STATISTICAL STRUCTURES ON ALMOST NORDEN MANIFOLDS

LEILA SAMEREH, ESMAEIL PEYGHAN AND ION MIHAI

Abstract. A Norden manifold is a complex manifold endowed with a pair of Norden metrics. We consider dual connections on such manifolds and study their geometric properties. One example of a complex statistical connection is provided.

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1. Introduction

The study of statistical manifolds provides techniques to investigate the intrinsic properties of probability distributions. They play important roles not only in probability and statistics, but also in wider areas of information sciences, such as machine learning, signal processing, optimization.

For an open subset Θ of $\mathbb{R}^n$ and a sample space Ω with parameter $\theta = (\theta^1, \cdots, \theta^n)$, we call the set of probability density functions

$$S = \{p(x;\theta) : \int_\Omega p(x;\theta) = 1, \quad p(x;\theta) > 0, \quad \theta \in \Theta \subseteq \mathbb{R}^n\}$$

a *statistical model*. For a statistical model S, the Fisher information metric $g(\theta) = [g_{ij}(\theta)]$ is given by

$$g_{ij}(\theta) = \int_\Omega \frac{1}{p(x;\theta)} \partial_i p(x;\theta) \partial_j p(x;\theta) \mathrm{d}x.$$

Rao was the first who studied the above metric in 1945 (see [12], [5] and [2]).

By a *statistical manifold* we mean a triple (M, g, ∇), where the manifold M is equipped with a statistical structure (g, ∇) containing a (pseudo) Riemannian metric g and a torsion-free linear connection ∇ on M such that the covariant derivative ∇g is symmetric.

On the other hand, Hermitian manifolds, in particular Norden manifolds, have been studied from various points of view. On Norden manifolds there exists a pair of Norden metrics; one can consider dual (conjugate) connections with respect to each of these metric tensors and their relations to dual connections relative to the almost complex structure. The study of statistical structures on these manifolds is in progress.

The main purpose of this note is to present certain results on almost Norden manifolds with statistical connections, mainly from our paper [13].

2. Preliminaries

Let (M, g) be a (pseudo) Riemannian manifold and ∇ a torsion-free linear connection.

The triple (M, g, ∇) is called a *statistical manifold* [1] if ∇g is symmetric, i.e., ∇g satisfies the Codazzi equation

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z) = (\nabla_Z g)(Y, X), \quad \forall X, Y, Z \in \Gamma(TM). \tag{2.1}$$

In this case, ∇ is called a statistical connection and the pairing (∇, g) a statistical structure on M (see [1]). The dual connection ∇^* of a linear connection ∇ is given by

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z), \qquad \forall X, Y, Z \in \Gamma(TM). \tag{2.2}$$

The *skewness operator* K, which is a tensor field of type (1,2) on M, is

$$K(X, Y) = \nabla_X Y - \nabla^*_X Y, \qquad \forall X, Y \in \Gamma(TM). \tag{2.3}$$

It is easy to see that K satisfies the following relations.

(i) $K(X, Y) = K(Y, X)$,
(ii) $g(K(X, Y), Z) = g(Y, K(X, Z))$,
(iii) $(\nabla_X g)(Y, Z) = -g(K(X, Y), Z)$, $\forall X, Y, Z \in \Gamma(TM)$.

It is known that if (M, g, ∇) is a statistical manifold, then (M, g, ∇^*) is a statistical manifold as well. Moreover, the Levi-Civita connection $\nabla^{(0)}$ is related to the dual connections by

$$\nabla^{(0)} = \frac{1}{2}(\nabla + \nabla^*). \tag{2.4}$$

Also we have

$$\nabla_X Y = \nabla^{(0)}_X Y + \frac{1}{2}K(X, Y).$$

In affine differential geometry, the dual connections are called *conjugate connections* (see [3, 10]).

3. Statistical Connections on Almost Norden Manifolds

Fei and Zhang proved an important result ([4], Theorem 2.13) about Klein transformation group of conjugate connections.

We will give some related properties of almost Norden statistical manifolds.

Let M be a $2n$-dimensional differentiable manifold, J an almost complex structure and g a pseudo-Riemannian metric compatible with J, i.e.,

$$J^2 X = -X, \qquad g(JX, JY) = -g(X, Y).$$

The couple (M, J) is said to be an almost complex manifold and the triple (M, J, g) is called an *almost Norden manifold*. The triple (M, J, g) is also called an almost complex manifold with Norden metric.

Obviously one has $g(JX, Y) = g(X, JY)$; it follows that the tensor $\tilde{g}$ defined by

$$\tilde{g}(X, Y) = g(X, JY),$$

is symmetric. The tensor $\tilde{g}$ is known as the associated (twin) metric of g. It is also a Norden metric, namely it satisfies

$$\tilde{g}(JX, JY) = -\tilde{g}(X, Y).$$

It is worth noting that the pseudo-Riemannian metrics g and $\tilde{g}$ are necessarily of neutral signature (n, n).

It is known [8] that on an almost Norden manifold (M, J, g), the Levi-Civita connection ∇ satisfies

$$g((\nabla_X J)Z, Y) = g((\nabla_X J)Y, Z).$$

An almost Norden manifold (M, J, g) with a statistical structure is called an *almost Norden statistical manifold*.

Let (M, J, g) be an almost Norden manifold and ∇ the Levi-Civita connection of g. If $(\nabla, \tilde{g})$ is a statistical structure, then (see [13])

$$\tilde{g}(J(\nabla_X J)Z, Y) = \tilde{g}(J(\nabla_Y J)Z, X).$$

Now, we consider an almost Norden statistical manifold $(M, J, \tilde{g}, \nabla)$ such that ∇ is the Levi-Civita connection of g.

Based on the assumption that $(M, \tilde{g}, \nabla)$ is a statistical manifold, it follows that

$$\nabla_X Y = \tilde{\nabla}^{(0)}_X Y + \frac{1}{2} K(X, Y),$$

where $\tilde{\nabla}^{(0)}$ is the Levi-Civita connection with respect to $\tilde{g}$. By the relation

$$(\nabla_X \tilde{g})(Y, Z) = (\nabla_X g)(JY, Z) + g((\nabla_X J)Y, Z)$$

and the above assumption that ∇ is the Levi-Civita connection of g, we have $(\nabla_X J)Y = (\nabla_Y J)X$. Therefore we can state the following theorem [13].

Theorem 3.1. *Let $(M, J, \tilde{g}, \nabla)$ be an almost Norden statistical manifold and ∇ the Levi-Civita connection of g. If the $(1, 2)$-symmetric tensor $K(X, Y) = J(\nabla_X J)Y$, then*

$$\nabla_X Y - \tilde{\nabla}^{(0)}_X Y = \frac{1}{2} J((\nabla_X J)Y).$$

The operator K on almost Norden statistical manifolds has the following properties.

(i) $(\nabla_X J)Y - (\nabla^*_X J)Y = K(X, JY) - J(K(X, Y)), \forall X, Y \in \Gamma(TM)$.

(ii) If $(\nabla_X J)Y = (\nabla_Y J)X$, then $(\nabla^*_X J)Y = (\nabla^*_Y J)X$ if and only if $K(X, JY) = K(Y, JX)$, $\forall X, Y \in \Gamma(TM)$.

We recall some results from [13].

Proposition 3.2. *Let $(M, J, \tilde{g}, \nabla)$ be an almost Norden statistical manifold and let ∇ be the Levi-Civita connection of g, then*

(i) $(\nabla_X J)Y = -J(K(X, Y)), \forall X, Y \in \Gamma(TM)$

(ii) $(\nabla^*_X J)Y = K(X, JY), \forall X, Y \in \Gamma(TM)$.

Corollary 3.3. *Let (M, J, g, ∇) be an almost Norden statistical manifold satisfying $\nabla J = 0$. Then $K(JX, Y) = K(JY, X), \forall X, Y \in \Gamma(TM)$.*

Proposition 3.4. *Let (M, J, g, ∇) be an almost Norden statistical manifold and ∇ the Levi-Civita connection of $\tilde{g}$. Then*

(i) $(\nabla_X J)Y = K(X, JY), \forall X, Y \in \Gamma(TM)$.

(ii) $(\nabla^*_X J)Y = -J(K(X, Y)), \forall X, Y \in \Gamma(TM)$.

Corollary 3.5. *Let $(M, J, \tilde{g}, \nabla)$ be an almost Norden statistical manifold and let $\nabla J = 0$ then $K(JX, Y) = K(JY, X), \forall X, Y \in \Gamma(TM)$.*

Next we investigate ∇-recurrent almost complex structures J on almost Norden statistical manifolds.

An almost complex structure J is called ∇-recurrent if for any $X, Y \in \Gamma(TM)$, $(\nabla_X J)Y = \tau(X)JY$, for some 1-form τ.

Proposition 3.6. [13] *Let (M, J, g, ∇) be an almost Norden statistical manifold with ∇-recurrent almost complex structures J. Then $K(JX, Y) = K(JY, X), \forall X, Y \in \Gamma(TM)$.*

We extend the notion of a statistical structure. Let M be a smooth manifold. Consider a tensor h of type $(0, 2)$ and a linear connection ∇. The triple (M, h, ∇) is called a *quasi statistical manifold* if $d^\nabla h = 0$ [9], where $d^\nabla h$ is defined by

$$(d^\nabla h)(X, Y, Z) := (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) + h(T^\nabla(X, Y), Z).$$

If h is a pseudo-Riemannian metric, we call (M, h, ∇) a statistical manifold admitting torsion [9].

On the other hand, the notion of a semi-symmetric linear connection on a differentiable manifold was introduced by Friedmann and Schouten [6].

Let (M, g) be a (pseudo) Riemannian manifold. A linear connection ∇ on M is said to be *semi-symmetric* if its torsion T^∇ is given by

$$T^\nabla(X, Y) = \pi(Y)X - \pi(X)Y,$$

for all $X, Y \in \Gamma(TM)$. π is a 1-form associated with the vector field P, i.e., $\pi(X) = g(X, P)$.

We have obtained in [13] a characterization of almost Norden statistical manifolds admitting torsion.

Proposition 3.7. *Let $(M, J, \tilde{g})$ be an almost Norden manifold. Consider a metrical structure (g, ∇) such that J is ∇-recurrent. Then $(M, \tilde{g}, \nabla)$ is a statistical manifold admitting torsion if and only if it is semi-symmetric.*

Let (M, J, g) be an almost Norden manifold, ∇ the Levi-Civita connection of g and $(\nabla, \nabla^*, \tilde{g})$ a dualistic structure. If J is ∇-recurrent, then for any $X, Y, Z \in \Gamma(TM)$, we have

(i) $\nabla^*_X Y = \nabla_X Y + \tau(X)Y$,
(ii) $(\nabla^*_X g)(Y, Z) = -2g(\tau(X)Y, Z)$,
(iii) $(\nabla^*_X g)(Y, Z) - (\nabla^*_X \tilde{g})(Y, Z) = -(\nabla_X \tilde{g})(Y, Z)$.

We denote by $\nabla^J_X Y = -J\nabla_X JY$, for any vector fields X, Y on M.

We recall the concept of projectively equivalent connections. Two linear connections ∇ and ∇^P on a differentiable manifold M are called projectively equivalent if there exists a 1-form τ such that

$$\nabla^P_X Y = \nabla_X Y + \tau(X)Y + \tau(Y)X, \quad \forall X, Y \in \Gamma(TM).$$

Proposition 3.8. [13] *Let (M, J) be an almost complex manifold and ∇ and ∇^P projectively equivalent linear connections on M. If $(\nabla_X J)Y = \tau(X)JY$, then*

$$\nabla^P_X Y = \nabla^J_X Y - J(\nabla_Y J)X, \forall X, Y \in \Gamma(TM),$$

$$R^P(X, Y)Z = R(X, Y)Z, \forall X, Y \in \Gamma(TM),$$

where R^P is the curvature tensor of ∇^P.

4. Kähler-Norden statistical manifolds

We give one example of a complex statistical connection [13] using a 1-form ρ given by $\rho(X) = \tilde{g}(X, \xi)$, where $\tilde{g}$ is the Norden metric and ξ is the dual vector field of ρ.

Since

$$\rho(J(X)) = (\rho \circ J)X = \tilde{g}(JX, \xi) = g(JX, J\xi) = -g(X, \xi),$$

and denoting by

$$K(X, Y) = \rho(JX)\rho(JY)J(\xi),$$

we obtain the following.

$$\begin{aligned}\tilde{g}(K(X, Y), Z) = \rho(JX)\rho(JY)g(J\xi, JZ) = (-g(X, \xi))(-g(Y, \xi))(-g(Z, \xi)) = \\ = -g(Y, \rho(JX)\rho(JZ)\xi) = \tilde{g}(Y, \rho(JX)\rho(JZ)J\xi) = \tilde{g}(K(X, Z), Y).\end{aligned}$$

Then, the connection ∇, defined by

$$\nabla_X Y = \overset{\tilde{g}}{\nabla}_X Y + \frac{1}{2}\rho(JX)\rho(JY)J(\xi), \tag{4.1}$$

is a statistical connection, where $\overset{\tilde{g}}{\nabla}$ is the Levi-Civita connection of $\tilde{g}$. Now considering the statistical connection (4.1), we investigate some of its properties.

Let (M, J, g) be an almost Norden manifold with the Levi Civita connection ∇ such that $\nabla J = 0$. The triple (M, J, g) is called a *Kähler-Norden manifold* [8].

New properties of Kähler-Norden manifolds and examples were studied in [11].

If in (reflili $\rho \otimes J(\xi) = \rho \circ J \otimes \xi$, then $\nabla J = 0$, then the above manifold is a Kähler-Norden manifold.

In this case $K(JY, X) = K(JX, Y), \forall X, Y \in \Gamma(TM)$.

Proposition 4.1. [13] *Let $(M, J, \tilde{g})$ be a Kähler-Norden statistical manifold with the statistical connection* (4.1). *Then the following conditions are equivalent.*

(i) $K(JY, X) = K(JX, Y), \forall X, Y \in \Gamma(TM)$,
(ii) $(\nabla_X J)Y = (\nabla_Y J)X, \forall X, Y \in \Gamma(TM)$,
(iii) $(\nabla^*_X J)Y = (\nabla^*_Y J)X, \forall X, Y \in \Gamma(TM)$.

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Department of Mathematics, Faculty of Science, Arak University, Arak, Iran
Email address: `l.samereh@yahoo.com`

Department of Mathematics, Faculty of Science, Arak University, Arak, Iran
Email address: `e-peyghan@araku.ac.ir`

Department of Mathematics, Faculty of Mathematics and Computer Science, Bucharest University, Romania
Email address: `imihai@fmi.unibuc.ro`