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SOME REMARKS ON F-PARTIAL ORDER AND PROPERTIES

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Abstract. In this paper, we investigate some properties an order induced by nullnorms on bounded lattices. We determine the relationship between the order induced by a nullnorm and the natural order on the lattice. Then, we give a necessary and sufficient condition making a bounded lattice L also a lattice with respect to the order $\preceq_F$.

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1. Introduction

Nullnorms and t-operators were introduced in [6], [16], respectively, which are also generalizations of the notions of t-norms and t-conorms. And then in [17], it is pointed out that nullnorms and t-operators are equivalent since they have the same block structures in $[0,1]^2$. Namely, if a binary operator F is a nullnorm then it is also a t-operator and vice versa.

In [13], nullnorms on bounded lattices were studied. Also, smallest nullnorm and the greatest nullnorm with zero element $a \in L \setminus \{0,1\}$ on L were obtained.

In [14], a partial order defined by means of t-norms on a bounded lattice was introduced

$x \preceq_T y :\Leftrightarrow T(\ell, y) = x$ for some $\ell \in L$,

where L is a bounded lattice, x, y of a bounded lattice L and T is a t-norm on L. This partial order $\preceq_T$ is called a T-partial order on L.

In [1], an order induced by nullnorms on bounded lattices was defined and discussed. Then, the set of incomparable elements with respect to the order induced by a nullnorm was defined. Also, by defining such an order, an equivalence relation on the class of nullnorms was defined and this equivalence was deeply investigated. Triangular norms and nullnorms were studied by many other authors [2], [3], [9], [10], [18], [19].

In the present paper, we investigate some properties an order induced by nullnorms on bounded lattices. The paper is organized as follows. We shortly recall some basic notions in Section 2. In Section 3, we determine with the examples the relationship between the order induced by a nullnorm and the order on the lattice. By an example, we show that a bounded lattice L need not be a lattice with respect to the order induced by $\preceq_F$. We prove that when the underlying t-norm and t-conorm of a nullnorm on a bounded lattice L are divisible, the order $\preceq_F$ coincides with the order on L. Finally, we give a necessary and sufficient condition making a bounded lattice L also a lattice with respect to the order $\preceq_F$.

2. Preliminaries

A bounded lattice $(L, \leq)$ is a lattice which has the top and bottom elements, which are written as 1 and 0, respectively, that is, there exist two elements $1, 0 \in L$ such that $0 \leq x \leq 1$, for all $x \in L$.

Definition 2.1. [15, 8] *Let $(L, \leq, 0, 1)$ be a bounded lattice. A triangular norm T (briefly t-norm) is a binary operation on L which is commutative, associative, monotone and has neutral element 1.*

Example 2.2. [15] *The following are the four basic t-norms T_M, T_P, T_L and T_D on $[0, 1]$ given by, respectively,*
$T_M(x, y) = min(x, y)$
$T_P(x, y) = xy$
$T_L(x, y) = max(x + y - 1, 0)$
$$T_D(x, y) = \begin{cases} 0 & , if\ (x, y) \in [0, 1)^2 \\ min(x, y) & , otherwise \end{cases}$$

$T_D < T_L < T_P < T_M$

Definition 2.3. [15] *Let $(L, \leq, 0, 1)$ be a bounded lattice. A triangular conorm S (briefly t-conorm) is a binary operation on L which is commutative, associative, monotone and has neutral element 0.*

Example 2.4. [15] *The following are the four basic t-conorms S_M, S_P, S_L and S_D on $[0, 1]$ given by, respectively,*
$S_M(x, y) = max(x, y)$
$S_P(x, y) = x + y - xy$
$S_L(x, y) = min(x + y, 1)$
$$S_D(x, y) = \begin{cases} 1 & , if\ (x, y) \in (0, 1)^2 \\ max(x, y) & , otherwise \end{cases}$$

$S_M < S_P < S_L < S_D$

Definition 2.5. [7] *A t-norm T on L is divisible if the following condition holds: $\forall\, x, y \in L$ with $x \leq y$ there is a $z \in L$ such that $x = T(y, z)$.*

A basic example of a non-divisible t-norm on an arbitrary lattice L (i.e., card$L > 2$) is the weakest t-norm T_W. Trivially, the infimum $T_\wedge$ is divisible: $x \leq y$ is equivalent to $x \wedge y = x$.

Proposition 2.6. [4] *Let T be a t-norm on $[0, 1]$. T is divisible if and only if T is continuous.*

Definition 2.7. [5] *Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, $a \leq b$, a subinterval $[a, b]$ of L is defined as*
$[a, b] = \{x \in L \mid a \leq x \leq b\}$
Similarly, $[a, b) = \{x \in L \mid a \leq x < b\}$, $(a, b] = \{x \in L \mid a < x \leq b\}$ and $(a, b) = \{x \in L \mid a < x < b\}$.

Definition 2.8. [6] *Let $(L, \leq, 0, 1)$ be a bounded lattice. A commutative, associative, non-decreasing in each variable function $F : L^2 \to L$ is called a nullnorm if there is an element $a \in L$ such that $F(x, 0) = x$ for all $x \leq a$, $F(x, 1) = x$ for all $x \geq a$.*

It can be easily obtained that $F(x, a) = a$ for all $x \in L$. So $a \in L$ is the zero (absorbing) element for F.

In the whole of the paper, for shortness, the set $[0, a) \times (a, 1] \cup (a, 1] \times [0, a)$ for $a \in L\backslash\{0, 1\}$ will be denoted by D_a, that is $D_a = [0, a) \times (a, 1] \cup (a, 1] \times [0, a)$ for $a \in L\backslash\{0, 1\}$.

Definition 2.9. [13] *An element $x \in L$ is called an idempotent element of a function $F : L^2 \to L$ if $F(x, x) = x$. The function F is called idempotent if all elements of L are idempotent.*

Definition 2.10. [14] *Let L be a bounded lattice, T be a t-norm on L. The order defined as following is called a $T-$ partial order (triangular order) for t-norm T:*

$$x \preceq_T y :\Leftrightarrow T(\ell, y) = x \text{ for some } \ell \in L.$$

Definition 2.11. [11] *Let L be a bounded lattice, S be a t-conorm on L. The order*

$$x \preceq_S y :\Leftrightarrow S(\ell, x) = y \text{ for some } \ell \in L.$$

Definition 2.12. [1] *Let $(L, \leq, 0, 1)$ be a bounded lattice and F be a nullnorm with zero element a on L. Define the following relation, for $x, y \in L$, as*

$$x \preceq_F y :\Leftrightarrow \begin{cases} \text{if } x, y \in [0, a] \text{ and there exists } k \in [0, a] \text{ such that } F(x, k) = y \text{ or,} \\ \text{if } x, y \in [a, 1] \text{ and there exists } \ell \in [a, 1] \text{ such that } F(y, \ell) = x \text{ or,} \\ \text{if } (x, y) \in L^* \text{ and } x \leq y. \end{cases} \tag{2.1}$$

where $I_a = \{x \in L \mid x \parallel a\}$ and $L^* = [0, a] \times [a, 1] \cup [0, a] \times I_a \cup [a, 1] \times I_a \cup [a, 1] \times [0, a] \cup I_a \times [0, a] \cup I_a \times [a, 1] \cup I_a \times I_a$.

The partial order $\preceq_F$ in (2.1) is called an F-partial order on L.

Lemma 2.13. [1] *Let $(L, \leq, 0, 1)$ be a bounded lattice. For all nullnorms F and all $x \in L$ it holds that $0 \preceq_F x$, $x \preceq_F x$ and $x \preceq_F 1$.*

Proposition 2.14. [1] *Let $(L, \leq, 0, 1)$ be a bounded lattice and F be a nullnorm on L. If $x \preceq_F y$ for any $x, y \in L$, then $x \leq y$.*

Proposition 2.15. [1] *Let $(L, \leq, 0, 1)$ be a bounded lattice and F be a nullnorm with zero element a. Then, $(L, \preceq_F)$ is a bounded partially ordered set.*

3. On F-partial order

Lemma 3.1. [1] *Let $(L, \leq, 0, 1)$ be a bounded lattice and F be a nullnorm with zero element a on L. The order $\preceq_F$ coincides with the order $\preceq_T$ ($\preceq_S$), when $a = 0$ ($a = 1$).*

In the paper, for any subset X of L, $\overline{X}_{\preceq_F}$ ($\underline{X}_{\preceq_F}$) denotes the set of the upper (lower) bounds of X with respect to $\preceq_F$. Also, for any $x, y \in L$, $x \wedge_F y$ ($x \vee_F y$) denotes the greatest (least) element of the lower (upper) bounds with respect to $\preceq_F$, if there exists.
Similar notations will be used for the orders $\preceq_T$ and $\preceq_S$.

Remark 3.2. *Let $(L, \leq, 0, 1)$ be a bounded lattice and F be a nullnorm with zero element a on L. Even if $(L, \leq, 0, 1)$ is a chain, the partially ordered set $(L, \preceq_F)$ may not be a chain. To illustrate this claim we shall give the following example.*

Example 3.3. *Consider $L = [0, 1]$ and take the nullnorm $\overline{F_a} : [0, 1]^2 \to [0, 1]$ with the zero element $a \in (0, 1)$ defined as follows:*

$$\overline{F_a}(x, y) = \begin{cases} min(x, y) & , (x, y) \in [a, 1]^2 \\ a & , (x, y) \in (0, a]^2 \cup D_a \\ max(x, y) & , otherwise \end{cases}$$

$\overline{F_a}$ is the maximum nullnorm on $[0, 1]$ by [9]. $\frac{2a}{3}$ and $\frac{3a}{4}$ are not comparable with respect to the $\preceq_{\overline{F_a}}$. If $\frac{3a}{4} \preceq_{\overline{F_a}} \frac{2a}{3}$, by Proposition 2.14 , it must be $\frac{3a}{4} \leq \frac{2a}{3}$, a contradiction. Suppose that $\frac{2a}{3} \preceq_{\overline{F_a}} \frac{3a}{4}$. Then, there exists an element $k \leq a$ such that $\overline{F_a}(\frac{2a}{3}, k) = \frac{3a}{4}$. Since $k \leq a$ and $\frac{2a}{3} \leq a$, by the definition of $\overline{F_a}$, we have that $a = \frac{3a}{4}$, that is, $a = 0$, a contradiction. So, $\frac{2a}{3}$ and $\frac{3a}{4}$ are not comparable with respect to the $\preceq_{\overline{F_a}}$. So L is not a chain with respect to the $\preceq_{\overline{F_a}}$.

Remark 3.4. *Let $(L, \leq, 0, 1)$ be a bounded lattice and F be a nullnorm with zero element a on L. Even if $(L, \leq, 0, 1)$ is a lattice, the partially ordered set $(L, \preceq_F)$ may not be a lattice. To illustrate this claim we shall give the following example.*

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Example 3.5. *Consider the function* $F := F_{(T^{nM},S,\frac{1}{5})} : [0,1]^2 \to [0,1]$ *defined as follows:*

$$F_{(T^{nM},S,\frac{1}{5})}(x,y) = \begin{cases} max(x,y) & ,(x,y) \in [0,\frac{1}{5}]^2 \\ \frac{1}{5} & ,((x,y) \in [\frac{1}{5},1]^2 \text{ and } x+y \leq 1) \text{ or } (x,y) \in D_{\frac{1}{5}} \\ min(x,y) & , otherwise \end{cases}$$

The function F *is a nullnorm with* $\frac{1}{5}$ *zero element by* [1]*. Then,* $([0,1], \preceq_F)$ *is not join-semilattice.*

Let $x, y \in [0,1]$ *be arbitrary. If* x *and* y *are comparable with respect to the* $\preceq_F$*, we have that* $x \vee_F y$ *is* x *or* y *with respect to the* $\preceq_F$*. Suppose that* x *and* y *are not comparable with respect to the* $\preceq_F$*. We claim that* $\frac{1}{4}$ *and* $\frac{1}{2}$ *are not comparable with respect to the* $\preceq_F$*.*

If $\frac{1}{2} \preceq_F \frac{1}{4}$*, then we have that* $\frac{1}{2} \leq \frac{1}{4}$*, a contradiction. If* $\frac{1}{4} \preceq_F \frac{1}{2}$*, then there exists an element* $\ell \geq \frac{1}{5}$ *such that* $F(\frac{1}{2}, \ell) = \frac{1}{4}$*. We have that*

$$F(\frac{1}{2}, \ell) = min(\frac{1}{2}, \ell) = \frac{1}{4}$$

by the definition of F*. It must be* $\ell = \frac{1}{4}$*. So, we have* $\frac{1}{2} + \frac{1}{4} < 1$*, a contradiction. Thus,* $\frac{1}{4}$ *and* $\frac{1}{2}$ *are not comparable with respect to the* $\preceq_F$*.*

Let $m \in \overline{\{\frac{1}{4}, \frac{1}{2}\}}_{\preceq_F}$*. Then we have that* $\frac{1}{4} \preceq_F m$ *and* $\frac{1}{2} \preceq_F m$*. Then, there exist* $\ell_1, \ell_2 \geq \frac{1}{5}$ *such that* $F(m, \ell_1) = \frac{1}{4}$ *and* $F(m, \ell_2) = \frac{1}{2}$*. Thus, it must be that* $m + \ell_1 > 1$ *and* $m + \ell_2 > 1$*. By the definition of the nullnorm* F*, we obtain that*

$$\frac{1}{4} = min(m, \ell_1) = F(m, \ell_1) \quad \text{and} \quad \frac{1}{2} = min(m, \ell_2) = F(m, \ell_2).$$

If $\frac{1}{4} = m = min(m, \ell_1)$*, it would be* $\frac{1}{2} = F(\ell_2, \frac{1}{4})$*. So, we have* $\frac{1}{2} \preceq_F \frac{1}{4}$*, a contradiction. If* $\frac{1}{2} = m = min(m, \ell_2)$*, it would be* $\frac{1}{4} = F(\ell_1, \frac{1}{2})$*. So, we have that* $\frac{1}{4} \preceq_F \frac{1}{2}$*, a contradiction. So, it must be*

$$\frac{1}{4} = \ell_1 = min(m, \ell_1) \quad \text{and} \quad \frac{1}{2} = \ell_2 = min(m, \ell_2).$$

Since $m + \ell_1 > 1$ *and* $m + \ell_2 > 1$*, we have that* $m > \frac{3}{4} > \frac{1}{2}$ *and so* $\overline{\{\frac{1}{4}, \frac{1}{2}\}}_{\preceq_F} \subseteq (\frac{3}{4}, 1]$*. Now, we want to show that* $\overline{\{\frac{1}{4}, \frac{1}{2}\}}_{\preceq_F} = (\frac{3}{4}, 1]$*. Let* $x \in (\frac{3}{4}, 1]$*. Now, let us prove that* $\frac{1}{4} \preceq_F x$ *and* $\frac{1}{2} \preceq_F x$*. For any* $x > \frac{3}{4}$*, since* $\frac{1}{2} = F(x, \frac{1}{2})$ *and* $\frac{1}{4} = F(x, \frac{1}{4})$*, we have that* $\frac{1}{4} \preceq_F x$ *and* $\frac{1}{2} \preceq_F x$*. So we showed that* $\overline{\{\frac{1}{4}, \frac{1}{2}\}}_{\preceq_F} = (\frac{3}{4}, 1]$*. Since there does not exist the least element of* $(\frac{3}{4}, 1]$ *with respect to the* $\preceq_F$*,* $([0,1], \preceq_F)$ *is not a join-semilattice.*

Remark 3.6. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice and* F *be a nullnorm with zero element* a *on* L*. If* $(L, \leq, 0, 1)$ *is a lattice, the partially ordered set* $(L, \preceq_F)$ *may be a lattice.*

Example 3.7. *Let* $(L = \{0, x, y, a, z, t, 1\}, \leq, 0, 1)$ *be a chain with* $0 < x < y < a < z < t < 1$*. Consider the function on* L *defined as follows:*

$$F(x,y) = \begin{cases} x \vee y & ,(x,y) \in [0,a]^2 \\ y & ,x = 1 \text{ and } y \geq a \\ x & ,y = 1 \text{ and } x \geq a \\ a & , otherwise \end{cases}$$

By [13]*, it can be easily seen that* F *is a nullnorm. Since* $y = x \vee y = F(x,y)$*, we have that* $x \preceq_F y$*. It is trivial that* $x \preceq_F a$*,* $y \preceq_F a$*,* $a \preceq_F z$*,* $a \preceq_F t$*. Also, it is clear that* $x \preceq_F z$*,* $x \preceq_F t$*,* $y \preceq_F z$*,* $y \preceq_F t$ *by the definition of* $\preceq_F$*. We claim that* $z \not\preceq_F t$*. Suppose that* $z \preceq_F t$*. Then there exists an element* $\ell \geq a$*,*

$$F(t, \ell) = z.$$

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If $\ell = 1$, then we have $t = z$, a contradiction. So, it must be $\ell \neq 1$. Thus, we obtained that $z = a$, a contradiction. So, it must be $z \not\preceq_F t$. The order $\preceq_F$ on L has its diagram as follows (see Figure 1).

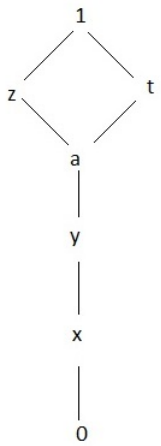

Figure 1: The order $\preceq_F$ on L

Proposition 3.8. [13] *Let $(L, \leq, 0, 1)$ be a bounded lattice, $a \in L \setminus \{0, 1\}$ and F be a nullnorm with zero element a on L. Then,*

(i) $S^* = F \mid_{[0,a]^2} \colon [0, a]^2 \to [0, a]$ is a t-conorm on $[0, a]$.
(ii) $T^* = F \mid_{[a,1]^2} \colon [a, 1]^2 \to [a, 1]$ is a t-norm on $[a, 1]$.

Proposition 3.9. *Let $(L, \leq, 0, 1)$ be a bounded lattice and F be a nullnorm on L with zero element $a \in L \setminus \{0, 1\}$ such that a is comparable with all elements of L. Then, $([0, a], \preceq_{S^*})$ and $([a, 1], \preceq_{T^*})$ are lattices if and only if $(L, \preceq_F)$ is a lattice.*

Proof. Suppose that $([0, a], \preceq_{S^*})$ and $([a, 1], \preceq_{T^*})$ are lattices.
• Let $x, y \in [0, a]$ be arbitrary. Since $([0, a], \preceq_{S^*})$ is a lattice $x \vee_{S^*} y$ and $x \wedge_{S^*} y$ exist. Let

$$x \vee_{S^*} y = m \in [0, a] \text{ and } x \wedge_{S^*} y = n \in [0, a].$$

Now, we want to show that $x \vee_F y = m$ and $x \wedge_F y = n$. Since $x \vee_{S^*} y = m$, we have that $x \preceq_{S^*} m$ and $y \preceq_{S*} m$. Then, there exist $k_1, k_2 \in [0, a]$ such that

$$S^*(x, k_1) = m \text{ and } S^*(y, k_2) = m.$$

Thus, we have that $F(x, k_1) = F \mid_{[0,a]^2} (x, k_1) = S^*(x, k_1) = m$ and $F(y, k_2) = F \mid_{[0,a]^2} (y, k_2) = S^*(y, k_2) = m$.
Thus, we obtained that $x \preceq_F m$ and $y \preceq_F m$, that is $m \in \overline{\{x, y\}}_{\preceq_F}$.

Let $t \in \overline{\{x, y\}}_{\preceq_F}$ be arbitrary. Then, we have $x \preceq_F t$ and $y \preceq_F t$. Since a is comparable with the elements of L, either $t \leq a$ or $a \leq t$. Let $t \leq a$. Then, there exist $\ell_1, \ell_2 \in [0, a]$ such that

$$t = F(x, \ell_1) = F \mid_{[0,a]^2} (x, \ell_1) = S^*(x, \ell_1) \text{ and } t = F(y, \ell_2) = F \mid_{[0,a]^2} (y, \ell_2) = S^*(y, \ell_2).$$

Then, $x \preceq_{S^*} t$ and $y \preceq_{S^*} t$, that is, it is obtained that $t \in \overline{\{x, y\}}_{\preceq_{S^*}}$. Since $x \vee_{S^*} y = m$, it must be $m \preceq_{S^*} t$. So, there exist an element $z \in [0, a]$ such that

$F(m, z) = F \mid_{[0,a]^2} (m, z) = S^*(m, z) = t$. So, it must be $m \preceq_F t$.
Let $a \leq t$. Since $m \in [0, a]$ and $t \in [a, 1]$, we have that $m \preceq_F t$, by the definition of $\preceq_F$. So, $x \vee_F y = m$. Similarly, it can be shown that $x \wedge_F y = n$.

• Let $x, y \in [a, 1]$. Since $([a, 1], \preceq_{T*})$ is a lattice similar arguments can be done in this case.
Finally, $x, y \notin [0, a]$ and $x, y \notin [a, 1]$. Since a is comparable with the elements of L, it must be either

$x \leq a \leq y$ or $y \leq a \leq x$. Suppose that $x \leq a \leq y$. Then, we have that $x \preceq_F y$, by the definition of $\preceq_F$. Thus,

$$x \vee_F y = y \text{ and } x \wedge_F y = x.$$

Similarly, $y \leq a \leq x$. We have that $x \vee_F y = x$ and $x \wedge_F y = y$. Thus, it is obtained that $x \vee_F y = x \vee y$ and $x \wedge_F y = x \wedge y$. Thus, $(L, \preceq_F)$ is a lattice.

Conversely, let $(L, \preceq_F)$ be a lattice. In this case $x \vee_F y$ and $x \wedge_F y$ exist for all $x, y \in L$. Let $x, y \in [0, a]$. Since $\preceq_F = \preceq_{S^*}$, $x \vee_{S^*} y$ and $x \wedge_{S^*} y$ exist. Let $x, y \in [a, 1]$. Since $\preceq_F = \preceq_{T^*}$, $x \vee_{T^*} y$ and $x \wedge_{T^*} y$ exist. So, it is obtained that $([0, a], \preceq_{S^*})$ and $([a, 1], \preceq_{T^*})$ are lattices. □

Proposition 3.10. [6, 16] *Let $F : [0,1]^2 \to [0,1]$ be a nullnorm with zero element $F(1,0) = k \notin \{0,1\}$. Then,*

$$F(x,y) = \begin{cases} kS(\frac{x}{k}, \frac{y}{k}) & , x,y \in [0,k] \\ k + (1-k)T(\frac{x-k}{1-k}, \frac{y-k}{1-k}) & , x,y \in [k,1] \\ k & , otherwise \end{cases}$$

where S is a t-conorm and T is a t-norm.

A nullnorm F with zero element k, underlying t-conorm S and underlying t-norm T will be denoted by $F =< S, k, T >$.

Proposition 3.11. *Let $F =< S, k, T >$ be a nullnorm with zero element $k \in (0,1)$. Then,*
i) $x \preceq_F y$ for $x, y \in [0, k]$ if and only if $\frac{x}{k} \preceq_S \frac{y}{k}$ for $x, y \in [0, k]$.
ii) $x \preceq_F y$ for $x, y \in [k, 1]$ if and only if $\frac{x-k}{1-k} \preceq_T \frac{y-k}{1-k}$ for $x, y \in [k, 1]$.

Proof. The proof is trivial. □

Proposition 3.12. [1] *Let $(L, \leq, 0, 1)$ be a bounded lattice, F be a nullnorm with zero element a on L and $a \in L \setminus \{0, 1\}$. Then, S^* and T^* are divisible if and only if $\preceq_F = \leq$.*

Proof. Let $x \preceq_F y$ for any $x, y \in L$. So, we have that $x \leq y$. Conversely let $x \leq y$.
• Firstly let $x, y \in [0, a]$. Since S^* is divisible t-conorm, it is clear that $\preceq_{S^*} = \leq$. So, we have that $x \preceq_{S^*} y$. Then, there exists an element $k \in [0, a]$ such that $S^*(k, x) = y$. Thus, it is obtained that

$$F(k, x) = F\mid_{[0,a]^2}(k, x) = S^*(k, x) = y.$$

This means that $x \preceq_F y$.
• Let $x, y \in [a, 1]$. Similarly, since T^* is divisible t-norm, it is clear that $\preceq_{T^*} = \leq$. So, we have that $x \preceq_{T^*} y$. Then there exists an element $\ell \in [a, 1]$ such that $T^*(y, \ell) = x$. Thus we have that

$$F(\ell, y) = F\mid_{[a,1]^2}(\ell, y) = T^*(\ell, y) = x.$$

So, it is obtained that $x \preceq_F y$.
• Finally let $x, y \notin [0, a]$ and $x, y \notin [a, 1]$. By the definition of the $\preceq_F$, we have that $x \preceq_F y$. Thus, if S^* and T^* are divisible, we have that $\preceq_F = \leq$.

Conversely suppose that $\preceq_F = \leq$. Let $x \leq y$ for $x, y \in [0, a]$. Then we have that $x \preceq_F y$. Since $x, y \in [0, a]$, we have that $x \preceq_{S^*} y$. Then there exists an element $k \in [0, a]$ such that $S^*(x, k) = y$. This show that S^* is divisible. Similarly it can be shown that T^* is divisible. □

Corollary 3.13. *Let $F : [0,1]^2 \to [0,1]$ be a nullnorm with zero element $a \in (0,1)$. Then, S^* and T^* are continuous if and only if $\preceq_F = \leq$ by Proposition 2.6. Moreover, if F is an idempotent nullnorm on a bounded lattice L, then we have that $S^* = S_M$ and $T^* = T_M$. So, we have unique idempotent nullnorm with zero element a as follows*

$$F(x,y) = \begin{cases} max(x,y) & , x,y \in [0,a] \\ min(x,y) & , x,y \in [a,1] \\ a & , otherwise \end{cases}$$

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Thus, the order $\preceq_F$ coincides with the order $\leq$.

Proposition 3.14. *Let L be a chain, F be a nullnorm with zero element $a \in L \setminus \{0,1\}$ and H_F be the set of all idempotent elements of F. Then, $(H_F, \preceq_F)$ is a chain.*

Proof. Let $x, y \in H_F$. Since L is a chain, it must be $x \leq y$ or $y \leq x$. Let $x \leq y$.

Let $x, y \in [0, a]$. Since $0 \leq x \leq y$, we have that

$$y = F(0, y) \leq F(x, y) \leq F(y, y) = y$$

by the monotonicity of F.
So, $F(x, y) = y$ and $x \preceq_F y$.

Let $x, y \in [a, 1]$. Since $x \leq y \leq 1$, we have that

$$x = F(x, x) \leq F(x, y) \leq F(1, x) = x$$

by the monotonicity of F.
So, $F(x, y) = x$ and $x \preceq_F y$.

Let $x, y \notin [0, a]$ and $x, y \notin [a, 1]$. Since $x \leq y$, it must be $x \leq a \leq y$. So, we have that $x \preceq_F y$ by the definition of $\preceq_F$. If $y \leq x$, similarly it can be shown that $y \preceq_F x$. So, $(H_F, \preceq_F)$ is a chain. □

4. Conclusions

We have discussed an order induced by nullnorms on a bounded lattice L. So, we have extended the S-partial order (T-partial order) to a more general form. We have determined the relationship between the order induced by a nullnorm and the order on the lattice. Moreover, we have given a necessary and sufficient condition making a bounded lattice L also a lattice with respect to the order $\preceq_F$.

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