A GEOMETRIC CHARACTERIZATION OF THE QUADRIC SURFACES OF REVOLUTION

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ABSTRACT. It is known (see [1] or [5]) that the ellipsoid of revolution

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}=1$$

satisfies the relation

$$k_{\mathrm{m}}=\frac{a^4}{b^2}\,k_{\mathrm{p}}^3,$$

where we denote the principal curvatures κ_1 and κ_2 of a rotational surface, whose curvature lines are the meridians (m) and the parallels (p), by k_{m} and k_{p} respectively.

It is proved in [8], and also in [9] in a different way, that any *closed* real analytic surface of revolution satisfying $k_{\mathrm{m}}=\mu\,k_{\mathrm{p}}^3$, for any *positive* constant $\mu>0$, is congruent to some ellipsoid of revolution. The aim of this paper is to motivate a generalization of the previous result, using our local approach to the study of rotational Weingarten surfaces given in [2], in order to characterize the non-degenerated quadric surfaces of revolution in terms of a cubic Weingarten relation.

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1. Motivation

Following [5] or [11], the *Weingarten surfaces* are those whose principal curvatures κ_1 and κ_2 satisfy a certain functional relation $\Phi(\kappa_1,\kappa_2)=0$, with Φ a smooth function. For example, the class of *linear Weingarten* surfaces, i.e. those ones verifying $a\,\kappa_1+b\,\kappa_2=c$, $a^2+b^2\neq 0$, $c\in\mathbb{R}$, includes the important family of constant mean curvature surfaces.

We emphasize the following classical result of S.S. Chern in 1945:

Theorem 1.1. [5] *Consider a Weingarten C^2-ovaloid, such that $\kappa_1=f(\kappa_2)$, with f a strictly decreasing smooth function. Then the ovaloid is a round sphere.*

Chern remarked (see also [1]) that any rotational ellipsoid in $\mathbb{R}^3$ satisfies $\kappa_1=\mu\,\kappa_2^3$, with $\mu>0$, and he used this as a counterexample in the sense that, for a characterization of spheres, one cannot modify the assumption *decreasing* to *increasing* in his theorem.

Classical analytic examples of *closed* Weingarten surfaces were found by H. Hopf in 1951:

Theorem 1.2. [7] *There are closed real analytic linear Weingarten surfaces of genus zero.*

Some years later, K. Voss proved in 1959 the following classical striking result:

Theorem 1.3. [10] *Any real analytic Weingarten surface of genus zero is rotational.*

Inspired by the previous Chern's results, S.T. Yau posed in 1982 (see [12]) the following problem:

Assume that the principal curvatures κ_1, κ_2, of a closed surface in $\mathbb{R}^3$ satisfy $\kappa_1 = \mu\,\kappa_2^3$, $\mu > 0$, in some order. Is the surface a rotational ellipsoid?

We should point out here that the *meridian* and *parallel* principal curvatures of a rotational surface do not make sense anymore for arbitrary surfaces; this explains the *in some order* comment for κ_1, κ_2.

This problem has a positive answer on real analytic surfaces, thanks to the following result of Kühnel and Steller in 2005:

Theorem 1.4. [8] *If a real analytic rotational genus zero surface satisfies $k_m = \mu\,k_p^3$, $\mu > 0$, where k_m (resp. k_p) denotes the principal curvature along meridians (resp. parallels), then it is a rotational ellipsoid.*

We point out that, under the condition of Yau's problem, the Gauss curvature of the surface must be non negative and thus it has genus zero. This fact, together with Voss' theorem and the above result, leads to a positive answer to Yau's question in the real analytic case.

However, the answer is negative for C^2 surfaces since I. Fernández and P. Mira constructed in [6] a counterexample considering a rotational example that starts at the plane and then bifurcates from it away from the axis. We must remark that this example satisfies the Weingarten condition $k_{\mathrm{p}} = \mu\,k_{\mathrm{m}}^3$, for some $\mu > 0$.

On the other hand, we must also emphasize that an alternative proof of Theorem 1.4 was later given by U. Simon as a consequence of the following local characterization of the non-degenerated quadric surfaces of revolution.

Theorem 1.5. [9] *Consider a non-degenerated rotational surface in $\mathbb{R}^3$ with non-vanishing Gauss curvature and without umbilics. The principal curvatures satisfy the relation $k_m = \mu\,k_p^3$, for some real non-zero constant $\mu \neq 0$, if and only if the surface is part of a non-degenerated quadric.*

Our aim in this paper is to prove the same characterization of the non-degenerated quadric surfaces of revolution without the hypothesis on the Gauss curvature and the umbilics, both necessary in Simon's proof because he used in a strong way techniques coming from differential affine geometry.

2. The geometric linear momentum of a rotational surface

In this section we deal with *rotational surfaces*, also called *surfaces of revolution*. They are surfaces globally invariant under the action of any rotation around a fixed line called *axis of revolution*. The rotation of a curve (called *generatrix*) around a fixed line generates a surface of revolution. The sections of a surface of revolution by half-planes delimited by the axis of revolution, called *meridians*, are special generatrices. The sections by planes perpendicular to the axis are circles called *parallels* of the surface.

We denote S_α the rotational surface in $\mathbb{R}^3$ generated by the rotation around the z-axis of a plane curve α in the xz-plane. That is, α is the generatrix curve that we can consider parametrized by arc-length, whose parametric equations are given by $x = x(s) > 0$, $y = 0$, $z = z(s)$, $s \in I \subseteq \mathbb{R}$. The function $x = x(s)$, $s \in I \subseteq \mathbb{R}$, represents the distance from the point $\alpha(s)$ to the z- axis of revolution. Then S_α is parametrized by

$$S_\alpha \equiv X(s,\theta) = (x(s)\cos\theta, x(s)\sin\theta, z(s))\,,\ (s,\theta) \in I \times (-\pi,\pi).$$

Given any plane curve α in the xz-plane, we introduced in [2, Section 2] the *geometric linear momentum* of α (with respect to the z-axis) as a smooth function assuming values in $[-1,1]$ that completely determines it (up to translations in the z-direction). It is defined by $\mathcal{K}(s) = \dot{z}(s)$, where the dot $\dot{}$ means derivation with respect to the arc parameter s. Geometrically, $\mathcal{K}$ controls the angle of the Frenet frame of the curve with the coordinate axes. Moreover, in physical terms, $\mathcal{K} = \mathcal{K}(s)$ may be described as the linear momentum (with respect to the z-axis) of a particle of unit mass with unit speed and trajectory $\alpha(s)$. We point out that $\mathcal{K}$ is well defined, up to the sign, depending on the orientation of α.

Remark 2.1. If the plane curve $\alpha = (x, z)$ is not necessarily parametrized by arc length, i.e. $\alpha = \alpha(t)$, t being any parameter, one can compute the geometric linear momentum $\mathcal{K} = \mathcal{K}(t)$ by means of

$$\mathcal{K}(t) = \frac{z'(t)}{|\alpha'(t)|},$$

where $'$ denotes derivation respect to t.

The importance of the geometric linear momentum $\mathcal{K}$ lies in the fact that it allows to determine by quadratures in a constructive explicit the plane curves $\alpha = (x, z)$ such that its curvature depends on the distance to the z-axis, that is, it is given as a function of x, i.e. $\kappa = \kappa(x)$. In this case, $\mathcal{K} = \mathcal{K}(x)$ satisfies $\mathcal{K}'(x) = \kappa(x)$ and the algorithm to recover the curve $\alpha = (x, z)$ involves the following computations (see [3] and [4]):

(i) Arc-length parameter s of $\alpha = (x, z)$ in terms of x, defined —up to translations of the parameter— by the integral:

$$s = s(x) = \int \frac{dx}{\sqrt{1 - \mathcal{K}(x)^2}}, \tag{2.1}$$

where $-1 < \mathcal{K}(x) < 1$, and inverting $s = s(x)$ to get $x = x(s)$.

(ii) z-coordinate of the curve —up to translations along z-axis— by the integral:

$$z(s) = \int \mathcal{K}(x(s))\, ds. \tag{2.2}$$

Alternatively, if we eliminate ds in the above integrals, we obtain:

$$z = z(x) = \int \frac{\mathcal{K}(x)dx}{\sqrt{1 - \mathcal{K}(x)^2}}. \tag{2.3}$$

Thus we can summarize the determining role of the geometric linear momentum in the next result.

Corollary 2.2. [2, Corollary 1] *Any plane curve $\alpha = (x, z)$, with x non-constant, is uniquely determined by its geometric linear momentum $\mathcal{K}$ as a function of its distance to z-axis, that is, by $\mathcal{K} = \mathcal{K}(x)$. The uniqueness is modulo translations in the z-direction. Moreover, the curvature of α is given by $\kappa(x) = \mathcal{K}'(x)$.*

It is obvious that if we translate the generatrix curve α of a rotational surface S_α along z-axis, we obtain a congruent surface to S_α. An immediate consequence of Corollary 2.2 is then the following key result:

Corollary 2.3. [2, Corollary 2] *Any rotational surface S_α, with generatrix curve $\alpha = (x, z)$, is uniquely determined, up to z-translations, by the geometric linear momentum $\mathcal{K} = \mathcal{K}(x)$ of its generatrix curve, being x non-constant.*

We can confirm the result established in Corollary 2.3 when we study the geometry of S_α through its first and second fundamental forms, I and II, since a direct computation, using that $\kappa(x) = \mathcal{K}'(x)$, shows that both can be expressed only in terms of the geometric linear momentum $\mathcal{K}$ and, of course, the non constant distance x from the surface to the axis of revolution:

$$I \equiv ds^2 + x^2 d\theta^2, \quad II \equiv \mathcal{K}'(x)ds^2 + x\mathcal{K}(x)d\theta^2.$$

Therefore we get the following expressions for the principal curvatures κ_1 and κ_2, whose curvature lines are the meridians (m) and the parallels (p) respectively of the rotational surface S_α:

$$\kappa_1 \equiv k_{\mathrm{m}} = \mathcal{K}'(x), \quad \kappa_2 \equiv k_{\mathrm{p}} = \frac{\mathcal{K}(x)}{x}. \tag{2.4}$$

Making use of Corollary 2.3, we can list the following characterizations of some simple surfaces of revolution:

Example 2.4. [2, Propositon 1]

(1) Any (horizontal) plane is uniquely determined by the geometric linear momentum $\mathcal{K} \equiv 0$.

(2) The circular cone with opening $\theta_0 \in (-\pi/2, \pi/2)$*, given by* $x^2 + y^2 = \cot^2\theta_0\, z^2$*, is uniquely determined by the geometric linear momentum* $\mathcal{K} \equiv \sin\theta_0$.

(3) The sphere of radius $R > 0$*, given by* $x^2+y^2+z^2 = R^2$*, is uniquely determined by the geometric linear momentum* $\mathcal{K}(x) = x/R$.

Now we can pay attention to *rotational Weingarten surfaces.* In general, we just simply write $\Phi(k_{\mathrm{m}}, k_{\mathrm{p}}) = 0$. But taking into account (2.4), we easily deduce that the above functional relation translates into a first-order differential equation $\hat{\Phi}(x, \mathcal{K}(x), \mathcal{K}'(x)) = 0$ for the geometric linear momentum $\mathcal{K} = \mathcal{K}(x)$ determining S_α according Corollary 2.3. This will be the simple idea in the proof of our main result in next section.

3. Cubic rotational Weingarten surfaces

It is known (see [1]) that the ellipsoid of revolution (see Figure 1)

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1 \tag{3.1}$$

satisfies the relation

$$k_{\mathrm{m}} = \frac{a^4}{b^2}\, k_{\mathrm{p}}^3. \tag{3.2}$$

For our purposes, recalling Corollary 2.3, we need to compute the geometric linear momentum of the ellipsoid (3.1). We parametrize the generatrix semiellipse by $x = a\cos t$, $z = b\sin t$, $t \in [-\pi/2, \pi/2]$, and using Remark 2.1, it is not difficult to conclude that

$$\mathcal{K}(x) = \frac{bx}{\sqrt{a^4 - (a^2-b^2)x^2}}. \tag{3.3}$$

Then, using (2.4), we can check (3.2) easily.

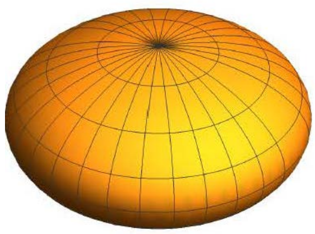

FIGURE 1. Ellipsoid of revolution

We proceed in the same way with the one-sheet hyperboloid of revolution (see Figure 2)

$$\frac{x^2+y^2}{a^2} - \frac{z^2}{b^2} = 1, \tag{3.4}$$

obtaining, from the generatrix hyperbola $x = a\cosh t$, $z = b\sinh t$, $t \in \mathbb{R}$, and Remark 2.1, that

$$\mathcal{K}(x) = \frac{bx}{\sqrt{(a^2+b^2)x^2 - a^4}} \tag{3.5}$$

and now we can check that the one-sheet hyperboloid of revolution satisfies the relation

$$k_{\mathrm{m}} = -\frac{a^4}{b^2} k_{\mathrm{p}}^3. \tag{3.6}$$

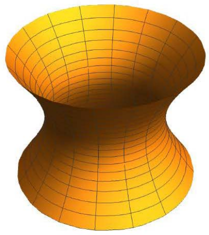

FIGURE 2. One-sheet hyperboloid of revolution

However, for the two-sheets hyperboloid of revolution (see Figure 3)

$$-\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}=1, \tag{3.7}$$

as now the generatrix hyperbola is $x = a\sinh t$, $z = \pm b\cosh t$, $t \geq 0$, we obtain from Remark 2.1 that

$$\mathcal{K}(x)=\frac{bx}{\sqrt{a^4+(a^2+b^2)x^2}} \tag{3.8}$$

and so we arrive at the following relation satisfied by the two-sheets hyperboloid of revolution:

$$k_{\mathrm{m}}=\frac{a^4}{b^2}\,k_{\mathrm{p}}^3, \tag{3.9}$$

that it is formally the same than the one (3.2) of the ellipsoid of revolution.

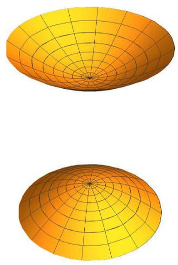

FIGURE 3. Two-sheets hyperboloid of revolution

Finally, for the paraboloid of revolution (see Figure 4)

$$z=\frac{x^2+y^2}{2a}, \tag{3.10}$$

we use the generatrix parabola $x = t$, $z = \frac{t^2}{2a}$, $t \geq 0$, and

$$\mathcal{K}(x)=\frac{x}{\sqrt{a^2+x^2}} \tag{3.11}$$

and then the paraboloid of revolution satisfies the relation

$$k_{\mathrm{m}}=a^2k_{\mathrm{p}}^3. \tag{3.12}$$

Now we are in a position to state our main result in this section characterizing all the quadric surfaces of revolution in terms of a cubic Weingarten relation.

Theorem 3.1. [2, Theorem 2] *The only rotational surfaces satisfying $k_m = \mu\, k_p^3$, $\mu \neq 0$, are the plane and the non-degenerated quadric surfaces of revolution.*

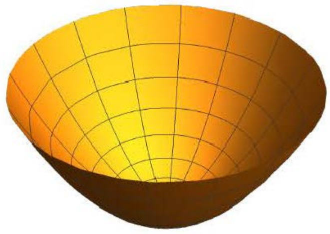

FIGURE 4. Paraboloid of revolution

Proof. Using (2.4), the cubic Weingarten relation $k_{\rm m} = \mu\, k_{\rm p}^3$ translates into the separable o.d.e.

$$\mathcal{K}'(x) = \mu\, \mathcal{K}(x)^3/x^3.$$

Its constant solution $\mathcal{K} \equiv 0$ leads to the plane (see Example 2.4). Its non-constant solution is given by

$$\mathcal{K}(x) = \pm \frac{x}{\sqrt{\mu + c\, x^2}},\ c \in \mathbb{R}. \tag{3.13}$$

We are going to identify the rotational surfaces uniquely determined, up to z-translations, by the one parameter family of geometric linear momenta (depending on c) given in (3.13) (see Corollary 2.3). There is no restriction if we only consider plus sign in (3.13).

We distinguish two cases according to the sign of μ.

- $\mu > 0$. We separate in turn three possibilities:
  (i) $c < 1$: Then $a^2 = \frac{\mu}{1-c}$ is well defined, and putting $b^2 = \frac{a^4}{\mu}$, we conclude that (3.13) is exactly (3.3) and Corollary 2.3 gives that we arrive at the ellipsoid of revolution (3.1). In particular, if $c = 0$, we obtain the sphere of radius $\sqrt{\mu}$ (see Example 2.4).
  (ii) $c > 1$: We now define $a^2 = \frac{\mu}{c-1}$ and $b^2 = \frac{a^4}{\mu}$. Then we obtain that (3.13) is exactly (3.8) and Corollary 2.3 concludes that we arrive at the two-sheets hyperboloid of revolution (3.7).
  (iii) $c = 1$: We define $a^2 = \mu$ and we conclude that (3.13) is exactly (3.11). We deduce from Corollary 2.3 that we arrive at the paraboloid of revolution (3.10).
- $\mu < 0$. Since $\mu + c\, x^2 > 0$ from (3.13) and taking into account that always $\mathcal{K}(x)^2 < 1$, we deduce that $(1-c)x^2 < \mu < 0$ and so $c > 1$. Then $a^2 = \frac{\mu}{1-c}$ is well defined, and putting $b^2 = -\frac{a^4}{\mu}$, we conclude that (3.13) is exactly (3.5) and Corollary 2.3 gives us the one-sheet hyperboloid of revolution (3.4).

This proves the result. ☐

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