COMPACT WEYL-PARALLEL MANIFOLDS

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ABSTRACT. By ECS manifolds one means pseudo-Riemannian manifolds of dimensions $n \geq 4$ which have parallel Weyl tensor, but not for one of the two obvious reasons: conformal flatness or local symmetry.

As shown by Roter [10, 2], they exist for every $n \geq 4$, and their metrics are always indefinite. The local structure of ECS manifolds has been completely described [3].

Every ECS manifold has an invariant called rank, equal to 1 or 2. Known examples of compact ECS manifolds [4, 6], representing every dimension $n \geq 5$, are of rank 1. When n is odd, some further, recently found examples are locally homogeneous [7]

We outline the proof of the author's result, joint with Ivo Terek [5], which states that a compact rank-one ECS manifold, if not locally homogeneous, replaced if necessary by a two-fold isometric covering, must be the total space of a bundle over the circle.

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1. The Olszak distribution

Given an ECS manifold (M, g), we define its *rank* to be the dimension $d \in \{1, 2\}$ of its *Olszak distribution* $\mathcal{D}$, which is a null parallel distribution on M. See [9] and [3, p. 119].

The sections of the Olszak distribution $\mathcal{D}$ are the vector fields v such that $g(v, \cdot) \wedge [W(v', v'', \cdot, \cdot)] = 0$ for all vector fields v', v''.

Lorentzian ECS manifolds have rank one: the Lorentz signature limits the dimensions of null distributions to at most 1.

2. Compact rank-one ECS manifolds

Examples of compact rank-one ECS manifolds have been found in all dimensions $n \geq 5$ [4, 6].

They are all geodesically complete, and none of them is locally homogeneous. Recently [7], locally homogeneous examples (which are necessarily incomplete) were exhibited in all *odd* dimensions $n \geq 5$.

It is an open question whether a compact ECS manifold may have rank 2, or be of dimension 4.

3. The global structure theorems

All known examples of compact ECS manifolds are diffeomorphic to nontrivial torus bundles over S^1, which reflects a general principle – here is our result [6, Theorem A].

Theorem 3.1. *Every non-locally-homogeneous compact rank-one ECS manifold, replaced if necessary by a two-fold isometric covering, is a bundle over the circle, with the leaves of* $\mathcal{D}^\perp$ *serving as the fibres.*

The proof is outlined in Sections 9 – 12.

One needs the two-fold isometric covering in Theorem 3.1 to make $\mathcal{D}^\perp$ transversally orientable.

It is not known whether the conclusion of Theorem 3.1 remains valid in the locally homogeneous case, although it does then hold if $\mathcal{D}^\perp$ is also assumed to have at least one compact leaf. The examples of locally homogeneous compact rank-one ECS manifolds, constructed in [7], are bundles over the circle.

A further result [6, Theorem B] pertains to universal coverings of compact rank-one ECS manifolds:

Theorem 3.2. *The leaves of $\widehat{\mathcal{D}}^\perp$ in the pseudo-Riemannian universal covering space $(\widehat{M}, \hat{g})$ of any compact rank-one ECS manifold are the factor manifolds of a global product decomposition of $\widehat{M}$.*

Our notation uses hatted versions of symbols such as $g, \mathcal{D}, \mathcal{D}^\perp, \nabla$ (the Levi-Civita connection) and Ric, standing for objects in a given manifold M, to represent their analogs in the universal covering $\widehat{M}$.

4. The dichotomy property for foliations

We refer to a codimension-one foliation $\mathcal{V}$ on a manifold M as having the *dichotomy property* when the following condition is satisfied:

Every compact leaf L of $\mathcal{V}$ has a neighborhood U in M such that the leaves of $\mathcal{V}$ intersecting $U \smallsetminus L$

(i) *are either all noncompact, or*

(ii) *they are all compact, and some neighborhood of L in M forming a union of compact leaves of $\mathcal{V}$ may be diffeomorphically identified with $\mathrm{I\!R} \times L$ so as to make $\mathcal{V}$ the L factor foliation.*

5. Examples of the dichotomy property

Transversal orientability implies the dichotomy property when

(a) M and $\mathcal{V}$ are real-analytic, or

(b) $\mathcal{V}$ has a finite number $r \geq 0$ of compact leaves.

For (a): any value of the leaf holonomy representation, sending a neighborhood of 0 in $\mathrm{I\!R}$ real-analytically into $\mathrm{I\!R}$, must equal Id if it agrees with Id on a nonconstant sequence tending to 0.

Case (b) trivially follows be default. Examples of (b) include the Reeb foliation on S^3, while for any $r \geq 0$ they obviously exist on T^2 and, consequently, on $T^2 \times K$, with any compact manifold K.

"Thickening" a compact leaf L satisfying (i) so as to replace it with the closure of a product-like $\mathcal{V}$-saturated neighborhood of L, one obtains a foliation without the dichotomy property.

The dichotomy property easily follows in the case where $\mathcal{V}$ is the horizontal distribution of a flat linear connection in an orientable (and hence trivial) real line bundle over a compact manifold L, with the total space M. Namely, the zero section L is then a compact leaf, and depending on whether the holonomy group of the connection is infinite or trivial, the bundle has no global parallel sections except L or, respectively, is trivialized by them.

6. The fibration lemma

Lemma 6.1. *Let a transversally-orientable codimension-one foliation $\mathcal{V}$ on a compact manifold M have the dichotomy property of Section* 4*, and some compact leaf L of $\mathcal{V}$ realize the option* (ii)*, so that a product-like $\mathcal{V}$-saturated neighborhood of L in M consists of compact leaves.*

Then the leaves of $\mathcal{V}$ are all compact, and constitute the fibres of a bundle projection $M \to S^1$.

This is [5, Theorem 4.1], and its proof uses the flow $\mathrm{I\!R} \times M \ni (\tau, x) \mapsto \phi(\tau, x) \in M$ of a C^∞ vector field $\mathcal{V}$. One fixes a point z of a leaf satisfying condition (ii) and applies a continuity argument to a maximal segment of the integral curve $\tau \mapsto \phi(\tau, z)$ which intersects compact leaves only.

Once we see that the maximal segment is defined on $(-\infty, \infty)$, while any two leaves can obviously be joined by a piecewise C^∞ curve, the smooth segments of which are integral curves of C^∞ vector fields nowhere tangent to $\mathcal{V}$, our claim becomes reduced to a well-known exercise [8, p. 49]: a transversally-orientable codimension-one foliation with compact leaves, on a compact manifold M, is tangent to the vertical distribution of a fibration $M \to S^1$.

7. The local structure of rank-one ECS manifolds

In coordinates t, s, x^i, where $i, j \in \{3, \ldots, n\}$, the following formula [10], using constants $g_{ij} = g_{ji}$ and $a_{ij} = a_{ji}$, along with a function f of the variable t,

$$\kappa\, dt^2 + dt\, ds + g_{ij} dx^i dx^j, \text{ with } \kappa = f g_{ij} x^i x^j + a_{ij} x^i x^j \tag{7.1}$$

defines a rank-one ECS metric if f is nonconstant, $\det[g_{ij}] \neq 0 = g^{ij} a_{ij}$ and $[a_{ij}] \neq 0$.

Conversely, at generic points (where Ric and $\nabla\mathrm{Ric}$ are nonzero), any rank-one ECS metric has the above form in suitable local coordinates. By lumping a rank-one ECS metrics together with a special narrow class of locally symmetric ones, and allowing f to possibly be constant, one gets rid of the genericity requirement [3, Theorem 4.1]: (7.1) always describes metrics of this more general type, and all such have, in suitable local coordinates, the form (7.1).

8. Proof of the global structure theorem: four steps

(I) We exhibit two functions $t, f : \widehat{M} \to \mathrm{I\!R}$ on the pseudo- Riemannian universal covering space $(\widehat{M}, \hat{g})$ of a fixed compact rank-one ECS manifold (M, g) in which $\mathcal{D}^\perp$ is transversally orientable, and introduce the space $\mathcal{S}$ of all continuous functions $\chi : \widehat{M} \to \mathrm{I\!R}$ such that the 1-form χdt is closed and projectable onto M, along with a linear operator $P : \mathcal{S} \to H^1(M, \mathrm{I\!R})$ given by $P\chi = [\chi dt]$, where χdt is treated as a closed 1-form on M, and closedness of a continuous 1-form means its local exactness.

(II) Using t, we prove the dichotomy property of $\mathcal{D}^\perp$.

(III) If $\dim \mathcal{S} < \infty$, local homogeneity follows.

(IV) When $\dim \mathcal{S} = \infty$, the operator P in (I) is noninjective, and a nontrivial function in its kernel leads, via Sard's theorem, to a compact leaf L of $\mathcal{D}^\perp$ realizing option (ii) of the dichotomy property, which allows us to use Lemma 6.1.

9. Step I: the functions t and f

We have $M = \widehat{M}/\Gamma$ for a group $\Gamma \approx \pi_1 M$ acting on $\widehat{M}$ freely and properly discontinuously via deck transformations so as to preserve $\hat{g}$ and the transversal orientation of $\widehat{\mathcal{D}}^\perp$.

The connection in $\widehat{\mathcal{D}}$ induced by the Levi-Civita connection $\widehat{\nabla}$ of $(\widehat{M}, \hat{g})$ is flat: due to the local-structure formula (7.1), $\widehat{\mathcal{D}}$ is spanned, locally, by the parallel gradient $\widehat{\nabla} t$.

Simple connectivity of $\widehat{M}$ allows us to drop the word 'locally' and choose a global surjective function $t : \widehat{M} \to I$ onto an open interval $I \subseteq \mathrm{I\!R}$ with parallel gradient $\widehat{\nabla} t$, spanning $\widehat{\mathcal{D}}$.

This surjective function $t : \widehat{M} \to I$ is, clearly, unique up to affine substitutions, and may be assumed, via an affine change, to coincide with the coordinate function t in the local-structure formula (7.1).

Also, (7.1) yields $\widehat{\mathrm{Ric}} = (2 - n) f\, dt \otimes dt$, thus defining $f : \widehat{M} \to \mathrm{I\!R}$, which is locally a function of t.

Consequently, any $\gamma \in \Gamma$ gives rise to $q, p \in \mathrm{I\!R}$ with $q > 0$, such that, for $(\,)\dot{} = d/dt$,

$$t \circ \gamma = qt + p, \quad \gamma^* dt = q\, dt, \quad f \circ \gamma = q^{-2} f, \quad \dot{f} \circ \gamma = q^{-3} f. \tag{9.1}$$

Closedness of a continuous 1-form ζ, such as $\chi dt \in \mathcal{S}$, means its being, locally, the differential of a C^1 function. The cohomology class $[\zeta] \in H^1(M, \mathrm{I\!R}) = \mathrm{Hom}(\pi_1 M, \mathrm{I\!R})$ then assigns to a homotopy class of piecewise C^1 loops at a fixed base point the integral of ζ over a representative loop.

This results in a well-defined linear operator $P : \mathcal{S} \to H^1(M, \mathrm{I\!R})$, where $P\chi = [\chi dt]$ and χdt is identified with the projected 1-form on M.

10. Step II: the dichotomy property of $\mathcal{D}^\perp$

The normal bundle of a fixed compact leaf L of $\mathcal{D}^\perp$ is canonically isomorphic, via g, to the line bundle $\mathcal{D}_L^*$ over L dual to $\mathcal{D}_L$ (the restriction of $\mathcal{D}$ to L).

The horizontal distribution of the flat linear connection in $\mathcal{D}_L^*$ arising from the one in the bundle $\mathcal{D}$ (spanned, locally, by the parallel gradients ∇t) corresponds to the distribution $\mathcal{D}^\perp$ under a suitable diffeomorphic identification Ψ of a neighborhood U of L in M with a neighborhood U' of the zero section L in the line bundle $\mathcal{D}_L^*$.

The final paragraph of Section 5, slightly modified, then implies the dichotomy property.

We obtain the required diffeomorphism Ψ using the flow $(\tau, x) \mapsto \phi(\tau, x)$ of a fixed smooth vector field on M, nowhere tangent to $\mathcal{D}^\perp$, and its lift $\hat{\phi}$ to $\widehat{M}$. The resulting integral-curve segments form the fibres of the tubular neighborhood U, and along these segments, pulled back to $\widehat{M}$, denoting by $\pi : \widehat{M} \to M$ the covering projection, we define Ψ by

$$\Psi(\phi(\tau, x)) = [t(\hat{\phi}(\tau, y)) - t(y)]\, \xi_y \circ (d\pi_y)^{-1} \in \mathcal{D}_x^* \subseteq \mathcal{D}_L^*, \text{ with } \pi(y) = x,$$

the parallel section ξ of the line bundle $\widehat{\mathcal{D}}^*$ over $\widehat{M}$ being dual to $\widehat{\nabla} t$ in the sense that $\xi(\widehat{\nabla} t) = 1$. Hence Ψ sends local t-levels to local sections parallel relative to the flat linear connection.

This construction is Γ-equivariant, and hence projects into M.

11. Step III: the case $\dim \mathcal{S} < \infty$

If $\dim \mathcal{S} = m < \infty$, (9.1) and the final line of the text in Section 7 give $|f|^{1/2}, |\dot{f}|^{1/3} \in \mathcal{S}$, while $\mathcal{S}$ is clearly closed under the m-argument operation $(\psi_1, \dots, \psi_m) \mapsto |\psi_1 \dots \psi_m|^{1/m}$. Simple set-theoretical reasons (see Appendix A) now cause $|\dot{f}|^{1/3}$ to equal a constant times multiple of $|f|^{1/2}$, making f globally a function of t, of the form $f = \varepsilon(t - b)^{-2}$ with real constants $\varepsilon \neq 0$ and b.

Combined with a result from algebraic geometry (Whitney's theorem), this implies local homogeneity of g. See Appendix B.

12. Step IV: the case $\dim \mathcal{S} = \infty$

Now $P : \mathcal{S} \to H^1(M, \mathrm{I\!R})$ is clearly noninjective.

Choosing $\chi \in \mathcal{F} \smallsetminus \{0\}$ with $P\chi = 0$, we see that χdt projects onto an exact 1-form on M, that is, onto $d\mu$ for some (nonconstant) C^1 function $\mu : M \to \mathrm{I\!R}$. As $\mathcal{D}^\perp = \operatorname{Ker} dt$ on $\widehat{M}$, this μ is constant along $\mathcal{D}^\perp$.

Sard's theorem normally applies to C^k mappings from an n-manifold into an m-manifold, for k, n and m with $k \geq \max(n - m + 1, 1)$, guaranteeing that the critical values form a set of zero measure. In our case, $\mu : M \to \mathrm{I\!R}$ is only of class C^1, and M can have any dimension $n \geq 4$.

However, the conclusion of Sard's theorem remains valid here [5, Remark 9.2], and so, due to compactness of M, the range $\mu(M)$ of μ contains an open interval formed by regular values of μ.

In fact, M is covered by finitely many connected open sets U each of which can be diffeomorphically identified with an open set $\widehat{U} \subseteq \widehat{M}$ such that the levels of $t : \widehat{U} \to \mathrm{I\!R}$ are all connected. This turns μ restricted to U into a function of t, allowing us to use Sard's theorem as stated above for $k = n = m = 1$.

Connected components L of regular levels of μ clearly realize option (ii) of the dichotomy property.

Appendix A

Here is an easy set-theoretical observation [5, Lemma 3.3]:

Lemma A.1. *Let a vector space $\mathcal{S}$ of functions $X \to \mathrm{I\!R}$ on a set X have a finite dimension $m > 0$ and be closed both under the absolute-value operation $\psi \mapsto |\psi|$ and under some m-argument operation Π sending $\psi_1, \dots, \psi_m$ to a function $\Pi(\psi_1, \dots, \psi_m) \geq 0$ having the same zeros as the product $\psi_1 \dots \psi_m$.*

Then some basis of $\mathcal{S}$ consists of nonnegative functions with pairwise disjoint supports.

By 'support' we mean *complement of the zero set.*

Applying the above lemma to our $\mathcal{S}$ we see that, on the set where $f \neq 0$, the ratio $|\dot{f}|^{1/3}/|f|^{1/2}$ is locally constant, which makes $|f|^{-1/2}$ (locally) linear as a function of t.

Hence $f \neq 0$ everywhere in $\widehat{M}$ (or else, at a boundary point of the zero set of f, the linear function $|f|^{-1/2}$ would be unbounded on a bounded interval of the variable t).

Thus, $f = \varepsilon(t-b)^{-2}$, as required, and t has the range $I \subseteq \mathrm{I\!R} \smallsetminus \{b\}$. Subjecting t to an affine substitution, we may assume that $b = 0$ and $I \subseteq (0,\infty)$, with $f = \varepsilon t^{-2}$.

Appendix B

Formula (7.1) easily implies that the Levi-Civita connection $\widehat{\nabla}$ induces a flat connection in the quotient bundle $\widehat{\mathcal{D}}^{\perp}/\widehat{\mathcal{D}}$ over $\widehat{M}$, with an $(n-2)$-dimensional pseudo-Euclidean space V of parallel sections.

The (parallel) Weyl tensor naturally gives rise to a nonzero traceless endomorphism $A : V \to V$, represented by the matrix $[a_{ij}]$ in formula (7.1).

Any $\gamma \in \Gamma$, acting on V as a linear isometry B, pushes this A forward onto $BAB^{-1} = q^2 A$, for q related to γ as in (9.1). Due to compactness of $M = \widehat{M}/\Gamma$,

(B.1) such q arising from all $\gamma \in \Gamma$ form an infinite subset of $(0,\infty)$, closed undertaking powers.

The set $\mathcal{J} = \{(q,B) \in \mathrm{I\!R} \times \mathrm{End}\, V : (BB^*, BAB^*) = (\mathrm{Id}, q^2 A)\}$ is an algebraic variety in $\mathrm{I\!R} \times \mathrm{End}\, V$. By Whitney's classical result [11], $\mathcal{J}$ has finitely many connected components, and hence so does the intersection $K = K' \cap (0,\infty)$, for the image K' of $\mathcal{J}$ under the projection $(q,B) \mapsto q$.

Thus, according to (B.1), $K = (0,\infty)$. Formula (7.1), with $f = \varepsilon t^{-2}$, now easily yields local homogeneity of $\hat{g}$. Cf. also [1].

References

[1] A. Derdziński, *On homogeneous conformally symmetric pseudo-Riemannian manifolds*, Colloq. Math. **40(1)** (1978), 167–185.

[2] A. Derdziński and W. Roter, *On conformally symmetric manifolds with metrics of indices* 0 *and* 1, Tensor (N. S.) **31(3)** (1977), 255–259.

[3] A. Derdzinski and W. Roter, *The local structure of conformally symmetric manifolds*, Bull. Belgian Math. Soc. **16(1)** (2009), 117–128.

[4] A. Derdzinski and W. Roter, *Compact pseudo-Riemannian manifolds with parallel Weyl tensor*, Ann. Glob. Anal. Geom. **37(1)** (2010), 73–90.

[5] A. Derdzinski and I. Terek, *The topology of compact rank-one ECS manifolds*, Proc. Edinb. Math. Soc. **66(3)** (2023), 789–809.

[6] A. Derdzinski and I. Terek, *New examples of compact Weyl-parallel manifolds*, Monatsh. Math. (online), DOI: 10.1007/s00605-023-01908-0.

[7] A. Derdzinski and I. Terek, *Compact locally homogeneous manifolds with parallel Weyl tensor*, preprint, https://arxiv.org/pdf/2306.01600.pdf.

[8] I. Moerdijk and J. Mrčun, *Introduction to Foliations and Lie Groupoids*, Cambridge University Press, 2003.

[9] Z. Olszak, *On conformally recurrent manifolds, I: Special distributions*, Zesz. Nauk. Politech. Śl., Mat.-Fiz. **68** (1993), 213–225.

[10] W. Roter, *On conformally symmetric Ricci-recurrent spaces*, Colloq. Math. **31(1)** (1974), 87–96.

[11] H. Whitney, *Elementary structure of real algebraic varieties*, Ann. of Math. (2) **66** (1957), 545–556.

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