ROTATIONALLY SYMMETRIC TRANSLATORS OF THE GAUSS CURVATURE FLOW

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Abstract. We completely classify in Euclidean 3−space the rotational translators of the flow by powers of the Gauss curvature. This classification is also extended to Lorentz-Minkowski 3−space.

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1. Introduction

Let $X : \Sigma \to \mathbb{R}^3$ be a smooth immersion of a strictly convex surface Σ in $\mathbb{R}^3$. The K^α-*flow*, $\alpha \in \mathbb{R} - \{0\}$, is a one-parameter family of smooth immersions $X(\cdot, t) : \Sigma \to \mathbb{R}^3$, $t \in [0, T)$, $X_0 = X$, such that

$$\frac{\partial X}{\partial t}(p,t) = -K^\alpha(p,t)N(p,t), \quad (p,t) \in \Sigma \times [0,T),$$

where $N(p, t)$ is the unit normal and $K(p, t)$ is the Gauss curvature at $X(p, t)$. The origin of this theory lies in the work of Firey [12]. See also [11, 20]. With regard to geometric analysis, it is of interest, see for example [1, 3, 21].

As particular solutions of K^α-flow, it is a mainstream to seek self-similar solutions, especially homothetic solutions (the surface moves by homothety) and translating solutions (the surface moves by translation). We will be interested in translating solutions of K^α-flow, which we call K^α-*translators* (see [10, 18, 22]).

Let $\vec{v} \in \mathbb{R}^3$ be a fixed unit vector, called the *speed* of the flow. We call a surface Σ a *translator* by K^α-flow with speed $\vec{v}$ if, up to a dilation of $\mathbb{R}^3$,

$$K^\alpha = \langle N, \vec{v} \rangle. \tag{1.1}$$

The particular case $\alpha = 1/4$ in equation (1.1) plays a key role due to that not only the interpretation in affine differential geometry (see [2, 8]) but also it is greatly simplified by a cancellation of terms comparing to the other values of α.

Our purpose is to give examples of K^α-translators. We will neglect those trivial examples of K^α-translators when the Gauss curvature K in equation (1.1) is constant on the surface. Because otherwise equation (1.1) describes a well-known class of surfaces in differential geometry, *constant angle surfaces* [9, 19].

In the case $\vec{v} = (0, 0, 1)$, equation (1.1) writes in nonparametric way $u = u(x, y)$ as

$$\left(\frac{\det D^2 u}{(1 + |Du|^2)^2} \right)^\alpha = \frac{1}{\sqrt{1 + |Du|^2}}. \tag{1.2}$$

If $\alpha = 1/4$ in (1.2), this equation is the Monge-Ampère equation $\det D^2 u = 1$ [14]. For any α (even if $\alpha = 1/4$), equation (1.2) is difficult to solve its generality in all. Hence, it is reasonable to assume some type symmetries on the surface where equation (1.1) (or (1.2)) converts into an ODE. In this paper we will assume that the surface is invariant under a uniparametric group of rotations.

When the ambient is the Lorentz-Minkowski space $\mathbb{R}^3_1$, equation (1.1) is still valid with the difference that the surface Σ is assumed to be spacelike and of positive Gauss curvature [4, 13, 17]. The K^α-translators in $\mathbb{R}^3_1$ can be employed as barriers in order to obtain C^1 and C^2 apriori estimates in the Dirichlet problem associated to the equation (1.1), such as it occurs for the prescribing Gauss curvature equation $K = \mathrm{ct}$ [7, 15, 16].

In this talk, presented at the The International Conference Riemannian Geometry and Applications–RIGA 2023, we will recall (without proofs) the results we obtained on rotational K^α-translators. We classify rotational K^α-translators in Euclidean space $\mathbb{R}^3$. This classification is also extended to Lorentz-Minkowski space $\mathbb{R}^3_1$. In $\mathbb{R}^3_1$, the family of surfaces of revolution is greater than of Euclidean case. According to the causal character of the rotation axis, three types of surfaces appear. In addition, the speed $\vec{v}$ has again three possible causal choices.

The proofs are included in the papers [5, 6].

2. Rotational K^α-translators in Euclidean setting

Let Σ be a surface of revolution in Euclidean space $\mathbb{R}^3$ with rotation axis L. After a rigid motion, we may assume that L is the z-axis. Then, a parametrization of Σ writes as

$$X(r,\theta) = (r\cos\theta, r\sin\theta, f(r)), \tag{2.1}$$

where $f : I \to \mathbb{R}$, $I \subset \mathbb{R}^+$, is a C^2-function.

Assume now that Σ is a K^α-translator with speed $\vec{v}$. A natural question is if there is a relation between $\vec{v}$ and the rotation axis. By [5, Proposition 1], as expectable, we understand that $\vec{v}$ must be parallel to the z-axis. Hence, it follows $\vec{v} = (0, 0, 1)$ after a symmetry about the xy-plane if necessary.

The following result completely classifies rotational K^α-translators.

Theorem 2.1. [5] *Let Σ be a K^α-translator. If Σ is a surface of revolution about the z-axis, then Σ is a circular cylinder of arbitrary radius or Σ parametrizes as* (2.1) *where*

$$f(r) = \begin{cases} \pm\int^r \left(\frac{1}{m}e^{t^2} - 1\right)^{1/2} dt, m > 0, & \alpha = \frac{1}{2} \\ \pm\int^r \left(\left(m - \frac{2\alpha-1}{2\alpha}t^2\right)^{\frac{2\alpha}{1-2\alpha}} - 1\right)^{1/2} dt, m \in \mathbb{R}, & \alpha \neq \frac{1}{2}. \end{cases} \tag{2.2}$$

Furthermore, the maximal domain of the function $f(r)$ is

(1) $[\sqrt{\log m}, \infty)$, *if $\alpha = 1/2$.*

(2) $[\sqrt{\frac{2\alpha}{2\alpha-1}m}, \infty)$, *if $\alpha \in (0, 1/2)$.*

(3) $[\sqrt{\frac{2\alpha}{2\alpha-1}(m-1)}, \sqrt{\frac{2\alpha}{2\alpha-1}m})$, *if $\alpha \notin [0, 1/2]$. In this case, we have*

$$\lim_{r\to\sqrt{\frac{2\alpha}{2\alpha-1}(m-1)}} f'(r) = 0, \qquad \lim_{r\to\sqrt{\frac{2\alpha}{2\alpha-1}m}} f(r) = \infty.$$

In all these cases, we understand that if in the radicand in the left-end of the interval is negative, then the value of this end is 0.

As addressed in Sect. 1, the particular value $\alpha = 1/4$ is special and one can explicitly obtain the solutions of equation (2.2).

Corollary 2.2. [5] *Rotational $K^{1/4}$-translators form a uniparametric family of surfaces parametrized by* (2.1)*, where*

$$f(r) = \frac{1}{2}\left(r\sqrt{m + r^2 - 1} + (m-1)\log\left(\sqrt{m + r^2 - 1} + r\right)\right) + c, \quad m, c \in \mathbb{R}.$$

The maximal domain of f is $[\sqrt{1-m},\infty)$ if $m<1$ and $[0,\infty)$ if $m\geq 1$. For the value $m=1$, f is the parabola $f(r)=r^2/2$, the graphic of $f(r)$ orthogonally intersects the rotation axis and Σ is a paraboloid.

In what follows, we study the situation that the generating curves of K^α-translators meet the rotation axis at a right angle independently from the the value of α.

Corollary 2.3. [5] *For each α, there are rotational K^α-translators whose generating curves intersect orthogonally the rotation axis. These surfaces are unique up to vertical translations. Furthermore,*

(1) If $\alpha\in(0,\frac{1}{2}]$, the maximal domain of f is $[0,\infty)$, $\lim_{r\to\infty}f(r)=\infty$ and

$$f(r)=(1-2\alpha)\left(\frac{1-2\alpha}{2\alpha}\right)^{\frac{\alpha}{1-2\alpha}}r^{\frac{1}{1-2\alpha}}+o(r^{\frac{1}{1-2\alpha}}).$$

(2) If $\alpha\notin[0,\frac{1}{2}]$, the maximal domain is $[0,\sqrt{\frac{2\alpha}{2\alpha-1}})$, with

$$\lim_{r\to\sqrt{\frac{2\alpha}{2\alpha-1}}}f(r)=\infty.$$

As we will see later, this result cannot be extended in the Lorentzian setting.

3. Rotational K^α-translators in Lorentzian setting

In this section, we first describe the parametrizations of the surfaces of revolution of $\mathbb{R}^3_1$. As pointed out in Sect. 1, there are three types of surfaces of revolution depending on the causal character of the rotation axis L.

Let $B=\{e_1,e_2,e_3\}=\{(1,0,0),(0,1,0),(0,0,1)\}$ be the usual basis of $\mathbb{R}^3$. After a rigid motion, we can assume that $L=\mathrm{sp}\{e_3\}$ (timelike), $L=\mathrm{sp}\{e_1\}$ (spacelike) or $L=\mathrm{sp}\{e_2+e_3\}$ (lightlike).

(1) The axis is timelike, $L=\mathrm{sp}\{e_3\}$. The generating curve is $r\mapsto(r,0,f(r))$, $r>0$, where f is a smooth function and the parametrization of the surface is

$$X(r,\theta)=(r\cos\theta,r\sin\theta,f(r)). \tag{3.1}$$

The surface is spacelike (resp. timelike) if $f'^2<1$ (resp. $f'^2>1$).

(2) The axis is spacelike, $L=\mathrm{sp}\{e_1\}$. The generating curve can be included in the xz-plane or in the xy-plane. In the first case, if the curve is $r\mapsto(f(r),0,r)$ $r>0$, the parametrization of the surface is

$$X(r,\theta)=(f(r),r\sinh\theta,r\cosh\theta). \tag{3.2}$$

The surface is spacelike (resp. timelike) if $f'^2>1$ (resp. $f'^2<1$). If the generating curve is included in the $xy-$plane and if $r\mapsto(f(r),r,0)$, $r>0$, the parametrization of the surface is

$$X(r,\theta)=(f(r),r\cosh\theta,r\sinh\theta).$$

The surface is always timelike.

(3) The axis is lightlike, $L=\mathrm{sp}\{e_2+e_3\}$. The generating curve is $r\mapsto(0,f(r)+r,f(r)-r)$, $r>0$, and the surface is

$$X(r,t)=(2rt,f(r)+r-rt^2,f(r)-r-rt^2). \tag{3.3}$$

The surface is spacelike (resp. timelike) if $f'>0$ (resp. $f'<0$).

Let Σ be a rotational K^α-translator in $\mathbb{R}^3_1$ with speed $\vec{v}$. As in the Euclidean case, the vector $\vec{v}$ must be parallel to the rotation axis L independently from the causal character of L. See [6, Propositions 3.1, 3.6, 3.10].

3.1. **The axis is timelike.** The first result classifies all rotational K^α-translators with timelike axis, distinguishing the case that $\alpha = 1/4$.

Theorem 3.1. [6] *Any rotational K^α-translator with timelike axis is parametrized as* (2.1)*, where*

$$f(r) = \begin{cases} \pm\int^r \left(1 - \frac{1}{m^2}e^{t^2}\right)^{1/2} dt, m > 1, & \alpha = \frac{1}{2} \\ \pm\int^r \left(1 - \left(m - \frac{2\alpha-1}{2\alpha}t^2\right)^{\frac{2\alpha}{1-2\alpha}}\right)^{1/2} dt, m \in \mathbb{R}, & \alpha \neq \frac{1}{2}. \end{cases}$$

The maximal domain of the above function $f(r)$ is:

(1) *Case $\alpha = 1/2$: $(0, \sqrt{\log m^2})$, where $m > 1$.*
(2) *Case $\alpha \in (0, 1/2)$: $(\sqrt{\frac{2\alpha}{2\alpha-1}m}, \sqrt{\frac{2\alpha}{1-2\alpha}(1-m)})$ if $m < 0$ or $(0, \sqrt{\frac{2\alpha}{1-2\alpha}(1-m)})$ if $0 \leq m < 1$.*
(3) *Case $\alpha \notin [0, 1/2]$: $(0, \sqrt{\frac{2\alpha}{2\alpha-1}(m-1)})$ and $m > 1$.*

Corollary 3.2. [6] *Rotational $K^{1/4}$-translators with timelike axis parametrize as* (3.1)*, where*

$$f(r) = \pm\frac{1}{2}\left(r\sqrt{1-m-r^2} - (m-1)\tan^{-1}\left(\frac{r}{\sqrt{1-m-r^2}}\right)\right),$$

where $r \in (0, \sqrt{1-m})$ and $m < 1$.

As in Sect. 2, we investigate the situation that the generating curve intersects the rotation axis. This means that the generating curve $r \mapsto (r, 0, f(r))$ is defined at the limit at $r = 0$. In the following, if the rotation axis is timelike, we will see that Corollary 2.3 is now invalid. Furthermore, we notice the existence of a different important behaviour.

Corollary 3.3. [6] *There are no rotational K^α-translators with timelike axis intersecting orthogonally the rotation axis. On the other hand, if $\alpha \in (0, 1/2)$, there are rotational K^α-translators with timelike axis intersecting the rotation axis at a conical point.*

3.2. **The axis is spacelike.** We first recall the classification results.

Theorem 3.4. [6] *Any rotational K^α-translator with spacelike axis is parametrized as* (3.2)*, where*

$$f(r) = \begin{cases} \pm\int^r \left(1 + \frac{1}{m^2}e^{-t^2}\right)^{1/2} dt, m > 0, & \alpha = \frac{1}{2} \\ \pm\int^r \left(1 + \left(m + \frac{2\alpha-1}{2\alpha}t^2\right)^{\frac{2\alpha}{1-2\alpha}}\right)^{1/2} dt, m \in \mathbb{R}, & \alpha \neq \frac{1}{2}. \end{cases}$$

The maximal domain of the above function $f(r)$ is:

(1) *Case $\alpha = 1/2$. The domain is $(0, \infty)$.*
(2) *Case $\alpha \in (0, 1/2)$. The domain is $(0, \sqrt{\frac{2\alpha}{1-2\alpha}m})$, where $m > 0$.*
(3) *Case $\alpha \notin [0, 1/2]$. The domain is $(0, \infty)$ if $m \geq 0$ or $(\sqrt{\frac{2\alpha}{1-2\alpha}m}, \infty)$ if $m < 0$.*

Corollary 3.5. [6] *Rotational $K^{1/4}$-translators with spacelike axis parametrize as* (3.2)*, where*

$$f(r) = \pm\frac{1}{2}\left(r\sqrt{1+m-r^2} + (1+m)\tan^{-1}\left(\frac{r}{\sqrt{1+m-r^2}}\right)\right),$$

where $r \in (0, \sqrt{1+m})$ and $m > -1$.

The following results differs of Corollary 2.3.

Corollary 3.6. [6] *There are no rotational K^α-translators with spacelike axis and intersecting orthogonally the rotation axis.*

3.3. **The axis is lightlike.**

Theorem 3.7. [6] *Any rotational K^α-translator with lightlike axis is parametrized as* (3.3)*, where*

$$f(r)=\begin{cases} m\int^r e^{4t^2}dt, m>0, & \alpha=\frac{1}{2} \\ \int^r \left(\frac{2(1-2\alpha)}{\alpha}t^2+m\right)^{\frac{2\alpha}{1-2\alpha}}dt, m\in\mathbb{R}, & \alpha\neq\frac{1}{2}. \end{cases}$$

The maximal domain of the above function $f(r)$ is:

(1) *Case $\alpha=1/2$. The domain is $(0,\infty)$.*
(2) *Case $\alpha\in(0,1/2)$. The domain is $(0,\infty)$ if $m\geq 0$ or $(\sqrt{\frac{\alpha}{2(2\alpha-1)}m},\infty)$ if $m<0$.*
(3) *Case $\alpha\notin[0,1/2]$. The domain is $(0,\sqrt{\frac{\alpha}{2(2\alpha-1)}m})$ where $m>0$.*

Corollary 3.8. [6] *Rotational $K^{1/4}$-translators with lightlike axis parametrize as* (3.3)*, where*

$$f(r)=\frac{4}{3}r^3+mr,$$

where $r\in(0,\infty)$ if $m>0$ and $r\in(\sqrt{-m}/2,\infty)$ otherwise.

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