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ALGEBRAIC EQUATIONS WITH LINEAR SHIFT OPERATORS ON SEQUENCES

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Dedicated to the memory of the great Teacher, Nicolae Popescu.

ABSTRACT. In this note we recall (see also [8]) the structure of all recurrent sequences which satisfy a fixed recurrence relation, with entries in a perfect field. As a consequence of these considerations we give a reasonable proof for the known result that the Hadamard product of two recurrent sequences is also a recurrent sequence.

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Introduction

Let K be a perfect field and let $\overline{K}$ be a fixed algebraic closure of K. Let S be the K-vector space of all sequences $f : \mathbb{N} \to K$ with entries in K and let $\overline{S} = S \otimes_K \overline{K}$ be the $\overline{K}$-vector space of all sequences $g : \mathbb{N} \to \overline{K}$ with entries in $\overline{K}$. Let $T : S \to S$, $T(f)(n) = f(n+1)$, $n \in \mathbb{N}$ be the usual simple shift operator on S. We also denote by T the extension of the simple shift operator to $\overline{S}$. For any $a_1, a_2, ..., a_k$, k elements of K, the operator

$$L = T^k + a_1 T^{k-1} + ... + a_k I,$$

where I is the identity operator on S, is called a k-order linear shift operator on S. We denote by $\overline{L}$ its canonical extension to $\overline{S}$. A recurrent sequence f of S is an element in $\ker L$ for a linear shift operator L. In Theorem 1.5 and Remark 1.10 we recall (see also [8]) the structure of $\ker L$ and of $\ker \overline{L}$ as a vector subspaces of S and of $\overline{S}$ respectively. In Propositions 1.7, 1.8, 1.9 we also recall (see also [8]) the structure of all solutions of the inhomogenous equation $L(f) = g$, where $f, g \in \overline{S}$. For a fixed recurrent sequence $f \in S$, a minimal linear shift operator of f is a linear shift operator L_0 with a minimal order k_0 such that $f \in \ker L_0$. In Proposition 2.2 we prove that if L is any other linear shift operator with $f \in \ker L$, then L is a multiple (relative to the extension by linearity of the multiplication $T^2 = T \circ T$) of L_0 and that this minimal linear shift operator L_0 is unique (see also [7]).

In Theorem 2.5 we prove that (with respect to the above multiplication "$\circ$") if $L = L_1 \circ L_2 \circ ... \circ L_h$ is a factorization of L into linear shift operators (over K) of orders greater or equal to 1, then $\ker L = \sum_{i=1}^{h} \ker L_i$ and this sum is a direct one.

In Theorem 2.7 we give the structure of $\ker L$ in language of the kernels of the irreducible factors of L in the factorial ring $K[T]$ (see also [7]).

In Section 3 we give a criterion to say when an element from $\ker \overline{L}$ is an element in $\ker L$ (Theorem 3.1). As a consequence of this criterion we prove that the Hadamard product between two recurrent sequences is again a recurrent sequences (Corollary 3.3). This result was proved in many other papers (see [1]). However, our methods used to study some arithmetical properties of the recurrent sequences over different extensions of fields can be used for other purposes.

During the study which follows, we used some ideas of the following basic works: [1] and [3]-[10].

1. Notation, definitions and basic results

In this section we rewrite some basic facts from [8] in our context with slight modifications. Sometimes, when the proofs of our results are relevant for what follows, we give them. Usually, for the proofs of the known results we send the reader to [8] for instance.

Let K be a commutative field (finite or not) and let $\overline{K}$ be a fixed algebraic closure of it. Let S be the set of all sequences $\{a_n\}_{n\in\mathbb{N}}$ with entries a_n, $n \in \mathbb{N}$, in K, i.e. the set of all functions $f : \mathbb{N} \to K$. When we speak of $f \in S$, where $f : \mathbb{N} \to K$, we mean that the sequence $\{f(n)\}_{n\in\mathbb{N}} \in S$. It is clear how S, become (infinite dimensional) vector space over the field K. In fact all operations are componentwise operations.

The linear operator $T : S \to S$, $T(f) = g$, where $g(n) = f(n + 1)$ is called *the simple shift operator of* S. If we write $\{a_n\}$ we usually mean $\{a_n\}_{n\in\mathbb{N}}$. We denote $T^k = \underbrace{T \circ T \circ ... \circ T}_{k-times}$, so $T^k(f) = g$, where $g(n) = f(n + k)$ for $n = 0, 1, ...$.

It is easy to see that $KerT^k = \{(x_n)_n : x_k = x_{k+1} = ... = 0\}$, i.e. $\dim_K KerT^k = k$. Let now $a_1, a_2, ..., a_k$ be k fixed elements in K and let us denote by $I : S \to S$ the identity operator on S: $I(f) = f$. The linear operator

$$L = T^k + a_1T^{k-1} + ... + a_{k-1}T + a_kI \tag{1.1}$$

defined on S is called a *k-order linear shift operator on* S. The equation

$$L(f) = g \tag{1.2}$$

is called a *k-order (linear) algebraic equation with shift operators.* If $g = 0$, the equation (1.2) is said to be *homogeneous*; otherwise it is called an *inhomogeneous* equation with shift operators.

Let us denote by $Sol = \ker L$, the vector subspace of S consisting of all solutions $f : \mathbb{N} \to K$ of the equation $L(f) = 0$. If $\{x_n\}_{n\in\mathbb{N}} \in Sol$, then

$$x_{n+k} = -a_1x_{n+k-1} - a_2x_{n+k-2} + ... + a_{k-1}x_{n+1} + a_kx_n, \tag{1.3}$$

for any $n = 0, 1, ...$. Such a sequence is called a *recurrent sequence over* K and k is called a *period of* $\{x_n\}_{n\in\mathbb{N}}$. It is clear that a recurrent sequence of period k is completely determined by the first k terms $x_0, ..., x_{k-1}$ of it. The problem is to find the general term x_n of such a sequence and the structure of all of them. We remark that if $a_k \neq 0$, there exists a unique recurrent sequence $\{x_n\}_{n\in\mathbb{N}}$ in S which satisfies the relation

$$x_{n+k} = -a_1x_{n+k-1} - a_2x_{n+k-2} + ... + a_{k-1}x_{n+1} + a_kx_n$$

for any $n \in \mathbb{N}$ and with $x_0, ..., x_{k-1}$ given in K.

We can assume that $a_k \neq 0$, otherwise, denoting $y_n = x_{n+1}$, $n = 0, 1, ...$, we see that the recurrence relation (1.3) becomes:

$$y_{n+k-1} = -a_1y_{n+k-2} - a_2y_{n+k-3} + ... + a_{k-1}y_n, \tag{1.4}$$

i.e. the sequence $\{y_n\}_{n\in\mathbb{N}}$ belongs to the kernel of the shift operator

$$L_1 = T^{k-1} + a_1T^{k-2} + ... + a_{k-1}I.$$

If $a_{k-1} = 0$, we go on to diminish the period by substituting the sequence $\{y_n\}_{n\in\mathbb{N}}$ with $\{z_n\}_{n\in\mathbb{N}}$, where $z_n = y_{n+1}$, etc. If all $a_1, ..., a_k$ are zero, the initial sequence $\{x_n\}_{n\in\mathbb{N}}$ is a constant sequence, a trivial case in our study. If $k = 3$, $a_1 = 1$ and $a_2 = 0$, then, in (1.4) $x_{n+3} = x_{n+2} + x_{n+1}$ and the sequence $y_n = x_{n+1}$ is a Fibonacci type sequence ([6]). This means that the first three terms x_0, x_1, x_2 are "free" and $x_3 = x_2 + x_1$, $x_4 = x_3 + x_2$, etc., i.e. from the second rank on, the sequence is a Fibonacci sequence.

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The term x_0 is free of any recurrence relation, i.e. it is not involved in such a relation. This is an example of a *mixed recurrence sequence*, i.e. of a sequence of the form:

$$(x_0, x_1, ..., x_{l-1}, 0, 0, ...) + (0, 0, ..., 0, y_l, y_{l+1}, ...), \tag{1.5}$$

where $x_0, x_1, ..., x_{l-1}$ are arbitrary elements in K and

$$(0, 0, ..., 0, y_l, y_{l+1}, ...)$$

is a recurrent sequence of period, say $k > l$, which is in the kernel of a shift operator of the form:

$$M = (T^{k-l} + a_1 T^{k-l-1} + ... + a_{k-l} I) \circ T^l, \ a_{k-l} \neq 0.$$

The sequence $z_n = y_{l+n}$, $n = 0, 1, ...,$ is a recurrent sequence which is in the kernel of the shift operator $T^{k-l} + a_1 T^{k-l-1} + ... + a_{k-l} I$. This is why the study of the kernel of a shift operator can be reduced to the case when $a_k \neq 0$.

We associate with the equation $L(f) = 0$ a polynomial equation in a variable r :

$$P(r) = r^k + a_1 r^{k-1} + ... + a_{k-1} r + a_k = 0, \tag{1.6}$$

where $P(r) \in K[r]$ is a polynomial of degree k with coefficients in the initial field K and $a_k \neq 0$, i.e. $r = 0$ is not a root for $P(r)$. This polynomial is called the *characteristic polynomial of the operator* L. The polynomial equation $P(r) = 0$ is called the *characteristic equation of* L. If $r_1 \neq 0$ is a solution (in $\overline{K}$) of the characteristic equation, then the sequence (in fact, a geometrical progression of ratio r_1) $\left\{x_n^{[1]}\right\}_{n \in \mathbb{N}}$, where $x_n^{[1]} = r_1^n$, is a solution of the shift operator equation $L(f) = 0$. If the characteristic polynomial is irreducible over K, then all its solutions are not zero, except the trivial case $P(r) = r$, which is not considered here.

Proposition 1.1. *(see also* [8]*, 2.3) With the above notation and definitions, if $r_1, r_2, ..., r_t$ are distinct roots (in $\overline{K}$) of the characteristic equation $P(r) = 0$, then their corresponding sequences $\left\{x_n^{[j]}\right\}_{n \in \mathbb{N}}$, $j = 1, 2, ..., t$, defined above, are linear independent elements in $\overline{S}$, over $\overline{K}$. Here $\overline{S} = S \otimes_K \overline{K}$, i.e. the $\overline{K}$-vector space of all sequences $\{x_n\}_{n \in \mathbb{N}}$, where $x_n \in \overline{K}$ for all $n \in \mathbb{N}$.*

Lemma 1.2. *(see also* [8]*, 2.3) Let $P_1(x), ..., P_l(x) \in \overline{K}[x]$ be $l \geq 2$ nonzero polynomials with coefficients in $\overline{K}$ and let $s_1, ..., s_l$, $l \geq 2$, be l distinct elements in $\overline{K}$. Let $\left\{x_n^{[j]} = s_j^n\right\}_{n \in \mathbb{N}}$, $j = 1, ..., l$ be the corresponding geometrical progression sequences defined by $s_1, ..., s_l$. Then the sequences $\left\{y_n^{[j]} = P_j(n) x_n^{[j]}\right\}_{n \in \mathbb{N}}$, $j = 1, ..., l$ are linear independent elements in the vector space $\overline{S} = S \otimes_K \overline{K}$.*

Proposition 1.3. *(see also* [8]*, 2.3) Let $r_1 \neq 0$ (for instance if $P(r)$ is irreducible) be a root of algebraic multiplicity $m_1 > 1$ of the characteristic polynomial $P(r)$ from (1.6) and let $P_1(x) \in \overline{K}[x]$ be a polynomial of degree $< m_1$. Then the sequence $\{x_n = P_1(n) r_1^n\}_{n \in \mathbb{N}}$ is in $\ker L$ and the sequences $\left\{x_n^{[j]} = n^j r_1^n\right\}_n$, $j = 0, 1, ..., m_1 - 1$ are linear independent elements in $\ker L$.*

Let

$$P(r) = (r - r_1)^{m_1} (r - r_2)^{m_2} ... (r - r_l)^{m_l}$$

be the factorization of the characteristic polynomial $P(r)$ (see (1.6)) into linear factors, where $r_1, ..., r_l$ are the distinct roots of $P(r) = 0$ in $\overline{K}$ and $m_1 + ... + m_l = k$ is the degree of $P(r)$.

Let $\mathcal{B}_i = \left\{(r_i^n)_n, (n r_i^n)_n, ..., (n^{m_i - 1} r_i^n)_n\right\}$, $i = 1, 2, ..., l$ be l subsets of elements in $\ker L$ (see Proposition 1.3).

Proposition 1.4. *(see also* [8]*) With the above notation and notions, the set $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup ... \cup \mathcal{B}_l$ of elements in $\ker L$ is linear independent over $\overline{K}$.*

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Theorem 1.5. *(see also [8]) The set $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup ... \cup \mathcal{B}_l$ is a basis for the vector subspace $\ker L$. In particular, $\dim \ker L = k$, the degree of the characteristic polynomial $P(r)$. Moreover, any element $\{x_n\}_n$ of $\ker L$ is a linear combination of the form:*

$$\{x_n\}_n = \left\{A_0^{[n]}\right\}_n x_0 + \left\{A_1^{[n]}\right\}_n x_1 + ... + \left\{A_{k-1}^{[n]}\right\}_n x_{k-1}, \tag{1.7}$$

where $\left\{A_0^{[n]}\right\}_n, \left\{A_1^{[n]}\right\}_n, ..., \left\{A_{k-1}^{[n]}\right\}_n$ is a canonical basis (over K) in $\ker L$ which corresponds to the particular values

$$(1,0,0,...,0),(0,1,0,...,0),...,(0,0,0,...,0,1)$$

respectively, given to $(x_0, x_1, x_2, ..., x_{k-1})$. In particular, the sequence $\{x_n\}_n$ is obviously completely determined by the first k values $x_0, x_1, x_2, ..., x_{k-1}$ of it.

Proof. Since $\mathcal{B}$ is linear independent (see Proposition 1.4) and since it has k elements, it is sufficient to prove that $\dim \ker L \geq k$. We shall construct k generators in $\ker L$ and the statement of the theorem will be completely proved at that moment. For this, let $\{x_n\}_n$ be a sequence in $\ker L$. Thus,

$$x_{n+k} = -a_1 x_{n+k-1} - a_2 x_{n+k-2} - ... - a_{k-1} x_{n+1} - a_k x_n \tag{1.8}$$

for any $n = 0, 1, ...$ (see (1.1)). For $n = 0$, we get

$$x_k = -a_1 x_{k-1} - a_2 x_{k-2} - ... - a_{k-1} x_1 - a_k x_0. \tag{1.9}$$

Let us denote: $A_{k-1}^{[k]} = -a_1$, $A_{k-2}^{[k]} = -a_2, .., A_0^{[k]} = -a_k$. Thus, (1.9) can be rewritten as:

$$x_k = A_0^{[k]} x_0 + A_1^{[k]} x_1 + ... + A_{k-2}^{[k]} x_{k-2} + A_{k-1}^{[k]} x_{k-1}. \tag{1.10}$$

For $n = 1$ in (1.8) we get:

$$\begin{aligned} x_{k+1} &= -a_1 x_k - a_2 x_{k-1} - ... - a_{k-1} x_2 - a_k x_1 = \\ &= -a_1(-a_1 x_{k-1} - a_2 x_{k-2} - ... - a_{k-1} x_1 - a_k x_0) - \\ &\quad -a_2 x_{k-1} - ... - a_{k-1} x_2 - a_k x_1 = \\ &= A_0^{[k+1]} x_0 + A_1^{[k+1]} x_1 + ... + A_{k-2}^{[k+1]} x_{k-2} + A_{k-1}^{[k+1]} x_{k-1}, \end{aligned}$$

where $A_0^{[k+1]} = a_1 a_k$, $A_1^{[k+1]} = a_1 a_{k-1} - a_k$, $A_2^{[k+1]} = a_1 a_{k-2} - a_{k-1}, ...$, $A_{k-2}^{[k+1]} = a_1 a_2 - a_3$, $A_{k-1}^{[k+1]} = a_1^2 - a_2$. Assume that we just constructed $A_0^{[j]}$, $A_1^{[j]}, ..., A_{k-1}^{[j]}$, $j = k, 1, ..., k+n-1$. Let us construct $A_0^{[k+n]}$, $A_1^{[k+n]}, ..., A_{k-1}^{[k+n]}$. Since

$$x_{k+n-1} = A_0^{[k+n-1]} x_0 + A_1^{[k+n-1]} x_1 + ... + A_{k-2}^{[k+n-1]} x_{k-2} + A_{k-1}^{[k+n-1]} x_{k-1},$$

where $A_0^{[k+n-1]}$, $A_1^{[k+n-1]}, ..., A_{k-1}^{[k+n-1]}$ are polynomials in $a_1, a_2, ..., a_k$, and since

$$\begin{aligned} x_{k+n} &= -a_1 x_{k+n-1} - a_2 x_{k+n-2} - ... - a_{k-1} x_{n+1} - a_k x_n = \\ &= -a_1 \left[A_0^{[k+n-1]} x_0 + A_1^{[k+n-1]} x_1 + ... + A_{k-2}^{[k+n-1]} x_{k-2} + A_{k-1}^{[k+n-1]} x_{k-1}\right] - \\ &\quad -a_2 \left[A_0^{[k+n-2]} x_0 + A_1^{[k+n-2]} x_1 + ... + A_{k-2}^{[k+n-2]} x_{k-2} + A_{k-1}^{[k+n-2]} x_{k-1}\right] - \\ &\quad -... - a_{k-1} \left[A_0^{[n+1]} x_0 + A_1^{[n+1]} x_1 + ... + A_{k-2}^{[n+1]} x_{k-2} + A_{k-1}^{[n+1]} x_{k-1}\right] - \\ &\quad -a_k \left[A_0^{[n]} x_0 + A_1^{[n]} x_1 + ... + A_{k-2}^{[n]} x_{k-2} + A_{k-1}^{[n]} x_{k-1}\right] = \\ &= A_0^{[k+n]} x_0 + A_1^{[k+n]} x_1 + ... + A_{k-2}^{[k+n]} x_{k-2} + A_{k-1}^{[k+n]} x_{k-1}, \end{aligned}$$

where

$$A_t^{[k+n]} = -a_1 A_t^{[k+n-1]} - a_2 A_t^{[k+n-2]} - ... - a_k A_t^{[k+n-k]}, \ t = 0, 1, ..., k-1. \tag{1.11}$$

We see from this last formula that the sequences $\left\{A_t^{[n]}\right\}_{n\in\mathbb{N}}$, where $A_t^{[j]}=0$, $j=0,1,...,k-1$, except $j=t$, when $A_t^{[t]}=1$, belong to $\ker L$ and they make up a generator system for $\ker L$ over K. Since $\dim\ker L\geq k$, we obtain that $\dim\ker L=k$. From

$$x_{k+n}=A_0^{[k+n]}x_0+A_1^{[k+n]}x_1+...+A_{k-2}^{[k+n]}x_{k-2}+A_{k-1}^{[k+n]}x_{k-1} \tag{1.12}$$

we also see that $\left\{A_0^{[n]}\right\}_n,...,\left\{A_{k-1}^{[n]}\right\}_n$ is a canonical basis of $\ker L$. It is obtained by making $x_0=1$, $x_1=0,...,x_{k-1}=0$; $x_0=0,x_1=1$, $x_2=0,...,x_{k-1}=0$, etc. in (1.12). In fact, we do not need to prove that $\left\{A_0^{[n]}\right\}_n,...,\left\{A_{k-1}^{[n]}\right\}_n$ are elements in $\ker L$. It is sufficient to see that $\ker L$ is a subspace in the vector subspace (of S_+) generated by $\left\{A_0^{[n]}\right\}_n,...,\left\{A_{k-1}^{[n]}\right\}_n$. Since the dimension of $\ker L$ is at least k and from this last remark it is at most k, we easily derive that $\dim\ker L=k$. □

Remark 1.6. If we look at the recurrent sequence $\{x_n\}_{n\in\mathbb{N}}\in\ker L$ as a sequence in $\overline{S}=S\otimes_K\overline{K}$, it can be uniquely described as a sum of the form: $\{x_n\}_{n\in\mathbb{N}}=\sum_{i=1}^{l}\{P_i(n)r_i^n\}_{n\in\mathbb{N}}$, where $r_1,...,r_l$ are all the distinct roots of the characteristic polynomial $P(r)$ of the shift operator $L=T^k+a_1T^{k-1}+...+a_{k-1}T+a_kI$ and $P_i(r)\in K[r]$ are polynomials of degree at most m_i-1, m_i being the algebraic multiplicity of the root r_i, $i=1,2,...,l$. If we look at the recurrent sequence $\{x_n\}_{n\in\mathbb{N}}\in\ker L$ as a sequence in S, i.e. with entries in K, it can be uniquely described as a sum of the form: $\{x_n\}_{n\in\mathbb{N}}=\left\{A_0^{[n]}\right\}_n x_0+\left\{A_1^{[n]}\right\}_n x_1+...+\left\{A_{k-1}^{[n]}\right\}_n x_{k-1}$ (see 1.7) where, this time, $\left\{A_0^{[n]}\right\}_n,...,\left\{A_{k-1}^{[n]}\right\}_n$ is a basis of recurrent sequences in $\ker L$ with entries in K itself. The problem is that we cannot easily describe the general term of each of these sequences $\left\{A_j^{[n]}\right\}_n$, $j=0,1,...,l$. For instance, for the Fibonacci sequence

$$\{x_0,x_1,x_1+x_0,...,x_n=x_{n-1}+x_{n-2},...\}, \tag{1.13}$$

$x_0,x_1\in K$, the periodic sequences $\left\{A_0^{[n]}\right\}_n$, $\left\{A_1^{[n]}\right\}_n$ are also Fibonacci sequences obtained from the general formula (1.13) by making $x_0=1$, $x_1=0$ and $x_0=0$, $x_1=1$, respectively. Thus,

$$\left\{A_0^{[n]}\right\}_n=\{1,0,1,1,2,3,5,8,...\},$$

$$\left\{A_1^{[n]}\right\}_n=\{0,1,1,2,3,5,8,...\}.$$

In the next section we shall see how to construct new other bases for $\ker L$, starting from its basis over $\overline{K}$, more exactly over $K[r_1,...,r_l]$, the subfield of $\overline{K}$ generated by all roots of the characteristic polynomial $P(r)$.

A *Cauchy problem* for the linear algebraic shift operator homogeneous equation $L(f)=0$ (here L is a fixed proper ($a_k\neq 0$) linear shift operator) is that of finding a solution $f\in S$ of it, if we know the "initial" k values of $f: f(0)=y_0,...,f(k-1)=y_{k-1}$.

Proposition 1.7. *The above Cauchy problem for the equation $L(f)=0$ with the initial conditions $f(0)=y_0,...,f(k-1)=y_{k-1}$ has a unique solution over K, i.e. in S, namely*

$$f(n)=A_0^{[n]}y_0+A_1^{[n]}y_1+...+A_{k-1}^{[n]}y_{k-1},\ n=0,1,...,$$

where $\left\{A_0^{[n]}\right\}_n,...,\left\{A_{k-1}^{[n]}\right\}_n$ is the canonical basis constructed during the proof of Theorem 1.5 and during the discussion in Remark 1.6.

Proof. The proof is obvious in view of the proof of Theorem 1.5. □

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Proposition 1.8. *The inhomogeneous equation $L(f) = g$, with f unknown in S, g known in S and with the initial conditions*

$$f(0) = y_0^*, ..., f(k-1) = y_{k-1}^*,$$

has a unique solution in S.

Proof. Let $g = \{y_n\}_{n\in\mathbb{N}}$ be a given sequence in S, i.e. with entries in K. We define $f = \{x_n\}_{n\in\mathbb{N}}$ in the following way:

$$x_0 = y_0^*, ..., x_{k-1} = y_{k-1}^*, \tag{1.14}$$

But $L(f) = g$ is equivalent to

$$x_{n+k} + a_1 x_{n+k-1} + a_2 x_{n+k-2} + ... + a_{k-1} x_{n+1} + a_k x_n = y_n,$$

for any $n = 0, 1, ...$. So

$$x_{n+k} = y_n - a_1 x_{n+k-1} - a_2 x_{n+k-2} - ... - a_{k-1} x_{n+1} - a_k x_n,$$

for any $n = 0, 1, ...$. The solution is obviously unique. □

A particular solution for the above equation $L(f) = g$ is obtained by taking $y_0^* = 0, ..., y_{k-1}^* = 0$ in (1.14). It is of the form:

$$f_p = \{x_n^{[p]}\}_{n\in\mathbb{N}} = \left\{ \underbrace{0, 0, ..., 0}_{k-times}, y_0, y_1 - a_1 y_0, y_2 - a_1 y_1 + a_1^2 y_0 - a_2 y_0, ... \right\}, \tag{1.15}$$

We can use this last particular solution f_p to find the structure of all solutions of the equation $L(f) = g$.

Proposition 1.9. *Let $\left\{A_0^{[n]}\right\}_n, ..., \left\{A_{k-1}^{[n]}\right\}_n$ be the canonical basis of $\ker L$ constructed during the proof of Theorem 1.5 and let f_p be a particular solution of the inhomogeneous equation $L(f) = g$. Then any solution of this last equation in S, i.e. over K, is of the form:*

$$f = f_p + C_0 \left\{A_0^{[n]}\right\}_n + C_1 \left\{A_1^{[n]}\right\}_n + ... + C_{k-1} \left\{A_{k-1}^{[n]}\right\}_n, \tag{1.16}$$

where $C_0, ..., C_{k-1}$ are arbitrary elements in K. Moreover, with the notation and definitions used in Remark 1.6, any solution of the equation $L(f) = g$, where g is a sequence over $\overline{K}$ (or over $K[r_1, ..., r_l]$), is of the form:

$$f = f_p + \sum_{i=1}^{l} \{P_i(n) r_i^n\}_{n\in\mathbb{N}}, \tag{1.17}$$

where $P_i(x) \in \overline{K}$ (or $K[r_1, ..., r_l]$) is an arbitrary polynomial of degree $m_i - 1$, if m_i is the algebraic multiplicity of the root r_i of the characteristic polynomial $P(r)$, for any $i = 1, 2, ..., l$.

Proof. Let f_1 be a solution of $L(f) = g$. Then $L(f_1) = g$ and $L(f_p) = g$. Subtracting the last equality from the first one, we get: $L(f_1 - f) = 0$, i.e. $f_1 - f \in \ker L$, so it is of the form $C_0 \left\{A_0^{[n]}\right\}_n + C_1 \left\{A_1^{[n]}\right\}_n + ... + C_{k-1} \left\{A_{k-1}^{[n]}\right\}_n$ or, if we work over $\overline{K}$, of the form $\sum_{i=1}^{l} \{P_i(n) r_i^n\}_{n\in\mathbb{N}}$, etc. □

Remark 1.10. All the above theory can be extended from sequences of S to sequences $f \in \widetilde{S}$, i.e. to complete sequences $f : \mathbb{Z} \to K$. Let L be a shift operator as in formula (1.1). Then $f = \{x_n\}_{n\in\mathbb{Z}}$ belongs to $\ker L$ if and only if

$$x_{n+k} = -a_1 x_{n+k-1} - a_2 x_{n+k-2} - ... - a_{k-1} x_{n+1} - a_k x_n, \ a_k \neq 0,$$

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for any $n \in \mathbb{Z}$. This recurrence relation completely defines the whole sequence $\{x_n\}_{n\in\mathbb{Z}}$ if we have the "positive part" $f_+ = \{x_n\}_{n\in\mathbb{N}}$ of it. Indeed,

$$x_{-1} = -\frac{1}{a_k}\left[x_{k-1} + a_1x_{k-2} + a_2x_{k-3} - ... - a_{k-1}x_0\right],$$

$$x_{-2} = -\frac{1}{a_k}\left[x_{k-2} + a_1x_{k-3} + a_2x_{k-4} - ... - a_{k-1}x_{-1}\right],$$

and so on. This is why $\ker L$ for $L : \widetilde{S} \to \widetilde{S}$ has the same bases as those constructed in Theorem 1.5, $\dim\ker L = k$, etc. The only extension is made relative to n, which have to run all over $\mathbb{Z}$ this time. For instance, if $K = \mathbb{Q}$ and $L = T^2 - 5T + 6I = (T - 2I) \circ (T - 3I)$, $\ker L = \{C_1 2^n + C_2 3^n\}_{n\in\mathbb{Z}}$ and its basis is the set of the two infinite geometrical progression:

$$..., \frac{1}{2^3}, \frac{1}{2^2}, \frac{1}{2}, 1, 2, 2^2, 2^3, ...$$

of ratio 2, and

$$..., \frac{1}{3^3}, \frac{1}{3^2}, \frac{1}{3}, 1, 3, 3^2, 3^3, ...$$

of ratio 3.

2. Recurrent sequences and algebraic field extensions

We assume that our field K is a perfect field, i.e. any algebraic extension of it is separable (see [2]). This is equivalent to say that for any $\alpha \in \overline{K}$, the polynomial $f_\alpha(x) \in K[x]$ of minimal degree such that α is a root of it has only simple roots, i.e. $f'_\alpha(\alpha) \neq 0$ ([2]).

Definition 2.1. We say that a sequence $f : \mathbb{Z} \to K$, defined on the whole $\mathbb{Z}$ with values in the field K, is a recurrent sequence of a period k if there exists a linear shift operator

$$L = T^k + a_1T^{k-1} + ... + a_{k-1}T + a_kI, a_j \in K, j = 1, 2, ..., k, a_k \neq 0,$$

(see also 1.1) with $f \in \ker L$. If $f(n) = x_n$, $n \in \mathbb{Z}$, i.e. if $f = \{x_n\}_{n\in\mathbb{Z}}$, f is a recurrent sequence if and only if

$$x_{n+k} = -a_1x_{n+k-1} - a_2x_{n+k-2} - ... - a_{k-1}x_{n+1} - a_kx_n,\ a_k \neq 0,\ n \in \mathbb{Z}. \tag{2.1}$$

If L_0 has the least possible order with $f \in \ker L_0$ and if k_0 is this order, we call this k_0 the degree (or the period) of f and L_0 is called the minimal (shift) operator of f (see bellow the uniqueness of L_0).

The mapping

$$L = T^k + a_1T^{k-1} + ... + a_{k-1}T + a_kI \to P(r) = r^k + a_1r^{k-1} + ... + a_{k-1}r + a_k \in K[r] \tag{2.2}$$

defines a ring isomorphism between the commutative ring of all linear shift operators (with composition "$\circ$" for multiplication) and the polynomial ring $K[r]$ in the variable r. Here by the composition between two linear shift operators $L = T^k + a_1T^{k-1} + ... + a_kI$ and $M = T^l + b_1T^{l-1} + ... + b_lI$ we mean a "polynomial multiplication" in $K[T]$, i.e. $T^i \circ T^j = T^{i+j}$, etc. We say that L is *irreducible* if its characteristic polynomial is irreducible in $K[r]$. Thus L is irreducible if and only if L cannot be written as $L = L_1 \circ L_2$, where the orders k_1, k_2 of L_1 and L_2 respectively are greater then zero. For instance, if $K = \mathbb{Q}$, the rational number field, then $L = T^2 - 2I$ is irreducible (over $\mathbb{Q}$) but it is not irreducible over $K = \mathbb{Q}[\sqrt{2}]$: $L = (T - \sqrt{2}I) \circ (T + \sqrt{2}I)$ over $\mathbb{Q}[\sqrt{2}]$. Here $T : \widetilde{S} \to \widetilde{S}$ is the usual simple shift operator: $T(f) = g$, where $g(n) = f(n + 1)$. We also remark that T is invertible: $T^{-1}(f)(n) = f(n - 1)$. It is not invertible if we restrict it to S.

If $L = L_1 \circ L_2$, we say that L_1 and L_2 are factors of L or that L is divisible by L_1 and L_2. We also say that L is a multiple of L_1 or of L_2.

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Proposition 2.2. *Let $\{x_n\}_{n\in\mathbb{Z}}$ be a recurrent sequence of $\widetilde{S}$ (over K) and let L_0 be a minimal shift operator of $\{x_n\}_{n\in\mathbb{Z}}$. Then any other linear shift operator L such that $\{x_n\}_{n\in\mathbb{Z}} \in \ker L$ is a multiple of L_0, i.e. $L = L_0 \circ L_1$. In particular L_0 is unique with the property that it is minimal for $\{x_n\}_{n\in\mathbb{Z}}$.*

Proof. Let $P_0(r)$, $P(r)$ be the characteristic polynomials of L_0 and L respectively. The division algorithm of Euclid says that there exist two monic polynomials $Q(r)$ and $R(r)$ with

$$P(r) = P_0(r)Q(r) + aR(r) \tag{2.3}$$

$\deg R(r) < \deg P_0(r)$ and $a \in K$. Because of the above ring isomorphism, the formula (2.3) can be rewritten in language of shift operators:

$$L = L_0 \circ L_1 + aL_2, \tag{2.4}$$

where L_1 and L_2 are the unique shift operators which have as characteristic polynomials $Q(r)$ and $R(r)$ respectively. Since $\{x_n\}_{n\in\mathbb{Z}} \in \ker L \cap \ker L_0$, from (2.4) we get that $\{x_n\}_{n\in\mathbb{Z}} \in \ker L_2$ if $a \neq 0$. Since L_0 has the least order possible such that $\{x_n\}_{n\in\mathbb{Z}} \in \ker L_0$ and since the order of L_2 is less than the order of L_0, we see that $L_2 = 0$ and so, $L = L_0 \circ L_1$. If $a = 0$, we get the same result. If L_0, M_0 were two minimal shift operators for $\{x_n\}_{n\in\mathbb{Z}}$, we get $L_0 = M_0 \circ Q_0$. Since L_0 and M_0 have the same order k_0, the order of Q_0 is equal to zero, i.e. $Q_0(f) = bf$, where $b \in K$. But the characteristic polynomials of L_0 and M_0 are monic polynomials, thus $b = 1$ and so $Q_0 = I$, the identity operator. Hence $L_0 = M_0$. ☐

Remark 2.3. The minimal shift operator of a recurrent sequence $\{x_n\}_{n\in\mathbb{Z}}$ is not always irreducible. For instance, the minimal shift operator L of the recurrent sequence $f = (...1, 0, 1, 0, 1, ...1, 0, 1, ...)$ (over $\mathbb{Q}$), $x_0 = 1$, $x_1 = 0$, etc., is $T^2 - I$. But this one is not irreducible: $T^2 - I = (T - I) \circ (T + I)$ and, as it is obvious, $f \notin \ker(T - I)$ and $f \notin \ker(T + I)$.

Remark 2.4. The set $\widetilde{S}$ of all sequences $f : \mathbb{Z} \to K$ is a commutative group relative to the usual addition of functions: $(f + g)(n) = f(n) + g(n)$. The subset $\mathcal{A}$ of all recurrent sequences f of S is an additive subgroup of S. Indeed, if $L_f = T^k + a_1T^{k-1} + ... + a_kI$, $L_g = T^l + b_1T^{l-1} + ... + b_lI$, $a_k, b_l \neq 0$, are the minimal shift operators of f and g respectively, then for $L = L_f \circ L_g = L_g \circ L_f$ we obviously have that $f + g \in \ker L$. We shall also see in the next section that the Hadamard product $(fg)(n) = f(n)g(n)$ of two recurrent sequences f and g of $\widetilde{S}$ is again a recurrent sequence in $\widetilde{S}$ (see also [1] for some other cases).

Theorem 2.5. *Let L be a linear shift operator defined on $\widetilde{S}$ and let*

$$L = L_1 \circ L_2 \circ ... \circ L_h$$

be a factorization of L into linear shift operators (over K) of orders greater or equal to 1. Then $\ker L = \sum_{i=1}^{h} \ker L_i$ and this sum is a direct sum.

Proof. Since the composition between linear shift operators is commutative we see that $\ker L \supset \sum_{i=1}^{h} \ker L_i$. Let k be the order of L and k_i, $i = 1, 2, ..., h$ be the order of L_i. The isomorphism (2.2) says that $k = \sum_{i=1}^{h} k_i$ and Theorem 1.5 says that $\dim_K L_i = k_i$, $i = 1, 2, ..., h$. Thus $\ker L = \sum_{i=1}^{h} \ker L_i$ and the sum is direct. ☐

Let L be a k-order linear shift operator, $k > 0$ and let $P(r) \in K[r]$ be its corresponding characteristic polynomial. Let $P(r) = P_1^{m_1}(r) \cdot ... \cdot P_h^{m_h}(r)$ be the factorization of $P(r)$ into distinct monic irreducible polynomial $P_1(r), ..., P_h(r)$ over K. The isomorphism (2.2) says that L has the following unique factorization:

$$L = L_1^{m_1} \circ L_2^{m_2} \circ ... \circ L_h^{m_h}, \tag{2.5}$$

where L_i is an irreducible linear shift operator over K for any $i = 1, 2, ..., h$. Theorem 2.5 says that it is sufficient to construct a "special" basis in $\ker L_i^{m_i}$, $i = 1, 2, ..., h$. The following lemma will reduce such a construction to the case of $m_i = 1$, i.e. to the case of an irreducible linear shift operator.

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Lemma 2.6. *Let L be a linear shift operator of order at least 1 and let $f = \{x_n\}_{n\in\mathbb{Z}}$ be an element in $\ker L$. Then the sequence $\{nx_n\}_{n\in\mathbb{Z}}$ is an element of $\ker L^2$, i.e. $L(\{nx_n\}_{n\in\mathbb{Z}}) \in \ker L$. In general, if $g = \{y_n\}_{n\in\mathbb{Z}} \in \ker L^m$, then $\{ng(n)\}_{n\in\mathbb{Z}} \in \ker L^{m+1}$ for any $m = 1, 2, \ldots$.*

Proof. Let $r_1, ..., r_l$ be the distinct roots (in $\overline{K}$) of the characteristic polynomial $P(r)$ of L and let $m_1, m_2, ..., m_l$ be the algebraic multiplicities of $r_1, r_2, ..., r_l$ respectively. Then $r_1, ..., r_l$ are all the distinct roots of the characteristic polynomial $P^2(r)$ of L^2 with their algebraic multiplicities $2m_1, ..., 2m_l$ respectively. From Theorem 1.5 we see that f is a linear combination over $\overline{K}$ with elements of the form $\{n^j r_i^n\}_n$, $j = 1, 2, ..., m_i - 1$, $i = 1, 2, ..., l$. Thus $\{nx_n\}_n$ is a linear combination of $\{n^{j+1}r_i^n\}_n$, $j = 1, 2, ..., m_i - 1$, $i = 1, 2, ..., l$. Since $1 \le j \le m_i - 1$, we see that $2 \le j + 1 \le m_i \le 2m_i - 1$, so $\left\{n^{j+1}r_i^n\right\}_n \in \ker L^2$ and finally $\{nx_n\}_n \in \ker L^2$. Let us assume now that $g \in \ker L^m$ is a linear combination over $\overline{K}$ of $\{n^j r_i^n\}_n$, $j = 1, 2, ..., mm_i - 1$, $i = 1, 2, ..., l$. Thus, $\{ng(n)\}_n$ is a linear combination of $\{n^{j+1}r_i^n\}_n$, $j = 1, 2, ..., mm_i - 1$, $i = 1, 2, ..., l$. Since

$$2 \le j + 1 \le mm_i \le (m + 1)m_i - 1$$

we see that $\{n^{j+1}r_i^n\}_n \in \ker L^{m+1}$, i.e. $\{ng(n)\}_n \in \ker L^{m+1}$. □

Let $\widetilde{S}$ be the set of all sequences $f = \{x_n\}_{n\in\mathbb{Z}}$ with values in a perfect field K and let $\overline{\widetilde{S}} = \widetilde{S} \otimes_K \overline{K}$ be the set of all sequences $g = \{y_n\}_{n\in\mathbb{Z}}$ with values in $\overline{K}$. Let $L = T^k + a_1 T^{k-1} + ... + a_{k-1}T + a_k I$, $a_k \ne 0$, be an irreducible linear shift operator over K, defined on S, and let m be a natural number greater than zero. We can also view L over $\overline{K}$, i.e. we can also view it as a linear shift operator $\overline{L}$ defined on $\overline{\widetilde{S}}$. It is obvious that $\overline{L}$ is irreducible if and only if L has order 1. Thus, the tower of K-vector subspaces:

$$\ker L \subset \ker L^2 \subset \ker L^3 \subset ... \subset \ker L^m \tag{2.6}$$

can be viewed by tensorization with $\overline{K}$ over K as a new tower of $\overline{K}$-vector subspaces:

$$\underset{\ker L\otimes_k \overline{K}}{\underset{\parallel}{\ker \overline{L}}} \subset \underset{\ker L^2\otimes_k \overline{K}}{\underset{\parallel}{\ker \overline{L}^2}} \subset \underset{\ker L^3\otimes_k \overline{K}}{\underset{\parallel}{\ker \overline{L}^3}} \subset ... \subset \underset{\ker L^m\otimes_k \overline{K}}{\underset{\parallel}{\ker \overline{L}^m}} . \tag{2.7}$$

Let $P(r)$ be the characteristic polynomial of L and let $r_1, r_2, ..., r_l$ be all the roots of it (in $\overline{K}$). Since K is a perfect field, and since $P(r)$ is irreducible, $r_1, r_2, ..., r_l$ are distinct. Let $F = K[r_1, r_2, ..., r_l]$ be the least subfield of $\overline{K}$ which contains all the roots of $P(r)$. The extension of fields F/K is a normal extension, i.e. if $y \in F$, all the other roots of the minimal (irreducible) polynomial $P_y(r) \in K[r]$ of y are also in F. This means that F/K is a Galois extension, i.e. for any $\alpha \in F$ and for any K-embedding σ of F in $\overline{K}$, $\sigma(\alpha) \in F$ (see [2]). Theorem 1.5 says that recurrent sequences (infinite geometrical progressions) $\{r_1^n\}_{n\in\mathbb{Z}}, ..., \{r_l^n\}_{n\in\mathbb{Z}}$ is a basis of $\ker \overline{L}$ over $\overline{K}$. Lemma 2.6 and this last mentioned theorem say that $\{nr_1^n\}_{n\in\mathbb{Z}}, ..., \{nr_l^n\}_{n\in\mathbb{Z}}$ is a basis of $\ker \overline{L}^2 / \ker \overline{L}, ..., \{n^{m-1}r_1^n\}_{n\in\mathbb{Z}}, ..., \{n^{m-1}r_l^n\}_{n\in\mathbb{Z}}$ is a basis of $\ker \overline{L}^m / \ker \overline{L}^{m-1}$. This is equivalent to saying that the set of sequences

$$\{r_1^n\}_{n\in\mathbb{Z}}, ..., \{r_l^n\}_{n\in\mathbb{Z}}, \{nr_1^n\}_{n\in\mathbb{Z}}, ..., \{nr_l^n\}_{n\in\mathbb{Z}}, ... \tag{2.8}$$

$$..., \{n^{m-1}r_1^n\}_{n\in\mathbb{Z}}, ..., \{n^{m-1}r_l^n\}_{n\in\mathbb{Z}} \tag{2.9}$$

is a basis of $\ker \overline{L}^m$. Since $\dim_K \ker L = \dim_{\overline{K}} \ker \overline{L} = l$, first of all we are to search for a basis of $\ker L$ over K with l elements.

For this, since F/K is separable, let z be a primitive element of F/K, i.e. an element $z \in F$ such that $F = K[z]$ (see [2]). We know that $\{1, z, z^2, ..., z^{l-1}\}$ is a basis of the vector space F over the field K (see [2]). Let $G = Gal(F/K)$ be the Galois group of F/K and let $\sigma_1 = \text{identity}, \sigma_2, ..., \sigma_l$ be all the elements of G. We can assume that $r_2 = \sigma_2(r_1), ..., r_l = \sigma_l(r_1)$. So, for any fixed $n \in \mathbb{Z}$ we can write:

$$\begin{cases} r_1^n = C_1^{[n]} + C_2^{[n]} z + ... + C_l^{[n]} z^{l-1} \\ r_2^n = C_1^{[n]} + C_2^{[n]} \sigma_2(z) + ... + C_l^{[n]} \sigma_2(z)^{l-1} \\ \vdots \\ r_l^n = C_1^{[n]} + C_2^{[n]} \sigma_l(z) + ... + C_l^{[n]} \sigma_l(z)^{l-1} \end{cases}, \tag{2.10}$$

where $C_j^{[n]}$, $j = 1, 2, ..., l$ are elements in K. We can view (2.10) as a linear system in the unknowns $C_1^{[n]}, C_2^{[n]}, ..., C_l^{[n]}$. Since its determinant is a Vandermonde determinant with value $\Delta = \prod_{1 \le i < j \le l} [\sigma_j(z) - \sigma_j(z)] \neq 0$, because $z = \sigma_1(z), \sigma_2(z), ..., \sigma_l(z)$ are all the (distinct) roots of the minimal polynomial of z. Since

$$C_1^{[n]} = \frac{\Delta_1^{[n]}}{\Delta}, ..., C_l^{[n]} = \frac{\Delta_l^{[n]}}{\Delta},$$

where

$$\Delta_j^{[n]} = \begin{array}{c} \begin{array}{ccccccc} & & & j-th\,\mathrm{col} & & & \end{array} \\ \begin{vmatrix} 1 & z & \cdots & r_1^n & \cdots & z^{l-2} & z^{l-1} \\ 1 & \sigma_2(z) & \cdots & r_2^n & \cdots & \sigma_2(z)^{l-2} & \sigma_2(z)^{l-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \sigma_{l-1}(z) & \cdots & r_{l-1}^n & \cdots & \sigma_{l-1}(z)^{l-2} & \sigma_{l-1}(z)^{l-1} \\ 1 & \sigma_l(z) & \cdots & r_l^n & \cdots & \sigma_l(z)^{l-2} & \sigma_l(z)^{l-1} \end{vmatrix} \end{array} = \tag{2.11}$$

$$= d_{j1} r_1^n + d_{j2} r_2^n + ... + d_{jl} r_l^n,$$

where $d_{js} \in F$ for any $j, s = 1, 2, ..., l$. Thus $\left\{C_j^{[n]}\right\}_{n \in \mathbb{Z}} \in \ker L$ for any $j = 1, 2, ..., l$. Since $\Delta \neq 0$ and since $\{r_1^n\}_{n \in \mathbb{Z}}, ..., \{r_l^n\}_{n \in \mathbb{Z}}$ is a basis in $\ker \overline{L}$ we see that $\left\{C_1^{[n]}\right\}_{n \in \mathbb{Z}}, \left\{C_2^{[n]}\right\}_{n \in \mathbb{Z}}, ..., \left\{C_l^{[n]}\right\}_{n \in \mathbb{Z}}$ is also a basis in $\ker \overline{L}$. In particular they are linear independent over K. Since $\dim \ker L = l$, they are also a basis in $\ker L$.

Multiplying by n^t, $t = 1, 2, ..., m-1$ the j-th column in (2.11) and using that fact that the set of (2.8) is a basis of $\ker \overline{L}^m$, we see that the set $\left\{nC_2^{[n]}\right\}_{n \in \mathbb{Z}}, ..., \left\{nC_l^{[n]}\right\}_{n \in \mathbb{Z}}$ is a basis of $\ker L^2 / \ker L$, ..., $\left\{n^{m-1} C_1^{[n]}\right\}_{n \in \mathbb{Z}}$, $n^{m-1}\left\{C_2^{[n]}\right\}_{n \in \mathbb{Z}}, ..., n^{m-1}\left\{C_l^{[n]}\right\}_{n \in \mathbb{Z}}$ is a basis of $\ker L^m / \ker L^{m-1}$ (over K). Hence

$$\left\{C_1^{[n]}\right\}_{n \in \mathbb{Z}}, \left\{C_2^{[n]}\right\}_{n \in \mathbb{Z}}, ..., \left\{C_l^{[n]}\right\}_{n \in \mathbb{Z}}, \left\{nC_2^{[n]}\right\}_{n \in \mathbb{Z}}, ..., \left\{nC_l^{[n]}\right\}_{n \in \mathbb{Z}},$$

$$..., \left\{n^{m-1} C_1^{[n]}\right\}_{n \in \mathbb{Z}}, n^{m-1}\left\{C_2^{[n]}\right\}_{n \in \mathbb{Z}}, ..., n^{m-1}\left\{C_l^{[n]}\right\}_{n \in \mathbb{Z}} \tag{2.12}$$

is a basis of $\ker L^m$ over K.

Now we are ready to find a basis in $\ker L$ for an arbitrary k-order linear shift operator L defined on a perfect field K.

Theorem 2.7. *Let $L = T^k + a_1 T^{k-1} + ... + a_{k-1} T + a_k I$, $a_k \neq 0$ be an arbitrary k-order linear shift operator over a perfect field K and let $P(r)$ be the characteristic polynomial of L. Let $P(r) = P_1(r)^{m_1} P_2(r)^{m_2} ... P_s(r)^{m_s}$ be the factorization of $P(r)$ into products of irreducible factors (over K). Let $t_1, ..., t_s$ be the degrees of the irreducible polynomials $P_1, P_2, ..., P_s$ respectively. Let $L = L_1^{m_1} \circ L_2^{m_2} \circ ... \circ L_s^{m_s}$ be the corresponding factorization of the operator L into products of irreducible operators. Let $\mathcal{B}_j$ be the basis (over K), of the form (2.12) for $\ker L_j^{m_j}$. Then $\mathcal{B} = \cup_{j=1}^{s} \mathcal{B}_j$ is a basis of $\ker L$ over K.*

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Proof. From Theorem 2.5 we know that $\ker L = \sum_{j=1}^{s} \ker L_j^{m_j}$ and this sum is a direct sum. Hence $\mathcal{B} = \cup_{j=1}^{s} \mathcal{B}_j$ is a basis of $\ker L$, where

$$\mathcal{B}_j = \left\{C_{j1}^{[n]}\right\}_{n\in\mathbb{Z}}, \left\{C_{j2}^{[n]}\right\}_{n\in\mathbb{Z}}, ..., \left\{C_{jt_j}^{[n]}\right\}_{n\in\mathbb{Z}}, \left\{nC_{j1}^{[n]}\right\}_{n\in\mathbb{Z}}, ..., \left\{nC_{jt_j}^{[n]}\right\}_{n\in\mathbb{Z}},$$
$$..., \left\{n^{m-1}C_{j1}^{[n]}\right\}_{n\in\mathbb{Z}}, n^{m-1}\left\{C_{j2}^{[n]}\right\}_{n\in\mathbb{Z}}, ..., n^{m-1}\left\{C_{jt_j}^{[n]}\right\}_{n\in\mathbb{Z}}$$

is the corresponding basis of $\ker L^{m_j}$ constructed as above (see (2.12)). □

Example 2.8. Let $L = T^6 + 7T^4 + 16T^2 + 12I$ be a linear shift operator defined on the vector space of all two-sided sequences of rational numbers. Its characteristic polynomial is $P(r) = (r^2+2)^2(r^2+3)$, this factorization being a factorization into irreducible factors over $\mathbb{Q}$, the field of rational numbers. Let $L_1 = T^2 + 2I$ and $L_2 = T^2 + 3I$ be the corresponding irreducible shift operators which appear in the factorization of $L = L_1^2 \circ L_2$. First of all let us find a K-basis in $\ker L_1^2$. For this, we see that $F_1 = \mathbb{Q}[i\sqrt{2}]$ is the decomposition field of $r^2 + 2 = 0$. So, $r_1 = i\sqrt{2}$ and $r_2 = -i\sqrt{2}$. Since

$$r_1^n = C_{11}^{[n]} + C_{12}^{[n]} i\sqrt{2},$$

where

$$C_{11}^{[n]} = \begin{cases} 4, \text{ if } n = 4k \\ 0, \text{ if } n = 4k+1 \\ -2, \text{ if } n = 4k+2 \\ 0, \text{ if } n = 4k+3 \end{cases}, C_{12}^{[n]} = \begin{cases} 0, \text{ if } n = 4k \\ 1, \text{ if } n = 4k+1 \\ 0, \text{ if } n = 4k+2 \\ -2, \text{ if } n = 4k+3 \end{cases}, \tag{2.13}$$

the basis of $\ker L_1^2$ over $\mathbb{Q}$ is:

$$\mathcal{B}_1 = \left[\left\{C_{11}^{[n]}\right\}_n, \left\{C_{12}^{[n]}\right\}_n, \left\{nC_{11}^{[n]}\right\}_n, \left\{nC_{12}^{[n]}\right\}_n\right].$$

Now, let find a basis in $\ker L_2$. For this, we see that $F_2 = \mathbb{Q}[i\sqrt{3}]$ is the decomposition field of $r^2+3 = 0$. So, $s_1 = i\sqrt{3}$ and $s_2 = -i\sqrt{3}$ are its roots. Since

$$s_1^n = C_{21}^{[n]} + C_{22}^{[n]} i\sqrt{3},$$

where

$$C_{21}^{[n]} = \begin{cases} 9, \text{ if } n = 4k \\ 0, \text{ if } n = 4k+1 \\ -3, \text{ if } n = 4k+2 \\ 0, \text{ if } n = 4k+3 \end{cases}, C_{12}^{[n]} = \begin{cases} 0, \text{ if } n = 4k \\ 1, \text{ if } n = 4k+1 \\ 0, \text{ if } n = 4k+2 \\ -3, \text{ if } n = 4k+3 \end{cases}, \tag{2.14}$$

the basis of $\ker L_2$ over $\mathbb{Q}$ is:

$$\mathcal{B}_2 = \left[\left\{C_{21}^{[n]}\right\}_n, \left\{C_{22}^{[n]}\right\}_n\right].$$

Hence, the basis of $\ker L$ over K is:

$$\mathcal{B} = \left[\left\{C_{11}^{[n]}\right\}_n, \left\{C_{12}^{[n]}\right\}_n, \left\{nC_{11}^{[n]}\right\}_n, \left\{nC_{12}^{[n]}\right\}_n, \left\{C_{21}^{[n]}\right\}_n, \left\{C_{22}^{[n]}\right\}_n\right].$$

3. Hadamard products of recurrent sequences over a perfect field

Let K be a perfect field and let $\overline{K}$ a fixed algebraic closure of it. Let $G = Gal(\overline{K}/K)$ be the absolute Galois group of K, i.e. the group of all K-automorphisms σ of $\overline{K}$. Let $\widetilde{S}$ be the K-vector space of all two-sided infinite sequences $f : \mathbb{Z} \to K$ and let $\overline{\widetilde{S}} = \widetilde{S} \otimes_K \overline{K}$ be the $\overline{K}$-vector space of all two-sided infinite sequences $g : \mathbb{Z} \to \overline{K}$. Thus, $\widetilde{S} \subset \overline{\widetilde{S}}$. The main problem of this section is to decide when a recurrent sequence g of $\overline{\widetilde{S}}$ is an element of $\widetilde{S}$. As usual, L_f denotes the minimal linear shift operator of $f \in \widetilde{S}$ and $\overline{L}_f$ denotes the extension of L_f to $\overline{\widetilde{S}}$. We know from Theorem 1.5 that if M is a linear shift operator on

$\widetilde{\overline{S}}$ and if $P_M(r) = (r-r_1)^{m_1}(r-r_2)^{m_2}...(r-r_l)^{m_l} \in \overline{K}[r]$ is the factorization into prime factors of its characteristic polynomial, then $\ker M$ consists of all the recurrent sequences $g : \mathbb{Z} \to \overline{K}$ of the form

$$g(n) = \sum_{i=1}^{l} P_i(n) r_i^n, \tag{3.1}$$

where $P_i(r) \in \overline{K}[r]$ is a polynomial of degree at most $m_i - 1$ for any $i = 1, 2, ..., l$. We can fix it by giving the first $m = m_1 + ... + m_l = \deg P_M(r)$ values $g(0), ..., g(m-1)$ of g. Now, if $f : \mathbb{Z} \to K$ is a recurrent sequence of $\widetilde{S}$, and if $f \in \ker L$, where L is a linear shift operator on $\widetilde{S}$, then f is also an element of $\ker \overline{L}$, L being the natural extension of L to $\widetilde{\overline{S}}$. Let $P_L(r) = Q_1^{l_1}(r) Q_2^{l_2}(r)...Q_t^{l_t}(r) \in K[r]$ be the factorization of the characteristic polynomial $P_L(r)$ of L into irreducible monic distinct factors Q_i. Then $Q_i(r)$, $i = 1, 2, ..., t$ are coprime factors of $P_L(r)$. Let $Q_i(r) = (r - r_{i1})(r - r_{i2})...(r - r_{is_i})$, $s_i = \deg Q_i$ be the factorization in $\overline{K}$ of the irreducible (over K) polynomial Q_i. Thus

$$P_L(r) = \prod_{i=1}^{t} \prod_{j=1}^{s_i} (r - r_{ij})^{l_i}. \tag{3.2}$$

Since $\{r_{ij}\}$ is a set of $k = \deg P_L(r)$ distinct elements in $\overline{K}$,

$$f(n) = \sum_{i=1}^{t} \sum_{j=1}^{s_i} P_{ij}(n) r_{ij}^n, \tag{3.3}$$

where $P_{ij}(r) \in \overline{K}[r]$ are polynomials of degrees at most $l_i - 1$ for all $j = 1, 2, ..., s_i$. For any $\sigma \in G = Gal(\overline{K}/K)$ we define the known action of σ on a polynomial $H(r) = h_0 r^q + h_1 r^{q-1} + ... + h_{q-1} r + h_q \in \overline{K}[r]$:

$$\sigma(H)(r) = \sigma(h_0) r^q + \sigma(h_1) r^{q-1} + ... + \sigma(h_{q-1}) r + \sigma(h_q) \in \overline{K}[r].$$

Since $Q_i(r)$ is irreducible over K, then $\sigma(Q_i)(r) = Q_i(r)$, $i = 1, 2, ..., t$ (see [2]). Moreover, if $Q_i(r) = (r - r_{i1})(r - r_{i2})...(r - r_{is_i})$, then

$$\begin{aligned} \sigma(Q_i)(r) &= (r - \sigma(r_{i1}))(r - \sigma(r_{i2}))...(r - \sigma(r_{is_i})) = \\ &= (r - r_{i1})(r - r_{i2})...(r - r_{is_i}) = Q_i(r). \end{aligned} \tag{3.4}$$

Since for any $i = 1, 2, ..., t$,

$$\sigma\left(\sum_{j=1}^{s_i} P_{ij}(n) r_{ij}^n\right) = \sum_{j=1}^{s_i} P_{ij}(n) r_{ij}^n, \tag{3.5}$$

from (3.3) we see that $\sigma(f(n)) = f(n)$ for any $\sigma \in G$, what is known from the definition of G. Let us remark that the set of elements $\{P_{ij}(n) r_{ij}^n\}_{j=1,2,...,s_i}$ are all conjugates one to each other for any $i = 1, 2, ..., t$.

We say that $g \in \widetilde{\overline{S}}$ is a *K-regular recurrent sequence if* $g(n) = \sum_{i=1}^{l} P_i(n) r_i^n$ (see (3.1)), where r_i are distinct and $\deg P_i(r) = n_i$, and if we can write the set $\{r_1, ..., r_l\}$ as a union

$$\{r_1, ..., r_l\} = \cup_{i=1}^{t} \mathcal{O}(r_i)$$

of orbits relative to G relative to some $r_i \in \{r_1, ..., r_l\}$ such that for any $r_v = \sigma(r_i) \in \mathcal{O}(r_i)$ the corresponding $P_v(r) = \sigma(P_i(r))$; in particular the $P_v(r)$ have the same degree for all $r_v \in \mathcal{O}(r_i)$, $i = 1, 2, ..., t$.

In fact we just proved the following criterion.

Theorem 3.1. *Let $g : \mathbb{Z} \to \overline{K}$ be a recurrent sequence in $\widetilde{\overline{S}}$ with the minimal linear shift operator L_g and $\ker L_g = \left\{(y_n)_n,\ y_n = \sum_{i=1}^{l} P_i(n) r_i^n\right\}$, where $r_1, ..., r_l$ are the distinct roots of characteristic polynomial P_g of L_g with the algebraic multiplicities $m_1, ..., m_l$ respectively. Then g is in $\widetilde{S}$ if and only if g is K-regular.*

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Example 3.2. The sequence

$$f(n) = (3n - \sqrt{2}) \left(\frac{1+\sqrt{2}}{5} \right)^n + (3n + \sqrt{2}) \left(\frac{1-\sqrt{2}}{5} \right)^n, n \in \mathbb{Z},$$

considered over $\mathbb{Q}[\sqrt{2}]$ is in fact a sequence over $\mathbb{Q}$. Indeed, if $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, then $\sigma(f(n)) = f(n)$, so $f(n) \in \mathbb{Q}$. We see that f is a $\mathbb{Q}$-regular sequence with the above definition. In fact $f(n)$ is the sum of elements of the orbit of $(3n - \sqrt{2}) \left(\frac{1+\sqrt{2}}{5} \right)^n$.

Corollary 3.3. *(see also* [1]*) Let f and g be two recurrent sequences of $\widetilde{S}$, i.e. with entries in K and let $h = fg$, $h(n) = f(n)g(n)$ be the Hadamard product of f and g. Then h is a recurrent sequence with a period at most km, where k is the period of f and m is the period of g.*

Proof. From the above discussion, let $P_f(r) = Q_1^{l_1}(r)Q_2^{l_2}(r)...Q_t^{l_t}(r) \in K[r]$ be the factorization into irreducible polynomials in $K[r]$ of the characteristic polynomial $P_f(r)$ of the minimal shift operator L_f of f. Let $Q_i(r) = (r - r_{i1})(r - r_{i2})...(r - r_{is_i})$, $s_i = \deg Q_i$, be the factorization of Q_i in $\overline{K}$, $i = 1, 2, ..., t$. Let $P_g(r) = R_1^{u_1}(r)R_2^{u_2}(r)...R_v^{u_v}(r) \in K[r]$ be the factorization into irreducible polynomials in $K[r]$ of the characteristic polynomial $P_g(r)$ of the minimal shift operator L_g of g $(\deg P_g(r) - m)$. Let $R_i(r) = (r - q_{i1})(r - q_{i2})...(r - q_{iw_i})$, $w_i = \deg R_i$, be the factorization of R_i in $\overline{K}$, $i = 1, 2, ..., v$. Thus,

$$f(n) = f(n) = \sum_{i=1}^{t} \sum_{j=1}^{s_i} P_{ij}(n) r_{ij}^n,$$

as in (3.3) and

$$g(n) = \sum_{i=1}^{v} \sum_{j=1}^{w_i} S_{ij}(n) q_{ij}^n,$$

where $\deg S_{ij}(r)$ is at most $u_i - 1$ for any $i = 1, 2, ..., v$. Since a linear combination of recurrent sequences over K is also a recurrent sequence over K (see Remark 2.4), it remains to see that the sum of the elements of an orbit of an element $P_{ij}(n)S_{ab}(n)r_{ij}^n q_{ab}^n$ in the expression of $f(n)g(n)$ is invariant relative to any $\sigma \in G = Gal(\overline{K}/K)$, i.e. we prove that fg is K-regular. But this is obvious because f and g are K-regular. Thus fg is again a recurrent sequence over K. Counting the number of terms in the expression of fg, we see that the degree of fg is at most km. □

In many other papers ([3]-[5], [10]) we find more complicated proofs of this main result (see the references of [1]).

Remark 3.4. Theorem 3.1 also implies another interesting result. Let f and g be two recurrent sequences in S with their minimal linear shift operators L_f and L_g respectively. Then there exist a unique recurrent sequence $h \in S$ such that $\ker L_f \otimes_K \ker L_g = \ker L_h$, where L_h is the minimal linear shift operator of h. We do not know if this h has something in common with the Hadamard product fg. Is it a new interesting product of f and g?

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