odd, m is even or n is even, m is odd, we have
$f(n, \frac{m-n-3}{2}) = \frac{m^2+2mn+3m+n^2+3n}{8}$ and $f(n, \frac{m-n-1}{2}) = \frac{m^2+2mn+3m+n^2+3n+2}{8}$.
From the boundaries of y, we get $f(n, 0) = \frac{3m+2mn}{4}$ and $f(n, m) = 0$. But since $0 < |S| < n + m + 2$, we can ignore $f(n, m) = 0$.

Case 2.3. For $y = 0$, we have $f(x, 0) = \frac{3}{4}m + \frac{3}{4}n - \frac{3}{4}x + \frac{1}{2}mx + \frac{1}{2}x(n - x)$. Solving $f_x(x, 0) = 0$ gives $x = \frac{2m+2n-3}{4}$. If $n \ge m - \frac{3}{2}$, $\frac{2m+2n-3}{4} \in [0, n]$. We need to try both $x = \lfloor \frac{2m+2n-3}{4} \rfloor$ and $x = \lceil \frac{2m+2n-3}{4} \rceil$ depending on n and m.

i: If n is odd, m is odd or n is even,m is even, we have
$f(\frac{m+n}{2}, 0) = \frac{m^2+2mn+3m+n^2+3n}{8}$ and $f(\frac{m+n-2}{2}, 0) = \frac{m^2+2mn+3m+n^2+3n+2}{8}$.

ii: If n is odd,m is even or n is even,m is odd, we have
$f(\frac{m+n-1}{2}, 0) = \frac{m^2+2mn+3m+n^2+3n+2}{8}$ and $f(\frac{m+n-3}{2}, 0) = \frac{m^2+2mn+3m+n^2+3n}{8}$.
From the boundaries of x, we get $f(0, 0) = \frac{3m+3n}{4}$ and $f(n, 0) = \frac{3m+2mn}{4}$.

Case 2.4. For $y = m$, we have $f(x, m) = \frac{3}{4}n - \frac{3}{4}x + \frac{1}{2}m(n - x) + \frac{1}{2}x(n - x)$. Solving $f_x(x, m) = 0$ gives $x = \frac{2n-2m-3}{4}$. If $n \ge m + \frac{3}{2}$, $\frac{2n-2m-3}{4} \in [0, n]$. We need to try both $x = \lfloor \frac{2n-2m-3}{4} \rfloor$ and $x = \lceil \frac{2n-2m-3}{4} \rceil$ depending on n and m.

i: if n is odd, m is odd or n is even, m is even, we have
$f(\frac{n-m}{2}, m) = \frac{m^2+2mn+3m+n^2+3n}{8}$ and $f(\frac{n-m-2}{2}, m) = \frac{m^2+2mn+3m+n^2+3n+2}{8}$.

ii: if n is odd, m is even or n is even, m is odd, we have
$f(\frac{n-m-1}{2}, m) = \frac{m^2+2mn+3m+n^2+3n+2}{8}$ and $f(\frac{n-m-3}{2}, m) = \frac{m^2+2mn+3m+n^2+3n}{8}$.
From the boundaries of x, we get $f(0, m) = \frac{3n+2mn}{4}$ and $f(n, m) = 0$. But since $0 < |S| < n + m + 2$, we can ignore $f(n, m) = 0$.

By Case 1 and 2, the total influence number of $\overline{DS}_{n,m}$ is

$$\eta_T(\overline{DS}_{n,m}) = \begin{cases} \frac{m^2+2mn+3m+n^2+5n+1}{8} & \text{if } n \le m - \frac{1}{2}, \\ \frac{m^2+2mn+5m+n^2+3n+1}{8} & \text{if } n \ge m + \frac{1}{2}, \\ \frac{m^2+2mn+3m+n^2+3n+2}{8} & \text{if } n = m. \end{cases}$$

□

Lemma 4.7. *Consider a complement of path $\overline{P}_n$. Let S_1 and S_2 are total influence sets for $\overline{P}_n$. Let $X_1 \cup X_2 \cup ... \cup X_t$ be a partition of S_1 such that any element of X_i can not be consecutive to any element of X_j, where $i \ne j$. Let $|X_1| = x_1$, $|X_2| = x_2$, ..., $|X_t| = x_t$, where $t \ge 2$ and $x_1, x_2, ...x_t \ge 2$. Additionally, let each X_i consist of vertices having consecutive indices, where $i \in \{1, ..., t\}$. Let $S_2 = X$ such that $|X| = x_1 + x_2 + ... + x_t$ and X consist of vertices having consecutive indices. Then $\eta_T(S_2) > \eta_T(S_1)$.*

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Proof. For a complement of path graph with $V(\overline{P}_n) = \{v_1, v_2, ..., v_n\}$, let $u = v_1$ and $v = v_n$ be end vertices. Moreover, let $|V(\overline{P}_n) - V(S_1)| = |V(\overline{P}_n) - V(S_2)| = z$. We consider three cases, depending on u and v's membership in S.

Case i Let $u, v \in S$,
since $|S_1| = |S_2| = x_1 + x_2 + ... + x_t$. We can write the following equalities for $\eta_T(S_1)$ and $\eta_T(S_2)$.

$$\begin{aligned}\eta_T(S_1) =& 2\frac{1}{4} + 2\frac{1}{2}(z-1) + \frac{1}{2}(x_1-2)z + 2\frac{1}{4} + 2\frac{1}{2}(z-1) + \frac{1}{2}(x_2-2)z \\ &+ ... + 2\frac{1}{4} + 2\frac{1}{2}(z-1) + \frac{1}{2}(x_t-2)z \\ =& -\frac{t}{2} + (x_1 + x_2 + ... + x_t)z\frac{1}{2}. \end{aligned} \tag{4.1}$$

$$\begin{aligned}\eta_T(S_2) =& 2\frac{1}{4} + 2\frac{1}{2}(z-1) + \frac{1}{2}(x_1 + x_2 + ... + x_t - 2)z \\ =& -\frac{1}{2} + (x_1 + x_2 + ... + x_t)z\frac{1}{2}. \end{aligned} \tag{4.2}$$

By (4.1), (4.2) and $t \geq 2$, $\eta_T(S_1) < \eta_T(S_2)$.

Case ii Let $u, v \notin S$,
by the equivalence of the complementary sets, this case is equivalent to Case i and so $\eta_T(S_1) < \eta_T(S_2)$.

Case iii Let $u \in S, v \notin S$ (or $u \notin S, v \in S$),
let u (or v)$\in X_1$, we can write the following equalities for $\eta_T(S_1)$ and $\eta_T(S_2)$.

$$\begin{aligned}\eta_T(S_1) =& \frac{1}{2} + \frac{1}{4} + 2\frac{1}{2}(z-1) + \frac{1}{2}(x_1-2)z + 2\frac{1}{4} + 2\frac{1}{2}(z-1) + \frac{1}{2}(x_2-2)z \\ &+ ... + 2\frac{1}{4} + 2\frac{1}{2}(z-1) + \frac{1}{2}(x_t-2)z. \\ =& -\frac{t}{2} + \frac{1}{4} + \frac{1}{2}(x_1 + x_2 + ... + x_t)z. \end{aligned} \tag{4.3}$$

$$\begin{aligned}\eta_T(S_2) =& \frac{1}{2} + \frac{1}{4} + 2\frac{1}{2}(z-1) + \frac{1}{2}(x_1 + x_2 + ...x_t - 2)z. \\ =& -\frac{1}{4} + \frac{1}{2}(x_1 + x_2 + ... + x_t)z. \end{aligned} \tag{4.4}$$

By (4.3), (4.4) and $t \geq 2$, $\eta_T(S_1) < \eta_T(S_2)$.

We point out that for total influence number of $\overline{P}_n$, we must choose S_2 instead of S_1. □

Theorem 4.8. *For a complement of path $\overline{P}_n$ with $n \geq 7$, a set S is an η_T-set if and only if it contains exactly $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$ vertices having consecutive indices such that either u or v must be in these vertices. Furthermore,*

$$\eta_T(\overline{P}_n) = \begin{cases} \frac{n^2-2}{8} & \text{if } n \text{ is even}, \\ \frac{n^2-3}{8} & \text{if } n \text{ is odd}. \end{cases}$$

Proof. Let $X \cup Y$ be a partition of $V(\overline{P}_n)$. By Lemma 4.7, we know X consist of vertices having consecutive indices. Let $x = |X|$ and $y = |Y \cap S|$, where $Y = V(\overline{P}_n) - X$, $Y \cap S = \{v_i : \text{for } \forall \{v_i, v_j\} \in Y,$ $(v_i, v_j) \in E(\overline{P}_n)\}$ and any element of $Y \cap S$ can not be consecutive to any element of X. We consider three cases depending on u and v's membership in S.

Case 1. Let $u, v \in S$ and $f(x, y) =: \eta_T(S)$.

$$\begin{aligned}f(x,y) =& 2\frac{1}{4} + 2\frac{1}{2}(n-x-y-1) + \frac{1}{2}(x-2)(n-x-y) \\ &+ 2\frac{1}{4}y + \frac{1}{2}y(n-x-y-2). \end{aligned}$$

Bounds are $2 \leq x \leq n-1$ and $0 \leq y \leq \lceil\frac{n-4}{2}\rceil$. Solving the system $f_x(x,y) = 0$ and $f_y(x,y) = 0$ doesn't give a solution. So we must seek the maximum of the function at the boundaries of x and y.

Case 1.1. For $x = 2$, we have $f(2,y) = n - \frac{1}{2}y - \frac{1}{2}y(y-n+4) - \frac{5}{2}$. Solving $f_y(2,y) = 0$ gives $y = \frac{n-5}{2}$. If $n \geq 5$, $\frac{n-5}{2} \in [0, \lceil\frac{n-4}{4}\rceil]$.

i: if n is odd, we get $f(2, \frac{n-5}{2}) = \frac{n^2-2n+5}{2}$

ii: if n is even, we consider $y = \lfloor\frac{n-5}{2}\rfloor$ and $y = \lceil\frac{n-5}{2}\rceil$ but they are complements of each other and thus have the same value. $f(2, \frac{n-6}{2}) = f(2, \frac{n-4}{2}) = \frac{n^2-2n+4}{8}$.

Then searching at the boundaries of y, we have $f(2,0) = \frac{2n-5}{2}$ at $y = 0$ and have $f(2, \frac{n-3}{2}) = \frac{n^2-2n+1}{8}$ when n is odd, $f(2, \frac{n-4}{2}) = \frac{n^2-2n+4}{8}$ when n is even at $y = \lceil\frac{n-4}{2}\rceil$.

Case 1.2. For $x = n-1$, y just takes the value 0. For $y = 0$, we only have $f(n-1,0) = \frac{n-2}{2}$.

Case 1.3. For $y = 0$, we have $f(x,0) = n - x + \frac{1}{2}(n-x)(x-2) - \frac{1}{2}$. Solving $f_x(x,0) = 0$ gives $x = \frac{n}{2}$, $\frac{n}{2} \in [2, n-1]$, if $n \geq 4$.

i: if n is odd, we have $f(\frac{n-1}{2}, 0) = f(\frac{n+1}{2}, 0) = \frac{n^2-5}{8}$

ii: if n is even, we have $f(\frac{n}{2}, 0) = \frac{n^2-4}{8}$

Then searching at the boundaries of x, we have $f(2,0) = \frac{2n-5}{2}$ at $x = 2$, $f(n-1,0) = \frac{n-2}{2}$ at $x = n-1$.

Case 1.4. for $y = \lceil\frac{n-4}{2}\rceil$, x just takes the value 0. So we have $f(2, \frac{n-3}{2}) = \frac{n^2-2n+1}{8}$, when n is odd, $f(2, \frac{n-4}{2}) = \frac{n^2-2n+4}{8}$, when n is even.

By these subcases, for Case 1 the function is maximized at $x = \frac{n-1}{2}$ or $x = \frac{n+1}{2}$ and $y = 0$ when n is odd; $x = \frac{n}{2}$ and $y = 0$ when n is even.

Case 2. Let $u \in S, v \notin S$ or $u \notin S, v \in S$.

For the vertex u or v, we have tree subcases.

Case 2.1. Let u(or v)$\in X$ and $f(x,y) := \eta_T(S)$.

$$\begin{aligned} f(x,y) =& \frac{1}{4} + \frac{1}{2} + 2\frac{1}{2}(n-x-y-1) + \frac{1}{2}(x-2)(n-x-y) \\ &+ 2\frac{1}{4}y + \frac{1}{2}y(n-x-y-2). \end{aligned}$$

Bounds are $2 \leq x \leq n-1$ and $0 \leq y \leq \lceil\frac{n-4}{2}\rceil$. Solving the system of $f_x(x,y) = 0$ and $f_y(x,y) = 0$ doesn't give a solution and we must seek the maximum of the function at the boundaries of x and y.

Case 2.1.1. For $x = 2$, we have $f(2,y) = n - \frac{1}{2}y - \frac{1}{2}y(y-n+4) - \frac{9}{4}$. Solving the $f_y(2,y) = 0$ gives $y = \frac{n-5}{2}$ and $\frac{n-5}{2} \in [0, \lceil\frac{n-4}{2}\rceil]$, if $n \geq 5$.

i: if n is odd, we have $f(2, \frac{n-5}{2}) = \frac{n^2-2n+7}{8}$.

ii: if n is even, we have $f(2, \frac{n-6}{2}) = f(2, \frac{n-4}{2}) = \frac{n^2-2n+6}{8}$.

Then searching at the boundaries of y, we have $f(2,0) = \frac{2n-5}{2}$ at $y = 0$, $f(2, \frac{n-3}{2}) = \frac{n^2-2n+3}{8}$ when n is odd and $f(2, \frac{n-4}{2}) = \frac{n^2-2n+6}{8}$ when n is even at $y = \lceil\frac{n-4}{2}\rceil$. But this function is maximized at $y = \frac{n-5}{2}$ when n is odd, at $y = \frac{n-6}{2}$ or $y = \frac{n-4}{2}$ when n is even.

Case 2.1.2. For $x = n-1$, y just takes the value 0. So we only have $f(n-1,0) = \frac{2n-3}{4}$.

Case 2.1.3. For $y = 0$, we have $f(x,0) = n - x + (\frac{1}{2}x - 1)(n-x) - \frac{1}{4}$. Solving the $f_x(x,0) = 0$ gives $x = \frac{n}{2}$. $\frac{n}{2} \in [2, n-1]$, if $n \geq 4$.

i: if n is odd, we have $f(\frac{n-1}{2}, 0) = f(\frac{n+1}{2}, 0) = \frac{n^2-3}{8}$

ii: if n is even, we have $f(\frac{n}{2}, 0) = \frac{n^2-2}{8}$

Then searching at the boundaries of x, we have $f(2,0) = \frac{4n-9}{4}$ at $x = 2$ and $f(n-1,0) = \frac{2n-3}{4}$ at $x = n-1$.

Case 2.1.4. For $y = \lceil\frac{n-4}{2}\rceil$, x just takes the value 2. So we only have $f(2, \lceil\frac{n-4}{2}\rceil)$.

By Case 2.1, the function is maximized at $x = \frac{n-1}{2}$ or $x = \frac{n+1}{2}$ and $y = 0$ when n is odd; at $x = \frac{n}{2}$ and $y = 0$ when n is even.

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Case 2.2. Let u(or v)$\in Y$, $x \geq 2$ and $f(x,y) := \eta_T(S)$. We have

$$\begin{aligned} f(x,y) =& 2\frac{1}{4} + 2\frac{1}{2}(n-x-y-1) + \frac{1}{2}(x-2)(n-x-y) \\ &+ 2\frac{1}{4}(y-1) + \frac{1}{2}(y-1)(n-x-y-2) + \frac{1}{4} \\ &+ \frac{1}{2}(n-x-y-1). \end{aligned}$$

Boundaries are $2 \leq x \leq n-3$ and $1 \leq y \leq \lceil \frac{n-4}{2} \rceil$. Solving the system of $f_x(x,y) = 0$ and $f_y(x,y) = 0$ doesn't give a result and we must seek the maximum of the function at the boundaries of x and y.
Case 2.2.1. For $x = 2$, we have $f(2,y) = \frac{3}{2}n - y - (\frac{1}{2}y - \frac{1}{2})(y-n+4) - \frac{17}{4}$. Solving $f_y(2,y) = 0$ gives $y = \frac{n-5}{2}$. $\frac{n-5}{2} \in [1, \lceil \frac{n-4}{2} \rceil]$, if $n \geq 7$.
i: if n is odd, we have $f(2, \frac{n-5}{2}) = \frac{n^2-2n+7}{8}$.
ii: if n is even, we have $f(2, \frac{n-6}{2}) = f(2, \frac{n-4}{2}) = \frac{n^2-2n+6}{8}$.

Then searching at the boundaries of y, we have $f(2,1) = \frac{6n-21}{4}$ at $y = 1$, $f(2, \frac{n-3}{2}) = \frac{n^2-2n+3}{8}$ when n is odd and $f(2, \frac{n-4}{2}) = \frac{n^2-2n+6}{8}$ when n is even at $y = \lceil \frac{n-4}{2} \rceil$.
Case 2.2.2. For $x = n-3$, y just takes the value 1. So in this case, we only have $f(n-3, 1) = \frac{4n-11}{4}$.
Case 2.2.3. For $y = 1$, we have $f(x,1) = \frac{3}{2}n - \frac{3}{2}x - (\frac{1}{2}x - 1)(x-n+1) - \frac{9}{4}$. Solving the $f_x(x,1) = 0$ gives $x = \frac{n-2}{2}$ and $\frac{n-2}{2} \in [2, n-1]$, if $n \geq 6$.
i: if n is odd, we have $f(\frac{n-3}{2}, 1) = f(\frac{n-1}{2}, 1) = \frac{n^2-7}{8}$
ii: if n is even, we have $f(\frac{n-2}{2}, 1) = \frac{n^2-6}{8}$
Then searching at the boundaries of x, we have $f(2,1) = \frac{6n-21}{4}$ for $x = 2$ and $f(n-3, 1) = \frac{4n-1}{4}$ for $x = n-3$.
Case 2.2.4. For $y = \lceil \frac{n-4}{2} \rceil$, x just takes the value 2. Thus we have $f(2, \frac{n-4}{2}) = \frac{n^2-2n+6}{8}$ when n is even and $f(2, \frac{n-3}{2}) = \frac{n^2-2n+3}{8}$ when n is odd.

By Case 2.2, the function is maximized at $x = \frac{n-3}{2}$ or $x = \frac{n-1}{2}$ and $y = 1$ when n is odd; $x = \frac{n-2}{2}$ and $y = 1$ when n is even.
Case 2.3. Let u(or v)$\in Y$, $x = 0$ and $f(x,y) := \eta_T(S)$.

$$f(y) = \frac{1}{4} + \frac{1}{2}(n-y-1) + 2\frac{1}{4}(y-1) + \frac{1}{2}(y-1)(n-y-2).$$

Bound is $1 \leq y \leq \lfloor \frac{n}{2} \rfloor$. Solving $f_y(y) = 0$ gives $y = \frac{n-1}{2}$ and $\frac{n-1}{2} \in [1, \lfloor \frac{n}{2} \rfloor]$, if $n \geq 3$.
i: if n is odd, we have $f(\frac{n-1}{2}) = \frac{n^2-2n+3}{8}$.
ii: if n is even, we have $f(\frac{n-2}{2}) = f(\frac{n}{2}) = \frac{n^2-2n+2}{8}$.

Then searching at the boundaries of y, we have $f(1) = \frac{2n-3}{4}$ at $y = 1$ and having $f(\frac{n-1}{2}) = \frac{n^2-2n+3}{8}$ when n is odd, $f(\frac{n}{2}) = \frac{n^2-2n+2}{8}$ when n is even at $y = \lfloor \frac{n}{2} \rfloor$.

By Case 2.3, the function is maximized at $y = \frac{n-1}{2}$ when n is odd and at $y = \frac{n-2}{2}$ or $y = \frac{n}{2}$ when n is even.
Case 3. Let $u, v \notin S$.
Case 3.1. Let $x = |X|$, $y = |Y \cap S|$ and $x \geq 2$. This condition is equivalent to Case 1.
Case 3.2. Let $x = 0$, $y = |Y \cap S|$ and $f(y) := \eta_T(S)$. We have

$$f(y) = 2\frac{1}{4}y + \frac{1}{2}y(n-y-2)$$

. Boundary is $1 \leq y \leq \lceil \frac{n-2}{2} \rceil$. Solving $f_y(y) = 0$ gives $y = \frac{n-1}{2}$.
i: if n is odd, $f(\frac{n-1}{2}) = \frac{n^2-2n+1}{8}$
ii: if n is even, y can be $\frac{n}{2}$ or $\frac{n-2}{2}$. Since $\frac{n}{2} \notin [1, \lceil \frac{n-2}{2} \rceil]$, we get $f(\frac{n-2}{2}) = \frac{n^2-2n}{8}$.

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Then searching at the boundaries of y, we have $f(1) = \frac{n-2}{2}$ at $y = 1$ and having $f(\frac{n-1}{2} = \frac{n^2-2n+1}{8})$ when n is odd and $f(\frac{n-2}{2}) = \frac{n^2-2n}{8}$ when n is even at $y = \lceil \frac{n-2}{2} \rceil$.

From all the cases, the total influence number of $\overline{P}_n$ is $\frac{n^2-2}{8}$ when n is even and $\frac{n^2-3}{8}$ when n is odd. $\square$

Theorem 4.9. *For a complement of cycle graph $\overline{C}_n$ with $n \geq 6$, S is an η_T-set if and only if it contains exactly $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$ vertices having consecutive indices. Furthermore,*

$$\eta_T(\overline{C}_n) = \begin{cases} \frac{n^2-5}{8} & \text{if } n \text{ is odd,} \\ \frac{n^2-4}{8} & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $X \cup Y$ be a partition of $V(\overline{C}_n)$. By Case i in Lemma 4.7, we know X consist of vertices having consecutive indices. Let $x = |X|$ and $y = |Y \cap S|$, where $Y = V(\overline{C}_n) - X$ and $Y \cap S = \{v_i :$ for $\forall v_i, v_j \in Y$, $(v_i, v_j) \in E(\overline{C}_n)\}$ and any element of $Y \cap S$ can not be consecutive to any element of X. We consider two cases: first, $x \geq 2$ and second, $x = 0$. These two cases are comprehensive because we ignore other cases from Case i in Lemma 4.7.

Case 1. Let $x = |X|$, $y = |Y \cap S|$, $x \geq 2$ and $f(x, y) := \eta_T(S)$. This case is equivalent to Case 1 in the proof of $\overline{P}_n$. For this case, the maximum value is $f(\lfloor \frac{n}{2} \rfloor, 0) = f(\lceil \frac{n}{2} \rceil, 0) = \frac{n^2-5}{8}$, when n is odd and $f(\frac{n}{2}, 0) = \frac{n^2-4}{8}$, when n is even.

Case 2. Let $|X| = x$, $|Y \cap S| = y$, $x = 0$ and $f(y) := \eta_T(S)$. This yields the following equation:

$$f(y) = 2\frac{1}{4}y + \frac{1}{2}y(n - y - 2).$$

Bound is $1 \leq y \leq \lfloor \frac{n}{2} \rfloor$. Solving $f_y(y) = 0$ gives $y = \frac{n-1}{2} \in [0, \lfloor \frac{n}{2} \rfloor]$. We have $f(\frac{n-1}{2}) = \frac{n^2-2n+1}{8}$ when n is odd, $f(\frac{n}{2}) = \frac{n^2-2n}{8}$ when n is even.

After examinations at the boundaries of y, the function is maximized at $y = \frac{n-1}{2}$.

By Case 1 and 2, the function is maximized at $x = \lfloor \frac{n}{2} \rfloor$ or $x = \lceil \frac{n}{2} \rceil$. Hence, the total influence number of $\overline{C}_n$ is

$$\eta_T(\overline{C}_n) = \begin{cases} \frac{n^2-5}{8} & \text{if } n \text{ is odd,} \\ \frac{n^2-4}{8} & \text{if } n \text{ is even.} \end{cases}$$

$\square$

Corollary 4.10. *For a complement of wheel graph $\overline{W}_{1,n}$, with $n \geq 6$, a set S is an η_T-set if and only if it contains exactly $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$ vertices having consecutive. Furthermore,*

$$\eta_T(\overline{W}_{1,n}) = \begin{cases} \frac{n^2-5}{8} & \text{if } n \text{ is odd,} \\ \frac{n^2-4}{8} & \text{if } n \text{ is even.} \end{cases}$$

Proof. A complement of wheel graph $\overline{W}_{1,n}$ contains a complement of cycle graph with n vertices and an isolated vertex. Since the isolated vertex doesn't influence any vertices of $\overline{C}_n$, the proof is done similar to Theorem 4.9. $\square$

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