HOPF-RINOW THEOREM OF SUB-FINSLERIAN GEOMETRY

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Abstract. The sub-Finslerian geometry means that the metric F is defined only on a given subbundle of the tangent bundle, called a horizontal bundle. In the paper, a version of the Hopf-Rinow theorem is proved in the case of sub-Finslerian manifolds, which relates the properties of completeness, geodesically completeness, and compactness. The sub-Finsler bundle, the exponential map and the Legendre transformation are deeply involved in this investigation.

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1. Introduction

In the Riemannian and Finslerian geometry, there are two concepts of completeness. The first is the completeness in the sense of metric spaces, using the Riemannian metric. Secondly, a Riemannian or Finsler manifold M is called geodesically complete if any geodesic $\gamma(t)$ starting from $x \in M$ is defined for all values of $t \in \mathbb{R}$. On the other hand, the completeness in the Finsler geometry is divided into forward and backward geodesically completenesses, according to forward and backward distance metrics, resp.

Hopf-Rinow theorem is a basic theorem of complete Riemannian manifolds, which connects the completeness properties with compactness, and the exponential map. Its consequence says that any two points of a complete manifold can be connected by a length minimizing geodesic. In 1931, H. Hopf and W. Rinow showed their theorem only for surfaces, but the proof in higher dimensions is not significantly different. Hopf-Rinow theorem has been studied in detail in both Riemannian and Finslerian geometries in the literature, the best general references here are [5, 7], [10]. In the Finsler case forward geodesic completeness is involved, only.

After a development of the sub-Riemannian geometry as well as its generalization, namely sub-Finslerian geometry, the generalization of core theorems of Riemannian geometry has been started. Relating to our issue, Strichartz [13], Rifford [12] and Agrachev et al. [1] gave an extension for a sub-Riemannian case, while Bao et al. [5] showed the Finslerian version of Hopf-Rinow theorem. It turned out that in sub-Riemannian geometry, for general complete sub-Riemannian structures, the exponential mapping is not surjective. This is due to the fact that we may have abnormal minimizing curves and this is the case in the sub-Finslerian context, too.

To prove the statements of Hopf-Rinow theorem in the sub-Finsler setting, we need the following concepts and explanations. First, in Section 2 we review some of the standard facts of sub-Finsler geometry. In the third Section, we extend our discussion about the Legendre transformation (see [2]) to define the sub-Finsler manifold on the distribution $\mathcal{D}^*$ of the cotangent space, where we look more closely at a sub-Hamiltonian H defined on $\mathcal{D}^*$, induced by the sub-Finslerian metric F^*. Afterwards, we construct a sub-Finsler bundle, which plays a major role in the formalization of the sub-Hamiltonian in sub-Finsler geometry, in Section 4. Moreover, the sub-Finsler bundle allows an orthonormal frame for the sub-Finsler structure. In Section 5, we introduce the notion of an exponential map in sub-Finsler geometry. In the last section our main theorem is stated and proved.

2. Definitions and some properties of sub-Finsler manifolds

In this section we review some of the standard facts on the sub-Finsler metrics and set up the notations and the terminology which will play an essential role in this paper, for more details we refer the reader to [2, 3, 4].

Let M be a smooth n-dimensional manifold. A vector distribution, denoted as $\mathcal{D}$, of rank k on M is defined as a collection of vector subspaces $\mathcal{D}_x \subset T_xM$ such that for each point x on the manifold, the dimension of $\mathcal{D}_x$ is equal to k.

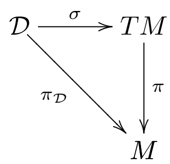
Furthermore, a vector distribution $\mathcal{D}$ is considered smooth if, for every point x_0 in M, there exists a neighborhood U_{x_0} of x_0 and a set of smooth vector fields $X_1, X_2, \ldots, X_k$ defined on U_{x_0} such that, for all x within U_{x_0}:

$$\mathcal{D}_x = \operatorname{span}\{X_1(x), \cdots, X_k(x)\}.$$

Definition 2.1. *Let M be an n–dimensional connected manifold. A sub-Finslerian structure on M is a triple $(\mathcal{D}, \sigma, F)$ where:*

(1) $(\mathcal{D}, \pi_{\mathcal{D}})$ *is a vector bundle on M.*

(2) $\sigma : \mathcal{D} \longrightarrow TM$ *is a morphism of vector bundles. In particular, the following diagram is commutative*



such that $\pi_{\mathcal{D}} : \mathcal{D} \longrightarrow M$ and $\pi : TM \longrightarrow M$ are the canonical projections.

(3) *A function $F : \widetilde{\mathcal{D}} \to \mathbb{R}$, where $\widetilde{\mathcal{D}} = \mathcal{D} \setminus \{0\}$, called a* sub-Finsler metric*, which satisfies the following properties:*
- *(***Positive definiteness***): $F_x(v) > 0$ for all $v \in \widetilde{\mathcal{D}}$, $x \in M$.*
- *(***Regularity***): F is smooth, i.e. C^∞ on $\widetilde{\mathcal{D}}$.*
- *(***Positive homogeneity***): $F_x(\lambda v) = \lambda F_x(v)$ for all $v \in \widetilde{\mathcal{D}}_x$ and $\lambda \in \mathbb{R}^+$.*
- *(***Strong convexity condition***): The Hessian matrix of F^2 with respect to the coordinates on the fibre is positive definite.*

One can replace the strong convexity condition by the following subadditivity property (in an equivalent terminology, a triangle inequality):

$$F_x(v + u) \leqslant F_x(v) + F_x(u), \text{ for all } v, u \in \widetilde{\mathcal{D}}.$$

A sub-Finsler manifold *is a smooth manifold M endowed with a sub-Finslerian structure, i.e. the triple $(\mathcal{D}, \sigma, F)$.*

Let $\mathcal{D}_x$ be the fiber over $x \in M$. The last condition of the sub-Finsler metric means that the matrix $\frac{\partial^2 F^2}{\partial v^i \partial v^j}(x, v)$ is positive definite for all $v = (v^1, \ldots, v^k) \in \mathcal{D}_x$. Equivalently, the corresponding indicatrix

$$I_x = \{v \mid v \in \mathcal{D}_x,\ F_x(v) = 1\}$$

is strictly convex.

The following technique describes the association between the sub-Finsler structure $(\mathcal{D}, \sigma, F)$ and a Finsler metric $\hat{F}$ on $\operatorname{Im}(\sigma) \subset TM$:

For each $u \in \mathrm{Im}(\sigma)_x \subset T_xM$ and $x \in M$, we have

$$\hat{F}_x(u) = \inf_v \{F_x(v)|\ v \in \mathcal{D}_x,\ \sigma(v) = u\}.$$

From now on we suppose that $\mathcal{D} \subset TM$, $\sigma : \mathcal{D} \longrightarrow TM$ is the inclusion $i : \mathcal{D} \longrightarrow TM$ and F is a sub-Finsler metric on $\mathcal{D}$.

As in the sub-Riemannian case, we call $\mathcal{D}$ the *horizontal distribution*. A piecewise smooth curve $\gamma : [0,T] \to M$ is called *horizontal*, or *admissible* if $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ for all $t \in [0,T]$, that is, $\gamma(t)$ is tangent to $\mathcal{D}$. The length of γ is defined as usual by

$$\ell(\gamma) = \int_0^T F(\dot{\gamma}(t))dt.$$

Equivalently, as in the Finslerian case, we observe that it suffices to minimize the *energy*

$$E(\gamma) = \frac{1}{2}\int_0^T F^2(\dot{\gamma}(t))dt.$$

instead of length $\ell(\gamma)$.

The length induces a sub-Finslerian distance $d(x,y)$ between two points x and y as in Finsler geometry:

$$d(x,y) = \inf\{\ell(\gamma)\ |\gamma : [0,T] \longrightarrow M \text{ horizontal},\ \gamma(0) = x, \gamma(T) = y\},$$

where we consider the infimum over all horizontal curves joining x and y. The distance is infinite if there is no such a horizontal curve between x and y. In addition, the horizontal curve $\gamma : [0,T] \to M$ is called a *length minimizing* (or simply a *minimizing*) geodesic, if it realizes the distance between its end points, that is, $\ell(\gamma) = d(\gamma(0), \gamma(T))$.

Chow theorem answers to the following question: given two points x and y in a sub-Finsler manifold, is there a horizontal curve that joins x and y?

In the case of an involutive distribution $\mathcal{D}$ the Frobenius theorem asserts that the set of the horizontal paths through S form a smooth immersed submanifold, the leaf through x, of dimension equal to the rank of distribution k. In this case, if $\mathcal{D}$ is involutive and y is not contained in the leaf through y, there is no any horizontal curve joining x and y.

A positive answer is given by the Chow theorem in the case of bracket generating distributions, which are the "contrary" of the involutive distributions.

Definition 2.2. [9] *A distribution $\mathcal{D}$ is said to be* bracket generating *if any local frame X_i of $\mathcal{D}$, together with all of its iterated Lie brackets spans the whole tangent bundle TM.*

Theorem 2.3. *(Chow's theorem* [9]*) If $\mathcal{D}$ is a bracket generating distribution on a connected manifold M then any two points of M can be joined by a horizontal path.*

Remark 2.4. *The problem of minimizing the length of a curve joining two given points x and y is equivalent to a time optimal problem: where the control bundle is $(\mathcal{D}, \pi_{\mathcal{D}}, M)$ and we are searching for such a curve $\gamma(t)$ and a control curve $v(t) \in \mathcal{D}_{\gamma(t)}$ minimizing the time T needed to connect x and y.*

3. Legendre transformation of sub-Finslerian geometry

Let $\mathcal{D}^*$ be a distribution of rank s on a smooth manifold M that assigns to each point $x \in U \subset M$ a linear subspace $\mathcal{D}^*_x \subset T^*_xM$ of dimension s, see [4]. In other words, $\mathcal{D}^*$ of rank s is a smooth subbundle of rank s of the cotangent bundle T^*M. Such a field of cotangent s-planes is spanned locally by s pointwise linear independent smooth differential 1-forms, namely,

$$\mathcal{D}^*_x = \mathrm{span}\{\alpha_1(x), \ldots, \alpha_s(x)\}, \qquad \alpha_i(x) \in \mathfrak{X}^*(M).$$

In addition, we refer to $\mathcal{D}^0_x$ as the annihilator of the distribution $\mathcal{D}$ (isomorphic to $\mathcal{D}$), of rank $n-k$, which is the set of all covectors that annihilates the vectors in $\mathcal{D}_x$, i.e.

$$\mathcal{D}^0_x = \{\alpha \in T^*_xM : \alpha(v) = 0\ \forall\ v \in \mathcal{D}_x\}. \tag{3.1}$$

In [2], we introduced the Legendre transformation of sub-Finsler geometry. Let us briefly recall it:

The *sub-Lagrange function* $L : \mathcal{D} \longrightarrow \mathbb{R}$, determined by F is given in the following way: $L = \frac{1}{2}F^2$. The fiber derivative of L defines the map

$$\mathcal{L}_L : \mathcal{D} \longrightarrow \mathcal{D}^*, \qquad \mathcal{L}_L(v)(w) = \frac{d}{dt}L_x(v + tw), \text{ where } v, w \in \mathcal{D}_x,$$

called the *Legendre transformation* of $(M, \mathcal{D}, F)$.

We denote by (x^i) the coordinate in a neighborhood $U \subset M$ with (x^i, v^a) in $\mathcal{D}|_U \subset TM$, and (x^i, p_a) in $\mathcal{D}^*|_U \subset T^*M$, respectively, where $i = 1, \ldots, n,\ a = 1, \ldots, k$. Then the relation of the distribution $\mathcal{D}$ of the tangent bundle and the distribution $\mathcal{D}^*$ of the cotangent bundle is given by the Legendre transformation in local coordinates as follows

$$\mathcal{L}_L(x^i, v^a) = (x^i, \frac{\partial L}{\partial v^a}).$$

Then the *sub-Hamiltonian* is given by

$$H : \mathcal{D}^* \longrightarrow \mathbb{R},$$

$$H = \iota_{\mathcal{L}_L^{-1}} - L \circ \mathcal{L}_L^{-1},$$

where $\iota_v(p) = \langle v, p \rangle = p(v)$ for any $v = \mathcal{L}_L^{-1}(p) \in \mathcal{D}$ and $p \in \mathcal{D}^*$. Moreover, locally given by,

$$H(x^i, p_a) = v^a p_a - L(x^i, v^a), \text{ where } p_a = \frac{\partial L}{\partial v^a}.$$

Secondly, using the fiber derivative of H, we define the Legendre transformation of the sub-Hamiltonian H in the following way:

$$\mathcal{L}_H : \mathcal{D}^* \longrightarrow \mathcal{D},$$

For any $p, q \in \mathcal{D}^*_x$, it holds

$$q(\mathcal{L}_H(p)) = \frac{d}{dt}H(x, p + tq).$$

This locally relates the distribution $\mathcal{D}^*$ of the cotangent bundle and the distribution $\mathcal{D}$ of the tangent bundle according to the next expression:

$$\mathcal{L}_H(x^i, p_a) = (x^i, \frac{\partial H}{\partial p_a}).$$

Naturally, $\mathcal{L}_L$ and $\mathcal{L}_H$ are inverses of each other:

$$\mathcal{L}_H \circ \mathcal{L}_L = 1_{\mathcal{D}}, \qquad \mathcal{L}_L \circ \mathcal{L}_H = 1_{\mathcal{D}^*}.$$

In other hand, for every $p \in \mathcal{D}^*_x$, one can define the sub-Finsler metric $F^* \in \widetilde{\mathcal{D}}^* \sim T^*M \setminus \mathcal{D}^0$ with help of the indicatrix I_x as follows:

$$F^*_x(p) := \sup_{w \in I_x} p(w) = \sup_{0 \neq v \in \mathcal{D}_x} p[\frac{v}{F_x(v)}].$$

Observed that $\widetilde{\mathcal{D}}^*$ is the subbundle of the cotangent bundle obtained by removing the zero cotangent vector from each fibre. In fact, F^* turns out to meet the same properties that mentioned in Definition 2.1, but on $\mathcal{D}^*$ instead of $\mathcal{D}$. Then

$$F^*(p) = F(v), \text{ where } p = \mathcal{L}_L(v), \quad \text{and} \quad H := \frac{1}{2}(F^*)^2,$$

see details in [5].

4. Sub-Finsler bundle

We define in this section a sub-Finsler vector bundle which will play a major role in the formalization of the sub-Hamiltonian in sub-Finsler geometry. Let us consider first the covector subbundle $(\mathcal{D}^*, \tau, M)$ with the projection $\tau : \mathcal{D}^* \longrightarrow M$, which is a subbundle of rank k $(= \dim \mathcal{D}^*)$ in the cotangent bundle of T^*M. The illustrious role in our consideration will play by the pullback bundle $\tau^*(\tau) = (\mathcal{D}^* \times \mathcal{D}^*, \mathrm{pr}_1, \mathcal{D}^*)$ of τ by τ as follows:

$$\mathcal{D}^* \times_M \mathcal{D}^* := \{(p, q) \in \mathcal{D}^* \times \mathcal{D}^* |\ \ \tau(p) = \tau(q)\},$$

$$\mathrm{pr}_1 : \mathcal{D}^* \times_M \mathcal{D}^* \longrightarrow \mathcal{D}^*,\ (p, q) \mapsto p.$$

Throughout, we call the above pullback bundle as the *sub-Finsler bundle* over $\mathcal{D}^*$. Now, if p is fixed, then

$$\begin{aligned}(\mathrm{pr}_1)^{-1}(p) &= \{(p, q) \in \mathcal{D}^* \times \mathcal{D}^* |\ \ q \in \mathcal{D}^*_{\tau(q)}\} \\ &= \{p\} \times \mathcal{D}^*_{\tau(p)},\end{aligned}$$

is a fiber of the sub-Finsler bundle over $p \in \mathcal{D}^*$.

We can introduce a Riemannian metric g^* on the sub-Finsler vector bundle induced by the sub-Hamiltonian H as follows:

$$\langle q, r\rangle_p = g^*_p(q, r) := \frac{\partial^2 H(p + tq + sr)}{\partial t \partial s}|_{t,s=0} \qquad \text{for all } q, r \in \mathcal{D}^*_{\tau(p)},$$

which locally means

$$g^{*ij} = \frac{\partial^2 H}{\partial p_i \partial p_j}.$$

Now the sub-Finsler bundle $\tau^*(\tau)$ allows k covector fields $X_1, X_2, \ldots, X_k$ which form an orthonormal frame with respect to the induced Riemannian metric g^*.

Notice that $X_i(p)$ is a covector field that depends on the position $x \in M$ and the direction $p \in \mathcal{D}^*$. Moreover, one can choose in a way that $X_i(p)$ is a homogeneous of degree zero in p, i.e. $X_i(tp) = t^0 X_i(p) = X_i(p)$. According to the above metric g^{*ij} on M which is homogeneous of degree zero, we could generate a new formalism of the sub-Hamiltonian function in the components p_i (induces naturally by the inner product, see [6])

$$H(x, p) = \frac{1}{2} \sum_{i,j=1}^{n} g^{*ij} p_i p_j, \tag{4.1}$$

such that this metric defined in the extended Finsler metric which was shown in [2]. We can write the sub-Hamiltonian function (4.1) in a more useful way using the orthonormality of X_i as follows

$$H(x, p) = \frac{1}{2} \sum_{i=1}^{k} \langle p, X_i(p)\rangle^2, \qquad p \in \mathcal{D}^*_x. \tag{4.2}$$

One can easily check the homogeneity of degree 2 in p of the sub-Hamiltonian function $H(x, p)$:

$$H(x, tp) = \frac{1}{2} \sum_{i=1}^{k} \langle tp, X_i(tp)\rangle^2 = \frac{t^2}{2} \sum_{i=1}^{k} \langle p, X_i(p)\rangle^2 = t^2 H(x, p). \tag{4.3}$$

The importance of $H(x, p)$ is to define sub-Finslerian geodesics. Our function $H(x, p)$ produces a system of sub-Hamiltonian differential equations, since it is a smooth function on $\mathcal{D}^*$. Such differential equations are in terms of canonical coordinates (x^i, p_i).

Definition 4.1. *The generated sub-Hamiltonian differential equations*

$$\dot{x}^i = \frac{\partial H}{\partial p_i}(x,p),$$
$$\dot{p}_i = -\frac{\partial H}{\partial x^i}(x,p), \quad i = 1,\dots,n,$$

are called normal geodesic equations.

Lemma 4.2. *If $\xi(t) := (x(t), p(t))$ is a solution of the sub-Hamiltonian system for all $t \in \mathbb{R}$, then there exists a constant $c \in \mathbb{R}$ such that $H(x(t), p(t)) = c$.*

Proof. Taking the derivative of $H(x(t), p(t))$ w.r.t. t, we get

$$\frac{d}{dt}H(x(t),p(t)) = \frac{\partial H}{\partial x^i}(x(t),p(t))\dot{x}(t) + \frac{\partial H}{\partial p_i}(x(t),p(t))\dot{p}(t).$$

Replacing $\dot{x}(t)$ and $\dot{p}(t)$ by the above sub-Hamiltonian differential equations in the Definition 4.1, we obtain

$$\begin{aligned}\frac{d}{dt}H(x(t),p(t))) &= \frac{\partial H}{\partial x^i}(x(t),p(t))\frac{\partial H}{\partial p_i}(x(t),p(t)) - \frac{\partial H}{\partial p_i}(x(t),p(t))\frac{\partial H}{\partial x^i}(x(t),p(t)) \\ &= 0.\end{aligned}$$

Therefore $H(x(t), p(t))$ is constant. □

Remark 4.3. *From Lemma 4.2, it follows that any solution $\xi(t) := (x(t), p(t))$ of the sub-Hamiltonian differential equations on $\mathcal{D}^*$ for a sub-Hamiltonian function $H(p)$ satisfies $H(x(t), p(t)) = c$. Let the projection $x(t) = \tau(\xi(t)) \in M$, so each sufficiently short subarc of $x(t)$ is a minimizer sub-Finslerian geodesic,* (see [11, Corollary 2.2])*. In addition, this subarc is the unique minimizer joining its end points.*

The projection curve $x(t)$ mentioned above is said to be the *normal sub-Finslerian geodesics* or simply the *normal geodesics.*

Remark 4.4. *In the sub-Finslerian geometry, not all the sub-Finslerian geodesics are normal (contrary to the Finsler geometry). This is due to the fact that the sub-Finslerian geodesics which are also a minimizing geodesic might not be solved the sub-Hamiltonian system. Those minimizer that are not normal geodesics called singular or abnormal geodesics (see* [9] *for more details).*

Moreover, we call the extremal pair $\xi(t) = (x(t), p(t))$ a *normal extremal* if it is a solution for the sub-Hamiltonian system, otherwise it is called an *abnormal extremal.*

Turning to the relationship between the normal geodesic and the locally length-minimizing horizontal curves, Calin et al. proved in [6] that any normal geodesic is a horizontal curve and a locally length-minimizing horizontal curve. After all, by using (4.2) one can generate the system of differential equations in terms of canonical coordinates (x, p) as follows:

$$\dot{x}^i = \frac{\partial H}{\partial p_i} = \sum_{j=1}^{k}\langle p, X_j(p)\rangle\ (\delta_i(X_j(p)) + \langle p, D_{p_i}X_j(p)\rangle), \tag{4.4}$$

$$\dot{p}_i = -\frac{\partial H}{\partial x^i} = -\sum_{j=1}^{k}\langle p, X_j(p)\rangle\langle p, D_{x^i}X_j(p)\rangle, \tag{4.5}$$

where δ_i is the i-th coordinate function.

5. Exponential map in sub-Finsler geometry

Let (M, d) be a general metric space, such that M is an n-dimensional manifold and the function $d : M \times M \longrightarrow \mathbb{R}^+ \cup \{\infty\}$, is called a metric if have the following properties: for all $x, y, z \in M$,

(i) $d(x, y) = 0$, with equality if and only if $x = y$;
(ii) $d(x, y) + d(y, z) \leq d(x, z)$.

If the function d is an asymmetric, then we can define the forward metric balls and forward metric spheres, with center $x \in M$ and radius $r > 0$ as follows:

$$B_x(r) = \{\ y \in M :\ d(x, y) < r\},$$
$$S_x(r) = \{\ y \in M :\ d(x, y) = r\}.$$

The cotangent balls and the cotangent spheres in $\mathcal{D}^*$ are defined as follows:

$$\mathcal{B}^*_x(r) = \{\ p \in \mathcal{D}^* :\ F^*_x(p) < r\},$$
$$\mathcal{S}^*_x(r) = \{\ p \in \mathcal{D}^* :\ F^*_x(p) = r\},$$

for any fix $x \in M$ and radius r.

A subset $U \subset M$ is said to be open if, for each point $x \in U$, there is a forward metric ball about x contained in U. Then we get the topology on M and all metric spaces are first countable and T_1-spaces. In general, we assume that the metric d of any metric space (M, d) is continuous with respect to the product topology on $M \times M$. Thus, every backward metric ball, i.e. $B^-_x(r) = \{\ y \in M :\ d(y, x) < r\}$, is open and the metric space is a Hausdorff (T_2) space. Hence the compact sets in such a space are closed.

As a result of the above, we immediately have the following

Proposition 5.1. *In a metric space (M, d) the following are equivalent:*

(i) *A sequence $\{x_k\}$ in (M, d) converges to $x \in M$ in the sense of topology.*
(ii) $\lim_{k\to\infty} d(x, x_k) = 0$.

Proposition 5.2. *Let x be any point in a (reversible) sub-Finslerian manifold M, and $\bar{B}_x(r)$ is a compact ball, for some $r > 0$. Then for any $y \in B_x(r)$ there is a minimizing geodesic from x to y, that is,*

$$d(x, y) = \min\{\ell(\gamma)\ |\gamma : [0, T] \longrightarrow M \text{ horizontal},\ \gamma(0) = x, \gamma(T) = y\}.$$

Proof. Fix $y \in B_x(r)$ and suppose that $\gamma_k : [0, T] \longrightarrow M$ is a minimizing sequence of horizontal paths with unit speed from x to y and such that

$$\lim_{k\to\infty} \gamma_k(0) = x, \quad \lim_{k\to\infty} \gamma_k(T) = y, \quad \lim_{k\to\infty} \ell(\gamma_k) = d(x, y).$$

For the reason that $d(x, y) < r$, we get $\ell(\gamma_k) \leq r$ for all $k \geq k_0$ large enough. Proposition 5.1 asserts that the metric d is continuous under the topology of the manifold and the reversibility of F holds on a compact set. Consequently, any sequence γ_k of curves which have uniformly bounded lengths has an uniformly convergent subsequence (Ascoli-Arzela theorem), we denote this subsequence by the same symbol, and a Lipschitz curve $\gamma : [0, T] \longrightarrow M$.

From above one can assume that $\gamma_k : [0, T] \longrightarrow M$ is a convergent subsequence of length minimizers parametrized by arc length (i.e. $F(\dot{\gamma}(t)) = 1$) on M such that such that $\gamma_k \longrightarrow \gamma$ uniformly on $[0, T]$. This gives that

$$\ell(\gamma_k) = d(\gamma_k(0), \gamma_k(T)),$$

which is due to the claim that γ_k is a minimizing geodesic. The sequence γ_k converges uniformly if for every $\epsilon > 0$ there is a natural number N such that for all $n \geq N$ and all $t \in [0, T]$ one has $d(\gamma_k(t), \gamma(t)) < \epsilon$. Further, the semicontinuity of the length implies that if $\lim_{k\to\infty} \gamma_k = \gamma$ then

$$\ell(\gamma) \leq \lim_{k\to\infty} \inf\ \ell(\gamma_k).$$

Now, by continuity of the distance, we obtain

$$\ell(\gamma) \leq \lim_{k\to\infty} \inf\ \ell(\gamma_k) = \lim_{k\to\infty} \inf\ d(\gamma_k(0), \gamma_k(T)) = d(\gamma(0), \gamma(T)).$$

This yields that γ is minimizing geodesic, i.e. $\ell(\gamma) = d(x, y)$. The horizontal property of γ follows in the same way as was done in [1], Theorem 3.41. $\square$

Next, we define the exponential map. For the general case, roughly speaking, if M is a smooth Finsler manifold, x a point in M and $u \in T_xM$. Then the exponential map is given by

$$\exp_x : T_xM \longrightarrow M,$$

such that $\exp_x(u) = \gamma_u(1)$ for the unique geodesic γ that starts at x and has initial speed vector u. Furthermore, in the dual space the exponential map for every $x \in M$ and $p \in T_x^*M$ defined by

$$\exp_x^* : T_x^*M \longrightarrow M,$$

such that $\exp_x^*(p) = \gamma_p(1)$ for the unique geodesic γ that starts at x and has initial speed vector $u = \mathcal{L}_L^{-1}(p)$, where L here is the Lagrangian of the Finsler manifold.

The exponential map is an essential object in sub-Finslerian geometry, parametrizing normal extremals through their initial covectors. We are going to define the exponential map in both of the distribution $\mathcal{D}, \mathcal{D}^*$ of the tangent and the cotangent bundles respectively.

Definition 5.3. *Let $\Omega_x \subset \mathcal{D}_x$ be the domain of the exponential map over $x \in M$ such that Ω_x given by*

$$\Omega_x = \{v \in \mathcal{D}_x |\ \xi \text{ is defined on the interval } [0, 1]\},$$

where $v = \mathcal{L}_H(p)$ by the Legendre transformation of sub-Hamiltonian H, and $\xi(t)$ is the normal extremal. Then the sub-Finsler exponential map *is defined as follows*

$$\exp_x : \Omega_x \subset \mathcal{D}_x \subset T_xM \longrightarrow M,\ v \mapsto \pi_{\mathcal{D}}(\mathcal{L}_H(\xi(1))).$$

We can do the same in the distribution $\mathcal{D}_x^*$. Let $\Omega_x^* \subset \mathcal{D}_x^*$ be the domain of the exponential map over $x \in M$ such that Ω_x^* given by

$$\Omega_x^* = \{p \in \mathcal{D}_x^* |\ \xi \text{ is defined on the interval } [0, 1]\}.$$

Consequently, the sub-Hamiltonian exponential map *is given by*

$$\exp_x^* : \Omega_x^* \subset \mathcal{D}_x^* \subset T_x^*M \longrightarrow M,\ p \mapsto \tau(\xi(1)),$$

where $\xi(t)$ is the same normal extremal as above. The set Ω_x^* contains the origin and star-shaped with respect to 0. Moreover, with the help of Legendre transformation it is fairly easy to see that

$$\exp_x(v) = \exp_x^*(p), \quad \text{where} \quad p = \mathcal{L}_L(v). \tag{5.1}$$

It follows that the normal sub-Finslerian geodesics $x(t) = \tau(\xi(t))$ satisfies

$$x(t) = \exp_x^*(tp), \quad \text{for all } t \in [0, T].$$

Theorem 5.4. *The exponential mapping $\exp_x^*$ is a local diffeomorphism on $\mathcal{D}_x^* \subset T_x^*M \backslash \{0\}$.*

Proof. In 4.3, we show the homogeneity of the sub-Hamiltonian function $H(x, p)$ with respect to p. So, for any constant $a > 0$, the curve $\xi(at) : (\epsilon/a, \epsilon/a) \longrightarrow M$ is the same geodesic satisfying the initial conditions $\tau(\xi_p(0)) = x$ and $\xi_p(0) = ap$, i.e.,

$$\tau(\xi_p(at)) = \tau(\xi_{ap}(t)).$$

Since the sub-Hamiltonian vector field

$$\vec{H}(x, p) = g^{ab}(x, p) p_b \frac{\partial}{\partial x^a} - \frac{1}{2} \frac{\partial g^{ab}}{\partial x^k}(x, p) p_a p_b \frac{\partial}{\partial p_k},$$

that introduced in [2], is smooth except for $p = 0$ where it is C^1. Then $\exp_x^*$ is C^∞ on $\mathcal{D}_x^* \subset T_x^*M \backslash \{0\}$, while it is C^1 at $p = 0$ and $d(\exp_x^*)|_0 = \text{id}$. Thus, $\exp_x^*$ is a local diffeomorphism. $\square$

By equation (5.1), one can get the following

Corollary 5.5. *The sub-Finsler exponential map* $\exp_x$ *is a* C^∞ *away from the zero section of* $\mathcal{D}$ *and only* C^1 *at the zero section such that for each* $x \in M$

$$d(\exp_x)_{|0} : \Omega_x \subset \mathcal{D}_x \longrightarrow \Omega_x \subset \mathcal{D}_x,$$

is the identity map at the origin $0 \in \mathcal{D}_x$.

Remark 5.6. *It is clear that in the case of sub-Finsler exponential map the following expressions holds:*

$$\exp_x^*[\mathcal{B}_x^*(r)] = B_x(r),$$
$$\exp_x^*[\mathcal{S}_x^*(r)] = S_x(r),$$

which are analogous to the Finslerian context, see Bao et al. [5] *for more details.*

Remark 5.7. *Turning to sub-Riemannian case, Strichartz in* [13] *stated that for bracket generating distributions the exponential map is a local diffeomorphism. This is due to the fact that the solutions of the sub-Hamiltonian system depend differentially on the initial data. But this is a difference from the Riemannian context, the exponential map is not a diffeomorphism at the origin just like the Finslerian case.*

6. Hopf-Rinow Theorem in sub-Finslerian geometry

In the following, one can see the explanation of the terms that will be used in Hopf-Rinow Theorem. A sub-Finsler manifold is said to be *forward complete* if every forward Cauchy sequence converges, and it is a *forward geodesically complete* if every normal geodesic $\gamma(t), t \in [0, T)$ parametrized to have unit speed, can be extended to a geodesic for all $t \in [0, \infty)$. A subset is said to be *forward bounded* if it is contained in some forward metric ball $B_x(r)$.

Theorem 6.1. *Let* $(M, \mathcal{D}, F)$ *be any connected sub-Finsler manifold, where* $\mathcal{D}$ *is bracket generating distribution. The following conditions are equivalent:*

(i) *The metric space* (M, d) *is forward complete.*
(ii) *The sub-Finsler manifold* $(M, \mathcal{D}, F)$ *is forward geodesically complete.*
(iii) $\Omega_x^* = \mathcal{D}_x^*$, *additionally, the exponential map is onto if there are no strictly abnormal minimizer.*
(iv) *Every closed and forward bounded subset of* (M, d) *is compact.*

Furthermore, for any $x, y \in M$ *there exists a minimizing geodesic* γ *joining* x *to* y, *i.e. the length of this geodesic is equal to the distance between these points.*

Proof. (i) $\Longrightarrow$ (ii) Let $\gamma(t) : [0, T) \longrightarrow M$ be a unit speed and maximally forward extended geodesic, $t \in [0, T)$. If we assume that $T \neq \infty$, and choose a sequence $\{t_i\} \longrightarrow T$ in $[0, T)$ then $\gamma(t_i)$ is forward Cauchy, since

$$d(\gamma(t_i), \gamma(t_j)) \leq |t_j - t_i|, \text{ for all } i \leq j.$$

Now, (i) makes it obvious that $\gamma(t_i)$ converges to $y \in M$. On one hand, let us define $\gamma(T)$ to be y. On the other hand, Lemma 4.1 in [13] told us that $\gamma(t)$ can be extended beyond $t = T$. This contradicts our assumption the fact that $T \neq \infty$. Thus, $T = \infty$ for sure, so we have the forward geodesically completeness.

(ii) $\Longrightarrow$ (iii) It is sufficient (for first part $\Omega_x^* = \mathcal{D}_x^*$) to prove that any normal extremal pair $\xi(t)$, starting from the initial conditions, is defined for all $t \in \mathbb{R}$. Suppose that the normal extremal is not extendable to the some interval $[0, T + \delta)$ for all $\delta > 0$ and suppose that it is defined on $[0, T)$. Let $\{t_i\}$ be any increasing sequence such that the limit of this sequence is T. Hence, the projection $x(t) = \tau(\xi(t))$ is a curve with unit speed defined on $[0, T)$, therefore, the sequence $\{t_i\}$ is a forward Cauchy sequence on M, since

$$d(x(t_i), x(t_j)) \leq |t_i - t_j|.$$

By completeness, it follows that the sequence $x(t_i)$ converges to some point $y \in M$. We suppose there are coordinates around the point y and an orthonormal frame $X_1, X_2, ..., X_k$ in small ball $\mathcal{B}_y^*(r)$ in the sub-Finsler bundle. Let us show that in the coordinates $\xi(t) = (x(t), p(t))$ the curve $x(t)$ is uniformly bounded. This grants a contradiction that the normal extremal is not extendable. In fact, for every $p \in \mathcal{D}^*$, we consider the following non-negative form (4.2) of the sub-Hamiltonian function H:

$$H(x,p) = \frac{1}{2}\sum_{i=1}^{k}\langle p, X_i(p)\rangle^2.$$

Then, the sub-Hamiltonian system has the form:

$$\dot{x}^i(t) = \frac{\partial H}{\partial p_i}(x(t),p(t)) = \sum_{j=1}^{k}\langle p(t), X_j(p(t))\rangle(\delta_i(X_j(p)) + \langle p, D_{p_i}X_j(p)\rangle),$$

$$\dot{p}_i(t) = -\frac{\partial H}{\partial x^i}(x(t),p(t)) = -\sum_{j=1}^{k}\langle p(t), X_j(p(t))\rangle\langle p(t), D_{x^i}X_j(p(t))\rangle,$$

for $t \in [T-\delta, T)$ with $\delta > 0$ small enough. Since $D_{\gamma(t)}X_i$ are given in a compact small ball $\bar{\mathcal{B}}^*_y(r)$, they are bounded, so there is a constant $\mathcal{C} > 0$ such that

$$|\dot{p}(t)| \leq \mathcal{C}|p(t)| \quad \forall t \in [T-\delta, T).$$

If we apply Gronwall's Lemma (see [12], p.122), it leads us to that $|p(t)|$ is uniformly bounded on a bounded interval. This contradicts our assumption that the normal extremal can not be extended beyond T.

(iii) $\Longrightarrow$ (iv) Assume that $\bar{A}$ is a closed and forward bounded subset of (M,d). Applying the bracket generating assumption, for every $y \in \bar{A}$, Proposition 5.2 asserts that there is a minimizing geodesic $\exp^*_x(tp_y)$, $0 \leq t \leq T$, from x to y. The set of all p_y is subset A of $\mathcal{D}^*_x$. Since $F^*_x(p_y) = d(x,y)$, and $d(x,y) \leq r$ for some r due to the forward boundedness of $\bar{A}$, the subset A is bounded and contained in the compact set $\mathcal{B}^*_x(r) \cup \mathcal{S}^*_x(r)$. By Remark 5.6, $\exp^*_x[\mathcal{B}^*_x(r) \cup \mathcal{S}^*_x(r)]$ is compact and contained in the closed set $\bar{A}$, then $\bar{A}$ it must be compact.

(iv) $\Longrightarrow$ (i) Let $\{x_i\}$ be a forward Cauchy sequence in M, and by the subadditivity it must be forward bounded. Choose $A := \{x_i | i \in \mathbb{N}\}$, then its closure $\bar{A}$ is still forward bounded under the manifold topology of M. Taking into account the assumption (iv), $\bar{A}$ should satisfy the compactness property, therefore, the sequence $\{x_i\}$ contains a convergent subsequence.

Let $\{x_k\}$ be a convergent subsequence, consider it converges to some $y \in \bar{A} \subset M$. In other hand, we need to check that $\{x_i\}$ converges to $y \in \bar{A} \subset M$. To do this, fix $\epsilon > 0$, since $\{x_i\}$ is forward Cauchy, there exist a positive number n_0 such that $j > i \geq n_0$, then

$$d(x_i, x_j) < \frac{\epsilon}{2}.$$

At the same time $\{x_k\}$ converge to y. So there is a positive number n_1 such that if $k \geq n_1$, then

$$d(x_k, y) < \frac{\epsilon}{2}.$$

One can assume that n is greater than n_0 and n_1. If needed, by expanding n further, there is no loss of generality in assuming that n indeed equals some index of the convergent subsequence. Then $d(x_n, y) \leq \dfrac{\epsilon}{2}$, so, for $i > n$, we get

$$d(x_i, y) \;\leq\; d(x_i, x_n) + d(x_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So, we have been shown that every forward Cauchy sequence is convergent. Hence (M,d) is forward complete.

At the end, we can use the same proof of Proposition 5.2 to verify that for every $x, y \in M$ there exists a length minimizing geodesic joining x and y, and it has to be normal geodesic by Remark 4.3. Finally, the property of compactness and completeness with help of Proposition 5.2, proves the second part of (iii).

□

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