AN EXTRINSIC AVERAGE VARIATIONAL METHOD $\Phi_{(i)}$-HARMONIC MAPS AND $\Phi_{(i)}$-SSU MANIFOLDS, $i = 1, 2, 3$

SHIHSHU WALTER WEI

Abstract. Employing an extrinsic average variational method in the calculus of variations ([42, 40]), we have found multiple large classes of new manifolds with geometric and topological properties in the setting of varied, coupled, generalized types of energy functionals and their associated harmonic maps $u : (M, g_M) \to (N, g_N)$. These newly found manifolds that involved with elementary symmetric functions $\sigma_i, i = 1, 2, 3$ of eigenvalues of the pullback metric u^*g_N with respect to the domain metric g_M have their interactions with geometry, topology, analysis, partial differential equations, calculus of variations, physics, and are briefly listed In Table 1. Whereas the method have been used, extended or generalized to other situations such as minimal submanifolds and rectifiable currents in a Riemannian manifold ([30, 24]), harmonic maps ([40, 23, 33]), Yang-Mills Fields ([41, 28]), p-harmonic maps ([56]), F-harmonic maps ([2]), Finsler geometry ([37]), etc, in this paper we illuminate the method and show how it works for the energy functional E. Generalizing the author's previous work that every stable harmonic map from an arbitrary compact Riemannian manifold into S^n or $S^n \times S^k$ for $n > 2, k > 2$ is constant and the work of R. Howard and S.W. Wei ([23]) on SU manifolds, we prove two new important results (Theorems 10.1 and 10.2).

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1. Philosophical Background

An ancient wisdom goes "The Tao of Heaven is to diminish superabundance, so as to supplement deficiency." This is due to a legendary sage Lao Tzu in his book Tao Te Ching. It is a natural and precious phenomenon that permeates or occurs in mathematics, astronomy, physics, engineering, psychology, real life, natural sciences, and medical sciences. In daily life, it recommends use our strength to "supplement" our limitation to achieve balance, optimality, harmony, or meeting challenges. In astronomy, it occurs astonishingly the Kepler's Second Law "equal time sweeps equal area". In physics, it involves with conservation laws (cf. [11]), the law of conservation of energy, angular momentum, etc. In mathematics

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it give rises to or interrelates the concept of *average*, balance, harmony, stable equilibrium, mean value property, symmetry, the least action principle, duality, etc. (cf. [32, 24, 57]).

From geometric function theoretic point of view, a harmonic function on the Euclidean space can be characterized as a function whose value at every point is equal to its *average* value around every ball (resp. sphere) centered at that point with arbitrary radius. We have the following elegant links.

Theorem 1.1. *Let* $f : \mathbb{R}^n \xrightarrow{C^2} R$. *Then for every point* $x_0 \in \mathbb{R}^n$ *and every ball* $B(x_0, r) \subset \mathbb{R}^n$,

$$f(x_0) = \frac{1}{\mathrm{Vol(B(x_0,r))}} \int_{B(x_0,r)} f(x)\, dx$$

$\Longleftrightarrow$

$$f(x_0) = \frac{1}{\mathrm{Vol(\partial B(x_0,r))}} \int_{\partial B(x_0,r)} f(x)\, dS.$$

$\Longleftrightarrow$

$$\text{On} \quad \mathbb{R}^n, \quad f \quad \textit{is} \quad \textit{harmonic} \quad \textit{i.e.,} \quad \Delta f = 0$$

or f *is a solution of the Laplace equation* $\Delta f = \mathrm{Div}(\nabla f) = \mathrm{trace}(\mathrm{Hess}\ f) = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} f = 0.$

$\Longleftrightarrow$

$$f \quad \text{is} \quad \text{a critical point of the energy functional E}$$

with respect to any compactly supported variation, where $E(f) = \int_{\mathbb{R}^n} |\nabla f|^2 dx$.

2. An Extrinsic Average Variational Method in the Calculus of Variations

Observing Mathematics and Nature are beautifully interwoven, frequently two sides of the same coin, and Nature is uncompromizingly efficient, S.W. Wei proposed an extrinsic, average variational method in the calculus of variations (cf. [40, 42]) as an approach to confront and resolve problems in global, nonlinear analysis, geometry and physics, by which the author pioneered the study of p-harmonic geometry (cf. e.g. [45, 49, 56]).

The method have been used, extended or generalized to other situations such as minimal submanifolds and rectifiable currents in a Riemannian manifold ([30, 24]), harmonic maps ([40, 23, 33]), Yang-Mills Fields ([41, 28]), p-harmonic maps ([56]), F-harmonic maps ([2]), Finsler geometry ([37]), etc.

More recently, employing the extrinsic average variational method ([42, 40]), we have found multiple large classes of new manifolds with geometric and topological properties in the setting of varied, coupled, generalized types of energy functionals and their associated harmonic maps $u : (M, g_M) \to (N.g_N)$. These newly found manifolds that involved with elementary symmetric functions $\sigma_i, i = 1, 2, 3$ of eigenvalues of the pullback metric u^*g_N with respect to the domain metric g_M have their interactions with geometry, topology, analysis, partial differential equations, calculus of variations, physics, and are briefly listed In Table 1.

Let $u : (M, g_M) \to (N, g_N)$ be a smooth map between two Riemannian manifolds M and N. Denote $e(u)$ the *energy density* of u, (resp. $e_p(u)$ the *p-energy density* of u) which is given by

$$e(u) = \frac{1}{2}\sum_{i=1}^m g_N\big(du(e_i), du(e_i)\big) = \frac{1}{2}|du|^2\,, \tag{2.1}$$
$$(\text{resp.} \quad e_p(u) = \frac{1}{p}\sum_{i=1}^m g_N\big(du(e_i), du(e_i)\big)^{\frac{p}{2}} = \frac{1}{p}|du|^p\,),$$

where $\{e_1, \cdots, e_n\}$ is a local orthonormal frame field on M, du is the differential of u, and $|du|$ is the Hilbert-Schmidt norm of du, determined by the metric g_M of M and the metric g_N of N. *The energy of* u, denoted by $E(u)$ and *the p-energy of* u, denoted by $E_p(u)$ are defined to be

$$E(u) = \int_M e(u)\,dv_g \quad \text{and} \quad E_p(u) = \int_M e_p(u)\,dv_g, \quad \text{respectively.} \tag{2.2}$$

A smooth map $u : M \to N$ is called *harmonic* (resp. *p-harmonic*) if u is a critical point of the energy functional E (resp. the p-energy functional E_p) with respect to any compactly supported variation, *E-stable or stable harmonic* if u is a local minimum of the energy functional $E(u)$, and *E-unstable or unstable harmonic* if u is not stable harmonic (resp. a *stable p-harmonic map* if u is a local minimum of the p–energy functional $E_p(u)$, *unstable p-harmonic or p-unstable* if u is not p-stable).

In this paper, we illuminate the method and show how it works for the energy functional E. Generalizing the author's previous work that every stable harmonic map from any compact Riemannian manifold into S^n or $S^n \times S^k$ for $n > 2, k > 2$ is constant. (cf. Theorem 2.3, or [40, Corollaries 3.1 and 3.2]), and the work of R. Howard and S.W. Wei on SU manifols [23, Remarks 2.11 and 5.5], we prove

Theorem 10.1 *Let $M_1^{m_1}, \cdots, M_\ell^{m_\ell}$ be p-SSU manifolds (cf. Definition 4.1). Then (i) the product manifold $M = M_1^{m_1} \times \cdots \times M_\ell^{m_\ell}$ is a p-SSU manifold of dimension $m = m_1 + \cdots + m_\ell$. Hence, M is p-SU. (ii) There is a neighborhood of the product metric g_0 of the product manifold $M = M_1^{m_1} \times \cdots \times M_\ell^{m_\ell}$ in the C^2 topology (if M is not compact we must use the strong C^2 topology (see [19] for the definition) such that for every g in this neighborhood, the Riemannian manifold (M, g) is a p-SSU manifold of dimension $m = m_1 + \cdots + m_\ell$. Hence, (M, g) is p-SU (cf. Definition 4.2).*

Theorem 10.2 *Let $M_1^{m_1}, \cdots, M_\ell^{m_\ell}$ be X-SSU manifolds, where X-SSU denotes one of the following: $\Phi_{(1)}$-SSU, Φ_S-SSU, $\Phi_{S,p}$-SSU, $\Phi_{(2)}$-SSU, and $\Phi_{(3)}$-SSU. Then (i) The product manifold $M = M_1^{m_1} \times \cdots \times M_\ell^{m_\ell}$ is an X-SSU manifold of dimension $m = m_1 + \cdots + m_\ell$. Hence, M is X-SU, i.e. M is the corresponding $\Phi_{(1)}$-SU, Φ_S-SU, $\Phi_{S,p}$-SU, $\Phi_{(2)}$-SU, or $\Phi_{(3)}$-SU. (ii) There is a neighborhood of the product metric g_0 of the product manifold $M = M_1^{m_1} \times \cdots \times M_\ell^{m_\ell}$ in the C^2 topology (if M is not compact we must use the strong C^2 topology) such that for every g in this neighborhood the Riemannian manifold (M, g) is an X-SSU manifold of dimension $m = m_1 + \cdots + m_\ell$. Hence, (M, g) is X-SU.*

The extrinsic average variational method in the calculus variations also gives (i) the first nonexistence theorem of stable *Yang-Mills fields* on product manifolds ([41]). (ii) the first classification of stable *rectifialbe currents* on product manifolds ([48]). (iii) the first nonexistence theorem of stable *harmonic maps* into product manifolds ([40]) (cf. Remark 10.3).

Approach I: Extrinsic Average Variations in the Target N

We assume M (resp. N) is isometrically immersed in the Euclidean space $\mathbb{R}^q$. Let $\overline{\nabla}$ be the standard flat connection on $\mathbb{R}^q$, ∇ (resp. ∇^N) the Riemannian connection on M (resp. N) and B (resp. B) the second fundamental form of M (resp. N) in $\mathbb{R}^q$. These are related by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y) \qquad \big(\text{resp}.\overline{\nabla}_{\mathsf{X}}\mathsf{Y} = \nabla^N_{\mathsf{X}}\mathsf{Y} + \mathsf{B}(\mathsf{X}, \mathsf{Y})\big), \tag{2.3}$$

where X, Y (resp. X, Y) are smooth vector fields on M (resp. N). Let M (resp. N and $\mathbb{R}^q$) be equipped with Reimannian metric $\langle\quad\rangle_M$ (resp. $\langle\quad,\rangle_N$ and $\langle\quad\rangle$). Define a selfadjoint linear map

$$Q_x^M : T_xM \to T_xM \quad (\text{resp.} \quad Q_y^N : T_yN \to T_yN)$$

by

$$\langle Q_x^M(X), X\rangle_M = \sum_{i=1}^m 2\langle B(X, e_i), B(X, e_i)\rangle - \langle B(X, X), B(e_i, e_i)\rangle, \tag{2.4}$$

where $x \in M$, $\{e_1, \cdots, e_m\}$ is an orthonormal basis for the tangent space T_xM to M at x.

$$\Big(\text{resp.} \quad (2.4') \qquad \langle Q_y^N(\mathsf{X}), \mathsf{X}\rangle_N = \sum_{i=1}^n 2\langle \mathsf{B}(\mathsf{X}, \mathsf{e}_i), \mathsf{B}(\mathsf{X}, \mathsf{e}_i)\rangle - \langle \mathsf{B}(\mathsf{X}, \mathsf{X}), \mathsf{B}(\mathsf{e}_i, \mathsf{e}_i)\rangle,$$

where $y \in N$, $\{\mathsf{e}_1, \cdots, \mathsf{e}_n\}$ is an orthonormal basis for the tangent space T_yN to N at y.)

Let $\{\mathsf{v}_1, \ldots, \mathsf{v}_n, \mathsf{v}_{n+1}, \ldots, \mathsf{v}_q\}$ be an orthonormal basis of $\mathbb{R}^q$ and let $\mathbf{x} : N^n \to \mathbb{R}^q$ be an isometrically immersion with second fundamental form B. As v_ℓ, $1 \le \ell \le q$, can be identified with a parallel and concircular vector field in $\mathbb{R}^q$ (i.e. $\overline{\nabla}_Z \mathsf{v}_\ell = 0 \cdot \tilde{Z} = 0$ for any $\tilde{Z}$ tangent to $\mathbb{R}^q$, where $Z = \tilde{Z}_{|_N}$, cf. [6]), this gives rise to a set of conservative vector fields

$$\{\mathsf{v}_1^T, \ldots, \mathsf{v}_n^T, \mathsf{v}_{n+1}^T, \ldots, \mathsf{v}_q^T\} \quad \text{on } N,$$

where $\mathsf{v}_1^T, \ldots, \mathsf{v}_q^T$ are conservative vector fields defined by $\mathsf{v}_\ell^T = \operatorname{grad}(\langle \mathsf{v}_\ell, \mathbf{x}\rangle)$.

Clearly, each v_ℓ^T generates a flow or a one-parameter group of diffeomorphisms $\varphi_t^{\mathsf{v}_\ell^T} : N \to N$. Further, given a smooth map $u : M^m \to N^n$ between two compact Riemannian manifolds, we can deform u in v_ℓ^T direction to obtain the variation $u_t = \varphi_t^{\mathsf{v}_\ell^T} \circ u$ of u with $u_0 = u$.

Let us consider the energy of $\varphi_t^{\mathsf{v}_\ell^T} \circ u$:

$$E(\varphi_t^{\mathsf{v}_\ell^T} \circ u) = \int_M \sum_{i=1}^m \langle d(\varphi_t^{\mathsf{v}_\ell^T} \circ u)e_i, d(\varphi_t^{\mathsf{v}_\ell^T} \circ u)e_i\rangle dV_M$$

where $d(\varphi_t^{\mathsf{v}_\ell^T} \circ u)$ is the differential of $\varphi_t^{\mathsf{v}_\ell^T} \circ u$, dV_M is the volume element of M. Thus, to each direction v_ℓ^T, the energy $E(\varphi_t^{\mathsf{v}_\ell^T}(u))$ via the variation $\varphi_t^{\mathsf{v}_\ell^T} \circ u$ is a smooth real valued function of t, and there corresponds to its rate of change of the energy in that direction v_ℓ^T to the second order, i.e. $\frac{d^2}{dt^2} E(\varphi_t^{\mathsf{v}_\ell^T}(u))\big|_{t=0}$. Therefore, to the set of the q vector fields $\{\mathsf{v}_1^T, \ldots, \mathsf{v}_n^T, \mathsf{v}_{n+1}^T, \ldots, \mathsf{v}_q^T\}$ on N, there correspond to the set of q second variations given by

$$\Big\{ \frac{d^2}{dt^2} E(\varphi_t^{\mathsf{v}_1^T}(u))\big|_{t=0}, \ldots, \frac{d^2}{dt^2} E(\varphi_t^{\mathsf{v}_n^T}(u))\big|_{t=0}, \\ \frac{d^2}{dt^2} E(\varphi_t^{\mathsf{v}_{n+1}^T}(u))\big|_{t=0}, \ldots, \frac{d^2}{dt^2} E(\varphi_t^{\mathsf{v}_n^T}(u))\big|_{t=0} \Big\}$$

and their average or sum: $\sum_{\ell=1}^q \frac{d^2}{dt^2} E(\varphi_t^{\mathsf{v}_\ell^T}(u))\big|_{t=0}$.

Let $\{\mathsf{e}_1, \ldots, \mathsf{e}_n\}$ be a local orthonormal frame field on N. It was shown by Howard and Wei in [23] that if the map u is a non-constant harmonic map and the second fundamental form B of N in $\mathbb{R}^q$ satisfies

$$\sum_{j=1}^n \big\{ 2\langle \mathsf{B}(\mathsf{X}, \mathsf{e}_j), \mathsf{B}(\mathsf{X}, \mathsf{e}_j)\rangle - \langle \mathsf{B}(\mathsf{X}, \mathsf{X}), \mathsf{B}(\mathsf{e}_j, \mathsf{e}_j)\rangle \big\} < 0 \tag{2.5}$$

for each tangent vector X to N at any point in N, then the average variation, or the sum satisfies

$$\begin{aligned} &\sum_{\ell=1}^q \frac{d^2}{dt^2} E(\varphi_t^{\mathsf{v}_\ell^T}(u))\big|_{t=0} \\ &= \int_M \sum_{i=1}^m \sum_{j=1}^n \big\{ 2\langle \mathsf{B}(du(e_i), \mathsf{e}_j), \mathsf{B}(du(e_i), \mathsf{e}_j)\rangle \\ &\qquad - \langle \mathsf{B}(du(e_i), du(e_i)), \mathsf{B}(\mathsf{e}_j, \mathsf{e}_j)\rangle \big\} \\ &< 0, \end{aligned} \tag{2.6}$$

by applying (2.5) in which $X = du(e_i)$ and summing it from $i = 1$ to m. Hence one of the terms must be < 0, Or the sum would be nonnegative, a contradiction, i.e.

$$\frac{d^2}{dt^2} E(\varphi_t^{\mathsf{v}_\ell^T}(u))\big|_{t=0} < 0 \quad \text{for some} \quad 1 \le \ell \le q. \tag{2.7}$$

This means that along one of the directions, v_ℓ^T, the variation decreases the energy of u, and hence u is not a local minimum of the energy functional E, i.e. u is not harmonic stable.

Remark 2.1. *If we only compute the the second variation of the energy along any single direction v_ℓ^T, we do not know the sign of*
$\frac{d^2}{dt^2}E(\varphi_t^{\mathsf{v}_\ell^T}(u))\big|_{t=0}$, *because of some troublesome terms involved. However, if we average the result* $\sum_{\ell=1}^q \frac{d^2}{dt^2}E(\varphi_t^{\mathsf{v}_\ell^T}(u))\big|_{t=0}$ *over the set of variation vector fields, then the troublesome terms are cancelled, we get* (2.6)*, from which we know the sign of the average is negative, under the above extrinsic condition* (2.5) *on N, and hence* (2.7) *holds.*

Remark 2.2. *One way to interpret* (2.6) *is that by the extrinsic average variational method, the set of "distingished" conservative vector fields*
$\{\mathsf{v}_1^T,\dots,\mathsf{v}_n^T,\mathsf{v}_{n+1}^T,\dots,\mathsf{v}_q^T\}$ *on N, "universally" decrease the energy E of "any" map into N.*

Let N be a complete hypersurface in $\mathbb{R}^{n+1}$ with principal curvature κ (resp. N' be a complete, hypersurface in $\mathbb{R}^{k+1}$ with principal curvature κ') and K_{min} be a function of N given by $K_{min}(x) =$ the minimum of all sectional curvatures of N at x(resp. K'_{min} be a function of N' given by K'_{min}=the minimum of all sectional curvature of N' at x'). In [40] S.W. Wei proved

Theorem 2.3 ([40])**.** *If*

$$\kappa^2 < (n-1)K_{min} \quad \textit{and} \quad (\kappa')^2 < (k-1)K'_{min}\,,$$

then any (weakly) stable harmonic maps from an arbitrary Riemannian manifold M into $N \times N'$ is constant.
In particular, any (weakly) stable harmonic maps from an arbitrary Riemannian manifold M into $S^n \times S^k$ or S^n for $n > 2, k > 2$ is constant.

The case when the target manifold is a single sphere S^n, for $n > 2$ is also due to P.F. Leung ([29]). Theorem 2.3 is generalized to Theorem 10.1 and extended to Theorem 10.2.

Approach II: Extrinsic Average Variations in the Domain M

Analogously, given a smooth map $u : M \to N$ between two compact Riemannian manifolds, we can deform u in v_ℓ^T direction to obtain the variation $u_t = u \circ \varphi_t^{v_\ell^T}$ of u with $u_0 = u$.

Let us consider the energy of $u \circ \varphi_t^{v_\ell^T}$:

$$E(u \circ \varphi_t^{v_\ell^T}) = \int_M \sum_{i=1}^q \langle d(u \circ \varphi_t^{v_\ell^T})e_i, d(u \circ \varphi_t^{v_\ell^T})e_i\rangle dV_M$$

where $d(u \circ \varphi_t^{v_\ell^T})$ is the differential of $u \circ \varphi_t^{v_\ell^T}$, $\{e_1,\dots,e_m\}$ is a local orthonormal frame field on M, and dV_M is the volume element of M . Thus, to each direction v_ℓ^T , the energy $E\big(u(\varphi_t^{v_\ell^T})\big)$ via the variation $u \circ \varphi_t^{v_\ell^T}$ is a smooth real valued function of t, and there corresponds to its rate of change of the energy in that direction v_ℓ^T to the second order, i.e. $\frac{d^2}{dt^2}E\big(u(\varphi_t^{v_\ell^T})\big)\big|_{t=0}$. Therefore, to the set of the q vector fields $\{v_1^T,\dots,v_m^T,v_{m+1}^T,\dots,v_q^T\}$ on M , there correspond to the set of q second variations given by

$$\Big\{\frac{d^2}{dt^2}E\big(u(\varphi_t^{v_1^T})\big)\big|_{t=0},\dots,\frac{d^2}{dt^2}E\big(u(\varphi_t^{v_m^T})\big)\big|_{t=0}$$
$$\frac{d^2}{dt^2}E\big(u(\varphi_t^{v_{m+1}^T})\big)\big|_{t=0},\dots,\frac{d^2}{dt^2}E\big(u(\varphi_t^{v_q^T})\big)\big|_{t=0}\Big\}$$

and their average or sum: $\sum_{\ell=1}^{q} \frac{d^2}{dt^2} E\big(u(\varphi_t^{v_\ell^T})\big)\big|_{t=0}$. Let $\{e_1, \dots, e_m\}$ be a local orthonormal frame field on M. It was shown by Howard and Wei in [23] that if the map u is non-constant and the second fundamental form B of M in $\mathbb{R}^q$ satisfies

$$\sum_{j=1}^{m} \big\{2\langle B(X, e_j), B(X, e_j)\rangle - \langle B(X, X), B(e_j, e_j)\rangle\big\} < 0 \tag{2.8}$$

for each tangent vector X to M at any point in M, then the average variation, or the sum satisfies

$$\begin{aligned} &\sum_{\ell=1}^{q} \frac{d^2}{dt^2} E\big(u(\varphi_t^{v_\ell^T})\big)\big|_{t=0} \\ &= \int_M \sum_{i=1}^{m} \sum_{j=1}^{m} |du(e_i)|^2 \{2\langle B((e_i), e_j), B(e_i, e_j)\rangle \\ &\qquad - \langle B(e_i, e_i), B(e_j, e_j)\rangle\big\} \\ &< 0, \end{aligned} \tag{2.9}$$

Hence one of the terms must be < 0, Or the sum would be nonnegative, a contradiction, i.e.

$$\frac{d^2}{dt^2} E\big(u(\varphi_t^{v_\ell^T})\big)\big|_{t=0} < 0 \quad \text{for some} \quad 1 \le \ell \le q. \tag{2.10}$$

This means that along one of the directions, v_ℓ^T, the variation decreases the energy of u, and hence u is not a local minimum of the energy functional E, i.e. $u : M \to N$ is not a stable harmonic map. or u is not stable.

Remark 2.4. *If we only compute the the second variation of the energy along any single direction* v_ℓ^T*, we do not know the sign of*
$\frac{d^2}{dt^2} E\big(u(\varphi_t^{v_\ell^T})\big)\big|_{t=0}$*, because of some troublesome terms involved. However, if we average the result*
$\sum_{\ell=1}^{q} \frac{d^2}{dt^2} E\big(u(\varphi_t^{v_\ell^T})\big)\big|_{t=0}$ *over the set of variation vector fields, then the troublesome terms are cancelled, we get* (2.9)*, from which we know the sign of the average is negative, under the above extrinsic condition* (2.8)*, on* M *and hence* (2.10) *holds.*

Remark 2.5. *One way to interpret* (2.9) *is that by the extrinsic average variational method, the set of "distinguished" conservative vector fields*
$\{v_1^T, \dots, v_n^T, v_{n+1}^T, \dots, v_q^T\}$ *on* M*, "universally" decrease the energy* E *of "any" map from* M*.*

Corollary 2.6 ([58])**.** *Every (weakly) stable harmonic maps from* $S^m, m > 2$ *into an any compact Riemannian manifold* N *is constant.*

Proof. We choose diagonalized orthonormal basis $\{e_1, \dots, e_m\}$ at a point in S^m, then

$$\sum_{j=1}^{m} \big\{2\langle B(X, e_j), B(X, e_j)\rangle - \langle B(X, X), B(e_j, e_j)\rangle\big\} = 2 - m < 0.$$

Hence, (2.9) holds, and the result follows. □

3. Averaging Second Variations

By an extrinsic average variational method, we derive the following average second variation formulas of the energy functional E:

An average second variational formula on the target for the energy of $u : M^n \to N^k$ (u is not necessarily harmonic) ([23]).

$$
\begin{aligned}
(3.1) \qquad & \sum_{\ell=1}^{q} \frac{d^2}{dt^2} E(\phi_t^{v_\ell^T} \circ u)\Big|_{t=0} \\
&= \int_M \sum_{i=1}^{m} \langle Q^N\big(du(e_i)\big), du(e_i)\rangle_N \, dv,
\end{aligned}
$$

where Q^N is as in (2.4).

An average second variational formula on the domain for the energy of a map $u : M^m \to N^n$ (u is harmonic)

$$
\begin{aligned}
(3.2) \qquad & \sum_{\ell=1}^{q} \frac{d^2}{dt^2} E(u \circ \phi_t^{\mathsf{v}_\ell^T})\Big|_{t=0} \\
&= \int_M \sum_{i=1}^{m} \langle du\big(Q^M(e_i)\big), du(e_i)\rangle_N \, dv,
\end{aligned}
$$

where Q^M is as in (2.4′).

4. Superstrongly Unstable (SSU) Manifolds ([43])

In contrast to an average method in PDE that we applied in [5] to obtain sharp growth estimates for warping functions in multiply warped product manifolds, we employ *an extrinsic average variational method* in the calculus of variations ([42, 40]), find a large class of manifolds of positive Ricci curvature that enjoy rich properties ([44, 41, 43, 56]).

Definition 4.1. *A Riemannian manifold M with its Riemannian metric $\langle\, ,\, \rangle_M$ is said to be a* **superstrongly unstable (SSU)** *manifold, if there exists an isometric immersion of M in $(\mathbb{R}^q, \langle \cdot \rangle)$ with its second fundamental form B, such that for every unit tangent vector v to M at every point $x \in M$, the following symmetric linear operator Q_x^M is negative definite.*

$$
(4.1) \qquad \langle Q_x^M(v), v\rangle_M = \sum_{i=1}^{m} \Big(2\langle B(v, e_i), B(v, e_i)\rangle - \langle B(v, v), B(e_i, e_i)\rangle \Big)
$$

*and M is said to be a p-***superstrongly unstable (p-SSU)** *manifold for $p \geq 2$ if the following functional is negative valued.*

$$
(4.2) \qquad F_{p,x}(v) = (p-2)\langle B(v, v), B(v, v)\rangle + \langle Q_x^M(v), v\rangle_M,
$$

where $\{e_1, \ldots, e_m\}$ is a local orthonormal frame on M.

Employing the extrinsic average variational method, we find the following.

Proposition 4.2. *A compact* SSU *manifold M cannot be the domain of any nonconstant stable harmonic map into any manifold. And a compact* SSU *manifold N cannot be the target of any nonconstant stable harmonic maps from any manifold.*

Proof. These follow at once from (2.9) (or (3.2)) and (2.6) (or (3.1)) respectively. □

Howard and Wei ([23]) (resp. Wei and Yau ([56])) introduce the following notions:

Definition 4.3. *A Riemannian manifold M is said to be strongly unstable* (SU) (*resp. p-strongly unstable* (p-SU)) *if M is neither the domain nor the target of any nonconstant smooth stable harmonic map, $(resp. stable p-harmonic map)$, and the homotopic class of maps from or into M contains a map of arbitrarily small energy E (resp. p-energy E_p).*

This concept leads to

Theorem 4.4. *Every compact superstrongly unstable* (SSU)*-manifold* (*resp. p-superstrongly unstable* (p-SSU)) *manifold is strongly unstable* (SU) (*resp. p-strongly unstable* (p-SU)) .

Proof. (SSU ⇒ SU) This follows from Proposition 4.2 and [23].
(resp. p-SSU ⇒ p-SU) This follows from [56, 45] □

Theorem 4.5 (Topological Vanishing Theorem)**.** *Suppose that M is a compact* SSU(resp. p-SSU) *manifold. Then M is* SU *and*

$$\begin{aligned} &\pi_1(M) = \pi_2(M) = 0 \\ &\big(\text{resp. } \pi_1(M) = \cdots = \pi_{[p]} = 0\big). \end{aligned} \tag{4.3}$$

Furthermore, the following three statements are equivalent:

(4.4)
(a) $\pi_1(M) = \pi_2(M) = 0$.
(b) *the infimum of the energy E is* 0 *among maps homotopic to the identity on* M .
(c) *the infimum of the energy E is* 0 *among maps homotopic to a map from* M .

That is,

$$\begin{aligned} \pi_1(M) = \pi_2(M) = 0 &\overset{[53]}{\Longleftrightarrow} \inf\{E(u') : u' \textit{is homotopic to } \mathrm{Id} \textit{ on } M\} = 0, \\ &\overset{[12]}{\Longleftrightarrow} \inf\{E(u') : u' \textit{is homotopic to } u : M \to \bullet\} = 0. \end{aligned} \tag{4.5}$$

Theorem 4.6 (Sphere Theorem)**.** *Suppose that M is a compact* SSU(resp. p-SSU) *manifold of dimension $m \le 5$ (* resp. $m \le 2p+1$)*. Then M is homeomorphic to S^m . If $m = 3$, then M is diffeomorphic to S^3 .*

Theorem 4.7 (Classification Theorem [33, 23])**.** *Let M be a compact irreducible symmetric space. The following statements are equivalent:*

(1) M is SSU.
(2) M is SU.
(3) M is U*; i.e.* Id_M *is an unstable harmonic map.*
(4) M is one of the following:

(4.6)
(i) *the simply connected simple Lie groups* $(A_l)_{l\ge 1}$, $B_2 = C_2$ and $(C_l)_{l\ge 3}$;
(ii) $SU(2n)/Sp(n)$, $n \ge 3$;
(iii) *Spheres* S^k, $k > 2$;
(iv) *Quaternionic Grassmannians* $Sp(m+n)/Sp(m) \times Sp(n), m \ge n \ge 1$;
(v) E_6/F_4;
(vi) *Cayley Plane* $F_4/Spin(9)$.

5. Varied Energy, Harmonicity, and Symmetry Invariants

We recall at any fixed point $x_0 \in M$, a symmetric 2-covariant tensor field α on (M, g_M) in general, or the pullback metric $u^* g_N$ in particular, has the eigenvalues λ relative to the metric g_M of M; i.e., the m real roots of the algebraic equation

$$\det(g_{ij}\lambda - \alpha_{ij}) = 0 \text{ where } g_{ij} = g_M(e_i, e_j),\ \alpha_{ij} = \alpha(e_i, e_j)\,,$$

and $\{e_1, \cdots e_m\}$ is a basis for $T_{x_0}(M)$. This gives rise to,

The algebraic invariants - the k-th elementary symmetric function of the eigenvalues of α at x_0, denoted by $\sigma_k(\alpha_{x_0}), 1 \le k \le m$ frequently have geometric meaning of the manifold M or the map u on M with analytic, topological and physical impacts.

Indeed, if we take α to be the second fundamental form of M in $\mathbb{R}^{m+1}$, then $\frac{1}{m}\sigma_1(\alpha)$, $\frac{2}{m(m-1)}\sigma_2(\alpha)$, and $\sigma_m(\alpha)$ are the mean curvature, scalar curvature, and the Gauss-Kronecker curvature of M respectively and are central themes of Yamabi problem ([1, 25, 35, 38]), special Lagrangian graphs ([21]), geometric aspects of the theory of fully nonlinear elliptic equations (e.g., [36]), and conformal geometry (e.g. [8],

[10]), etc. If we take α to be Schoulten tensor, then a study of $\sigma_2(\alpha)$ leads to a generalized Yamabe problem ([4]). In the study of prescribed curvature problems in PDE, the existence of closed starshaped hypersurfaces of prescribed mean curvature in Euclidean space was proved by A.E. Treibergs and S.W. Wei [39], solving a problem of F. Almgren and S.T. Yau [60]. While the case of prescribed Guass-Kronecker curvature was studied by V.I. Oliker [34] and P. Delanoë [9], the case of prescribed k-th mean curvature, in particular the intermediate cases, $2 \leq k \leq m-1$ were treated by L. Caffarelli, L. Nirenberg and J. Spruck [7].

These motivate us from the viewpoint of geometric mapping theory $u : (M^m, g) \to (N^n, h)$, taking $\alpha = u^*h$, in this paper, to extend the study of harmonic maps or $\Phi_{(1)}$-harmonic maps (cf [13]), Φ-harmonic maps or $\Phi_{(2)}$-harmonic maps (cf. [22]), to explore geometric properties of $\Phi_{(3)}$-harmonic maps by unified geometric analytic methods.

The first symmetric function σ_1, and $\Big($harmonic map $u : (M, g_M) \to (N, g_N)$ can be viewed as$\Big)$ $\Phi_{(1)}$-harmonic map.

A harmonic map u or a $\Phi_{(1)}$-harmonic map is a critical point of the energy functional, given by the integral of a half of an algebraic invariant - **the first elementary symmetric function σ_1, of engenvalues relative to the metric g_M,** or the trace of the pullback metric tensor u^*g_N, with respect to g_M, where $\{e_1, \cdots, e_m\}$ is an local orthonormal frame field on M. More precisely,

$$E(u) = \int_M \frac{1}{2}\sum_{i=1}^m g_N\big(du(e_i), du(e_i)\big)\, dv = \int_M \frac{1}{2}\big(\sigma_1(u^*)\big)\, dv. \tag{5.1}$$

A p-harmonic map can be viewed as a critical point of the p-energy functional $E_p(u)$, given by the integral of $\frac{1}{p}$ times σ_1 or the trace of the pullback metric tensor to the power $\frac{p}{2}$, i.e.,

$$E_p(u) = \int_M \frac{1}{p}\Big(\sum_{i=1}^m g_M\big(du(e_i), du(e_i)\big)\Big)^{\frac{p}{2}}\, dv = \int_M \frac{1}{p}\big(\sigma_1(u^*)\big)^{\frac{p}{2}}\, dv. \tag{5.2}$$

The 2^{nd} symmetric function σ_2 and $\Big($$\Phi$-Harmonic Map [22] can be viewed as$\Big)$$\Phi_{(2)}$-Harmonic Map.

In [22], Y.B. Han and S.W. Wei introduce the notions of Φ-energy density, Φ-energy, Φ-harmonic maps and stable Φ-harmonic maps that arise from the second symmetric function σ_2 of of engenvalues of a symmetric 2-covariant tensor field α on (M, g_M) relative to the metric g_M, where α is the pullback metric u^*g_N.

Let $u : (M, g_M) \to (N, g_N)$ be a smooth map between two Riemannian manifolds M and N.

Definition 5.1. *The Φ-energy density of u, denoted by $e_\Phi(u)$ is a quarter of the second symmetric function σ_2 of the engenvalues of the pullback metric u^*g_N , given by*

$$\begin{aligned} e_\Phi(u) &= \frac{1}{4}\sigma_2(\alpha), \text{ where } \alpha = u^*g_N \\ &= \frac{1}{4}\sum_{i,j=1}^m \Big(g_N\big(du(e_i), du(e_j)\big)\Big)^2. \end{aligned} \tag{5.3}$$

Thus, the Φ-energy density, similarly to the energy density depends on the metric g_M of M and the metric g_N of N.

The Φ-energy of u, denoted by $E_\Phi(u)$ is defined to be

$$E_\Phi(u) = \int_M e_\Phi(u)\, dv_g. \tag{5.4}$$

A smooth map $u : M \to N$ is called Φ-harmonic if u is a critical point of the Φ-energy functional E_Φ with respect to any compactly supported variation, Φ-stable if u is a local minimum of the Φ-energy functional $E_\Phi(u)$, and Φ-unstable if u is not Φ-stable.

We show that the extrinsic average variational method in the calculus of variations employed in the study of harmonic maps, p-harmonic maps, F-harmonic maps and Yang-Mills fields can be extended to the study of Φ-harmonic maps, and find Φ-superstrongly unstable (Φ-SSU) manifold.

Definition 5.2. *A Riemannian manifold M^m is said to be Φ-superstrongly unstable* (Φ-SSU) *if there exists an isometric immersion $\mathbb{R}^q$ such that, for all unit tangent vectors v to at every point $x \in M^m$, the following functional is always negative:*

$$F_{\Phi_x}(v) = \sum_{i=1}^{m} \big(4\langle B(v, e_i), B(v, e_i)\rangle - \langle B(v, v), B(e_i, e_i)\rangle\big), \tag{5.5}$$

where B is the second fundamental form of M^m in $\mathbb{R}^q$, and $\{e_1, \cdots, e_m\}$ is a local orthonormal frame on M near x.

Examples of Φ-SSU manifolds include hypersurfaces in Euclidean space with principal curvatures satisfying

$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_m < \frac{1}{3}(\lambda_1 + \cdots + \lambda_{m-1})$, m-dimensional elliptic paraboloid in $\mathbb{R}^{m+1}$, $\{(x_1, \cdots, x^m, y) : y = x_1^2 + \cdots + x_m^2\}$, the standard m-sphere S^m, for $m > 4$, certain ellipsoids, minimal submanifolds in ellipsoids and in convex hypersurfaces, arbitrary finite product of compact or noncompact manifolds (cf. [22, Theorems 5.1-5.4, Corollaries 5.1-5.2, and 5.4]). Indeed, examples of Φ-SSU manifolds are not limited to topological spheres or some unstable Yang-Mills fields in the sense of Bourguignon-Lawson-Simons [3, 30], Wei [41, 55], Kobayashi-Ohnita-Takeuchi (cf. [28]), (cf. also [22, Theorem 5.2]).

In [22] Y.B. Han and S.W. Wei introduced

Definition 5.3. *A manifold M is said to be Φ-Strongly Unstable* (Φ-SU) *if* (a) *M cannot be the target of any nonconstant stable Φ-harmonic map,* (b) *The homotopic class of any map into M contains elements of arbitrarily small Φ-energy,* (c) *M cannot be the domain of any nonconstant stable Φ-harmonic map, and* (d) *The homotopic class of any map from manifold M contains elements of arbitrarily small Φ-energy.*

and proved that

Theorem 5.4 ([22]). *Every compact Φ-superstrongly unstable* (Φ-SSU) *manifold must be Φ-strongly unstable* (Φ-SU).

Remark 5.5. *That Φ-SSU manifold is Φ-SU is an analog of* SSU *manifold being* SU *This can be viewed as $\Phi_{(2)}$-SSU manifold being $\Phi_{(2)}$-SU or $\Phi_{(1)}$-*SSU *manifold being $\Phi_{(1)}$-*SU.

6. Φ_S-harmonic maps, extended σ_2, and Φ_S-SSU manifolds ([15])

For a given map $u : (M, g_M) \to (N, g_N)$, the stress energy tensor S given by

$$S = e(u)g_M - u^* g_N \tag{6.1}$$

plays an important role in unifying the theory of harmonic maps and their generalizations. The norm of the stress energy tensor S given by

$$||S||^2 = \Sigma_{i,j=1}^m \Big(e(u)g_M(e_i, e_j) - u^* g_N(e_i, e_j)\Big)^2 = \frac{m-4}{16}|du|^4 + \sigma_2(u^* g_N)\,. \tag{6.2}$$

Associated with the stress energy tensor S, S.X. Feng, Y.B. Han, X. Li and S.W. Wei in [15], introduce the notion of the Φ_S-energy density $e_{\Phi_S}(u)$ of u, Φ_S-energy $E_{\Phi_S}(u)$ of u and Φ_S-harmonic maps, which are σ_2 version of the stress energy tensor S.

Definition 6.1. *The Φ_S-energy density $e_{\Phi_S}(u)$ of u is given by*

$$e_{\Phi_S}(u) = \big(||S||^2 = \big)\frac{m-4}{16}|du|^4 + \sigma_2(u^* g_N) = \frac{m-4}{4}e_4(u) + e_\Phi(u) \tag{6.3}$$

and the Φ_S-energy $E_{\Phi_S}(u)$ of u is defined to be the integral of Φ_S-energy density $e_{\Phi_S}(u)$ over M. Namely,

$$E_{\Phi_S}(u) = \int_M e_{\Phi_S}(u)\, dv = \frac{m-4}{4}E_4(u) + E_\Phi(u)\,, \tag{6.4}$$

where $E_4(u)$ and $E_\Phi(u)$ are 4-energy of u and Φ-energy of u respectively.

A smooth map u is said to be a Φ_S-*harmonic map* if it is a critical point of the Φ_S-energy functional E_{Φ_S} with respect to any smooth compactly supported variation of u, stable Φ_S-harmonic or simply Φ_S-stable if u is a local minimum of $E_{\Phi_S}(u)$, and Φ_S-unstable if u is not Φ_S-stable.

In [15], using the extrinsic average variational method in the calculus of variations, S.X. Feng, Y.B. Han, X. Li and S.W. Wei find a large class of manifolds, Φ_S-*superstrongly unstable* (Φ_S-SSU) *manifolds*,

Definition 6.2. *A Riemannian manifold N with $\dim N > 4$ is said to be a Φ_S-superstrongly unstable* (Φ_S-SSU) *manifold if there exists an isometric immersion of. N into $\mathbb{R}^q$ with its second fundamental form $\mathbf{B}$ such that, for all unit tangent vectors $\mathbf{x}$ to at every point $y \in N^n$, the following functional is always negative:*

$$\mathbf{F}_{\Phi_{S_\mathbf{y}}}(\mathbf{x}) = \sum_{\beta=1}^{n} \big(4\langle \mathbf{B}(\mathbf{x},\mathbf{e}_i), \mathbf{B}(\mathbf{x},\mathbf{e}_i)\rangle_{\mathbb{R}^q} - \langle \mathbf{B}(\mathbf{x},\mathbf{x}), \mathbf{B}(\mathbf{e}_i,\mathbf{e}_i)\rangle_{\mathbb{R}^q}\big), \tag{6.5}$$

where $\mathbf{B}$ is the second fundamental form of N^n in $\mathbb{R}^q$, and $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ is a local orthonormal frame on N near y.

and introduce the notion of Φ_S-strongly unstable (Φ_S-SU) manifolds

Definition 6.3. *A manifold M is said to be Φ_S-Strongly Unstable (Φ_S-SU) if (a) M is not the target of any nonconstant stable Φ_S-harmonic map, (b) The homotopic class of any map into M contains elements of arbitrarily small Φ_S-energy, (c) M is not the domain of any nonconstant stable Φ_S-harmonic map, and (d) The homotopic class of any map from M contains elements of arbitrarily small Φ_S-energy.*

and prove

Theorem 6.4. *Every compact Φ_S-superstrongly unstable* (Φ_S-SSU) *manifold is Φ_S-strongly unstable* (Φ_S-SU).

Remark 6.5. *This Φ_S-SSU manifold being Φ_S-SU is an analog of* SSU *manifold being* SU, *i.e., $\Phi_{(1)}$-*SSU *manifold being $\Phi_{(1)}$*-SU.

7. $\Phi_{S,p}$-harmonic maps, coupled σ_2 with σ_1, and $\Phi_{S,p}$-SSU Manifolds ([16])

We introduce the notion of $\Phi_{S,p}$-harmonic maps, which is a coupled generalized σ_2 version of the stress energy tensor S, and a σ_1 version of the pullback u^*g_N.

Just as we define the Φ_S-energy density $e_{\Phi_{S(u)}}$ and Φ_S-energy $E_{\Phi_{S(u)}}$ of a map $u : M \to N$, Φ_S-harmonic map, stable Φ_S-harmonic map, and unstable Φ_S-harmonic map that are associated with the stress energy tensor S, so do we introduce the notions of the $\Phi_{S,p}$-energy density $e_{\Phi_{S,p}}(u)$ and the $\Phi_{S,p}$-energy $E_{\Phi_{S,p}}(u)$ of a map $u : M \to N$, $\Phi_{S,p}$-harmonic map, stable $\Phi_{S,p}$-harmonic map, and unstable

$\Phi_{S,p}$-harmonic map. This is a natural L^2 version that involves with the p-th power of the norm of the the induced (0, 2)-tensor u^*g_N. When $p=2$, $e_{\Phi_{S,p}}$, $E_{\Phi_{S,p}}$, $\Phi_{S,p}$-harmonic maps and stable $\Phi_{S,p}$-harmonic maps become $e_{\Phi_{S,}}$, $E_{\Phi_{S,}}$, Φ_S-harmonic maps and stable Φ_S–harmonic maps respectively.

Definition 7.1. *The $\Phi_{S,p}$-energy density $e_{\Phi_{S,p}}(u)$ of u is given by*

$$e_{\Phi_{S,p}}(u) = \frac{m-2p}{2p^3}|du|^{2p} + \frac{1}{2p} m^{\frac{p}{2}-1}|u^*g_N|^p = \frac{m-2p}{p^2} e_{2p}(u) + m^{\frac{p}{2}-1} e_{\Phi_p}(u) \tag{7.1}$$

The $\Phi_{S,p}$-energy $E_{\Phi_{S,p}}(u)$ of u is given by

$$E_{\Phi_{S,p}}(u) = \int_M e_{\Phi_{S,p}}(u)\, dv = \frac{m-2p}{p^2} E_{2p}(u) + m^{\frac{p}{2}-1} E_{\Phi_p}(u) \tag{7.2}$$

where $E_{2p}(u)$ and $E_{\Phi_p}(u)$ are $2p$-energy of u and Φ_p-energy of u respectively.

Definition 7.2. *A smooth map u is said to be a $\Phi_{S,p}$-harmonic map (or a stress-energy stationary map) if it is a critical point of the $\Phi_{S,p}$-energy functional $E_{\Phi_{S,p}}$ with respect to any smooth compactly supported variation of u, stable $\Phi_{S,p}$-harmonic or simply $\Phi_{S,p}$-stable if u is a local minimum of $E_{\Phi_{S,p}}(u)$, and $\Phi_{S,p}$-unstable if u is not $\Phi_{S,p}$-stable.*

In [16], S.X. Feng, Y.B. Han, and S.W. Wei show that the extrinsic average variational method in the calculus of variations employed in the study of σ_1 and σ_2 versions of the pullback metric u^*g_N on M and stress-energy tensor can be extended to the study of a combined extended second symmetric function σ_2 version. In fact, we find a large class of manifolds, $\Phi_{S,p}$-superstrongly unstable ($\Phi_{S,p}$-SSU) manifolds,

Definition 7.3. *A Riemannian n-manifold N is said to be $\Phi_{S,p}$-supersrongly unstable* ($\Phi_{S,p}$-SSU) *if there exists an isometric immersion of N in $\mathbb{R}^q$ with its second fundamental form B such that, for all unit tangent vectors x to N at every point $y \in N$, the following functional is always negative-valued:*

$$\begin{aligned} \mathsf{F}_{\Phi_{S,p},y}(\mathsf{x}) =& 2(p-2)\langle \mathsf{B}(\mathsf{x},\mathsf{x}), \mathsf{B}(\mathsf{x},\mathsf{x})\rangle \\ &+ \sum_{\beta=1}^{n} \Big(4\langle \mathsf{B}(\mathsf{x},\mathsf{e}_\beta), \mathsf{B}(\mathsf{x},\mathsf{e}_\beta)\rangle_{\mathbb{R}^q} - \langle \mathsf{B}(\mathsf{x},\mathsf{x}), \mathsf{B}(\mathsf{e}_\beta,\mathsf{e}_\beta)\rangle_{\mathbb{R}^q} \Big). \end{aligned} \tag{7.3}$$

introduce the notion of $\Phi_{S,p}$-strongly unstable ($\Phi_{S,p}$-SU) manifolds

Definition 7.4. *A manifold M is said to be $\Phi_{S,p}$-Strongly Unstable* ($\Phi_{S,p}$-SU) *if (a) M is not be the target of any nonconstant stable $\Phi_{S,p}$-harmonic map, (b) The homotopic class of any map into M contains elements of arbitrarily small $\Phi_{S,p}$-energy, (c) M is not the domain of any nonconstant stable $\Phi_{S,p}$-harmonic map, and (d) The homotopic class of any map from M contains elements of arbitrarily small $\Phi_{S,p}$-energy.*

and prove

Theorem 7.5. *Every compact $\Phi_{S,p}$-superstrongly unstable* ($\Phi_{S,p}$-SSU) *manifold is $\Phi_{S,p}$-strongly unstable* ($\Phi_{S,p}$-SU).

Remark 7.6. *This $\Phi_{S,p}$-SSU manifold being $\Phi_{S,p}$-SU is an analog of Φ_S-SSU manifold being Φ_S-SU, or* $\Phi_{(2)}$-SSU *manifold being* $\Phi_{(2)}$-SU.

8. $\Phi_{(3)}$-harmonic maps, σ_3-symmetric functions, and $\Phi_{(3)}$-SSU manifolds([14])

We introduce unified notations and concepts of $\Phi_{(i)}$-harmonic maps which are σ_i version of the pullback u^*g_N, for $i=1,2,3$.

Definition 8.1. *Let $d_{(1)}u, d_{(2)}u$ and $d_{(3)}u$ be 1-forms with values in the pullback bundle $u^{-1}TN$ given by*

$$
\begin{aligned}
&d_{(1)}u(X) = du(X)\,, \\
&d_{(2)}u(X) = \sum_{j=1}^{m} h\big(du(X), du(e_j)\big) du(e_j)\,, \quad \text{and} \\
&d_{(3)}u(X) = \sum_{j,k=1}^{m} h\big(du(X), du(e_j)\big) h\big(du(e_j), du(e_k)\big) du(e_k)\,,
\end{aligned} \tag{8.1}
$$

respectively, for any smooth vector field X on (M,g), where $\{e_i\}$ is a local orthonormal frame field on (M,g), with the following corresponding norms

$$
\begin{aligned}
||d_{(1)}u||^2 &= \sum_{i=1}^{m} h\big(d_{(1)}u(e_i), du(e_i)\big) = \sum_{i=1}^{m} h\big(du(e_i), du(e_i)\big)\,, \\
||d_{(2)}u||^2 &= \sum_{i=1}^{m} h\big(d_{(2)}u(e_i), du(e_i)\big) = \sum_{i,j=1}^{m} h\big(du(e_i), du(e_j)\big) h\big(du(e_j), du(e_i)\big)\,, \quad \text{and} \\
||d_{(3)}u||^2 &= \sum_{i=1}^{m} h\big(d_{(3)}u(e_i), du(e_i)\big) = \sum_{i,j,k=1}^{m} h\big(du(e_i), du(e_j)\big) h\big(du(e_j), du(e_k)\big) h\big(du(e_k), du(e_i)\big).
\end{aligned}
$$

The $\Phi_{(1)}$-energy density $e_{\Phi_{(1)}}(u)$, $\Phi_{(2)}$-energy density $e_{\Phi_{(2)}}(u)$, and $\Phi_{(3)}$-energy density $e_{\Phi_{(3)}}(u)$ of u are given by

$$
\begin{aligned}
e_{\Phi_{(1)}}(u) &= \frac{||d_{(1)}u||^2}{2}, \\
e_{\Phi_{(2)}}(u) &= \frac{||d_{(2)}u||^2}{4}, \quad \text{and} \\
e_{\Phi_{(3)}}(u) &= \frac{||d_{(3)}u||^2}{6}, \quad \text{respectively}\,.
\end{aligned} \tag{8.2}
$$

The $\Phi_{(1)}$-energy $E_{\Phi_{(1)}}(u)$, $\Phi_{(2)}$-energy $E_{\Phi_{(2)}}(u)$, and $\Phi_{(3)}$-energy $E_{\Phi_{(3)}}(u)$ of u are given by

$$
\begin{aligned}
E_{\Phi_{(1)}}(u) &= \int_M e_{\Phi_{(1)}}(u) dv_g, \\
E_{\Phi_{(2)}}(u) &= \int_M e_{\Phi_{(2)}}(u) dv_g, \quad \text{and} \\
E_{\Phi_{(3)}}(u) &= \int_M e_{\Phi_{(3)}}(u) dv_g, \quad \text{respectively}\,.
\end{aligned} \tag{8.3}
$$

Definition 8.2. *For $i = 1, 2, 3$, a smooth map u is said to be a $\Phi_{(i)}$-harmonic map if it is a critical point of the $\Phi_{(i)}$-energy functional $E_{\Phi_{(i)}}$ with respect to any smooth compactly supported variation of u, stable $\Phi_{(i)}$-harmonic or simply $\Phi_{(i)}$-stable, if u is a local minimum of $E_{\Phi_{(i)}}(u)$, and $\Phi_{(i)}$-unstable if u is not $\Phi_{(i)}$-stable.*

Remark 8.3. (i) *The norm $||d_{(1)}u||$ is the Hibert-Schmid norm of the differential du, i.e., $||d_{(1)}u|| = |du|$.* (ii) *The $\Phi_{(1)}$-energy density $e_{\Phi_{(1)}}(u) = e(u)$ is the energy density of u.* (iii) *$\Phi_{(1)}$-harmonic map is ordinary harmonic map (cf. [13]).* (iv) *The $\Phi_{(2)}$-energy density $e_{\Phi_{(2)}}(u) = e_\Phi(u)$ is the Φ-energy density of u.* (v) *$\Phi_{(2)}$-harmonic map is Φ-harmonic map (cf. [22]).* (vi) *Definition 8.2 can be extended to $4 \le i \le m = \dim M$. Hence, for any integer $1 \le i \le m$, a smooth map u is said to be a $\Phi_{(i)}$-harmonic map if it is a critical point of the $\Phi_{(i)}$-energy functional $E_{\Phi_{(i)}}$ with respect to any smooth compactly supported variation of u, stable $\Phi_{(i)}$-harmonic or simply $\Phi_{(i)}$-stable, if u is a local minimum of $E_{\Phi_{(i)}}(u)$, and $\Phi_{(i)}$-unstable if u is not $\Phi_{(i)}$-stable.*

In fact, S.X. Feng, Y.B. Han, K. Jiang, and S.W. Wei show that the extrinsic average variational method in the calculus of variations employed in the study of σ_1 and σ_2 versions of the pullback metric u^*g_N on M can be extended to the study of the third symmetric function σ_3 version. The "distinguished" conservative vector fields worked in SSU manifolds to "universally" decrease the the energy E works in p-SSU, Φ-SSU, $\Phi_{(2)}$-SSU, Φ_S-SSU, $\Phi_{S,p}$-SSU, and $\Phi_{(3)}$-SSU manifolds. We introduce the notion of a $\Phi_{(3)}$-harmonic map and find a large class of manifolds, $\Phi_{(3)}$-superstrongly unstable ($\Phi_{(3)}$-SSU) manifolds,

Definition 8.4. *A Riemannian manifold M^m is said to be $\Phi_{(3)}$-superstrongly unstable ($\Phi_{(3)}$-SSU) if there exists an isometric immersion of M^m in $\mathbb{R}^q$ with its second fundamental form B such that for all unit tangent vectors v to M^m at every point $x \in M^m$, the following functional is negative valued.*

$$F_{\Phi_{(3)},x}(v) = \sum_{i=1}^{m} \big(6\langle B(v,e_i), B(v,e_i)\rangle_{\mathbb{R}^q} - \langle B(v,v), B(e_i,e_i)\rangle_{\mathbb{R}^q}\big), \tag{8.4}$$

where $\{e_1, \cdots, e_m\}$ is a local orthonormal frame field on M^m near x.

and introduce

Definition 8.5. *A Riemannian manifold M is $\Phi_{(3)}$-strongly unstable ($\Phi_{(3)}$-SU) if M is neither the domain nor the target of any nonconstant smooth $\Phi_{(3)}$-stable harmonic map (into or from any compact Riemannian manifold), and the homotopic class of maps from or into M contains a map of arbitrarily small $\Phi_{(3)}$-energy $E_{\Phi_{(3)}}$.*

and prove

Theorem 8.6 ([14])**.** *Every compact $\Phi_{(3)}$-superstrongly unstable ($\Phi_{(3)}$-SSU) manifold is $\Phi_{(3)}$-strongly unstable ($\Phi_{(3)}$-SU).*

Remark 8.7. *This $\Phi_{(3)}$-SSU manifold being $\Phi_{(3)}$-SU is an analog of SSU manifold being SU.*

9. Varied, Coupled, Generalized Harmonic maps, -Energy, with corresponding -SSU Manifolds and -SU ([52])

As in the philosophy of Lao-Tzu (Chapter 42 of Tao Te Ching),
"Ten Thousands things embrace polar opposites: Yin and Yang.
Integrating and balancing them through their generated "flow" achieve *harmony*".
(where "flow" is in English "Qi", which is pronounced the first syllable of "cheese")

Employing the extrinsic average variational method in [42, 40], we have found multiple large classes of new manifolds with geometric and topological properties in the setting of varied, coupled, generalized type of harmonic maps.

These newly found manifolds have their interactions with geometry, topology, analysis, partial differential equations, calculus of variations, physics, and are briefly listed in the following table. For more details, related ideas, techniques, we refer to [5], [15]-[14], [32]-[40], [52], [54], [56] and references within.

10. Product Manifolds

In Theorem 2.3 ([40]) we prove that in particular S^n or $S^n \times S^k$, for $n > 2, k > 2$ cannot be the target of any nonconstant stable harmonic maps.

The extrinsic average variational method can carry this idea and result to more general settings, These include from spheres, hypersurfaces in the Euclidean space, etc. to SSU manifolds, and the extension from the instability of a map to the infimum of variant energy of the map in its homotopy class. For example, we have the following

TABLE 1. An Extrinsic Average Variational Method

Mappings	Functionals	New manifolds found	Geometry	Topology
harmonic map or $\Phi_{(1)}$-harmonic map	energy functional E or $\Phi_{(1)}$-energy functional $E_{\Phi_{(1)}}$	SSU manifolds or $\Phi_{(1)}$-SSU manifolds	SU or $\Phi_{(1)}$-SU	$\pi_1 = \pi_2 = 0$ $\pi_1 = \pi_2 = 0$
$p-$ harmonic map	$p-energyfunctional$ E_p	$p-$ SSU manifolds	$p-$ SU	$\pi_1 = \cdots = \pi_{[p]} = 0$
Φ-harmonic map or $\Phi_{(2)}$-harmonic map	Φ-energy functional E_Φ or $\Phi_{(2)}$-energy functional $E_{\Phi_{(2)}}$	Φ-SSU manifolds or $\Phi_{(2)}$-SSU manifolds	Φ-SU or $\Phi_{(2)}$-SU	$\pi_1 = \cdots = \pi_4 = 0$ $\pi_1 = \cdots = \pi_4 = 0$
Φ_S-harmonic map	Φ_S-energy functional E_{Φ_S}	Φ_S-SSU manifolds	Φ_S-SU	$\pi_1 = \cdots = \pi_4 = 0$
$\Phi_{S,p}$-harmonic map	$\Phi_{S,p}$-energy functional $E_{\Phi_{S,p}}$	$\Phi_{S,p}$-SSU manifolds	$\Phi_{S,p}$-SU	$\pi_1 = \cdots = \pi_{[2p]} = 0$
$\Phi_{(3)}$-harmonic map	$\Phi_{(3)}$-energy functional $E_{\Phi_{(3)}}$	$\Phi_{(3)}$-SSU manifolds	$\Phi_{(3)}$-SU	$\pi_1 = \cdots = \pi_6 = 0$

Theorem 10.1. *Let $M_1^{m_1}, \cdots, M_\ell^{m_\ell}$ be p-SSU manifolds. Then (i) The product manifold $M = M_1^{m_1} \times \cdots \times M_\ell^{m_\ell}$ is a p-SSU manifold of dimension $m = m_1 + \cdots + m_\ell$. Hence, M is p-SU. (ii) There is a neighborhood of the product metric g_0 of the product manifold $M = M_1^{m_1} \times \cdots \times M_\ell^{m_\ell}$ in the C^2 topology (if M is not compact we must use the strong C^2 topology) such that for every g in this neighborhood the Riemannian manifold (M, g) is a p-SSU manifold of dimension $m = m_1 + \cdots + m_\ell$. Hence, (M, g) is p-SU.*

Proof. (i) By assumption, for each $1 < i < \ell$ we have an isometric immersion of $M_i^{m_i}$ into $\mathbb{R}^{q_i}$ with the second fundamental for B^{M_i} in such a way that each functional as in (4.2),

$$F_{p,x_i}^{M_i}(v) = (p-2)\langle B^{M_i}(v,v), B^{M_i}(v,v)\rangle + \langle Q_{x_i}^{M_i}(v), v\rangle_{M_i}, \tag{4.2$'$}$$

is negative valued. It follows that for the product immersion of $M = M_1^{m_1} \times \cdots \times M_\ell^{m_\ell}$ into $\mathbb{R}^q$ ($q = q_1 + \cdots + q_\ell$), $F_{p,x}(v)$ is also negative valued. This proves that M is a p-SSU manifold of dimension m. M is p-SU follows from Theorem 4.4.

To prove (ii), it is enough to show there is a neighborhood $\mathcal{U}$ of g_0 in the C^2 topology such that for every $g \in \mathcal{U}$ the set $\{V_1, \cdots V_\ell\}$ is still universally p-energy E_p decreasing on (M, g). Let $\mathfrak{M}(M)$ be the space of smooth Riemannian metrics on M with the strong C^2 topology and let $C^\infty(\bullet, M)$ be the space of smooth maps into M. Consider the function on $C^\infty(\bullet, M) \times \mathfrak{M}(M)$ given by

$$(u,g) \mapsto \sum_{i=1}^{\ell} \mathcal{E}_p(V_i, u, g) := \frac{1}{p}\sum_{i=1}^{\ell} \frac{d^2}{dt^2}\bigg|_{t=0} ||(\phi_t^{v_i^T} \circ u) * X||^p. \tag{10.1}$$

If this is continuous then

$$\mathcal{U} = \{g \in \mathfrak{M}(M) : \sum_{i=1}^{\ell} \mathcal{E}_p(V_i, u, g) < 0 \text{ for all } u \in C^\infty(\bullet, M)\}$$

is the required neighborhood of g_0. To show that the function given by (10.1) is continuous it is enough to show that for any smooth vector field V the function $(g, u) \mapsto \mathcal{E}_p(V, u, g)$ is continuous.

We note $\mathcal{E}_p(V, u, g)$ involves with $\mathcal{A}^V$ and $\nabla_V \mathcal{A}^V$. When $p = 2$, $\mathcal{E}_p(V, u, g)$ becomes

$$\mathcal{E}(V, u, g) = g(\mathcal{A}^V\mathcal{A}^V X, X) + g(\mathcal{A}^V X, \mathcal{A}^V X) + g(\nabla_V \mathcal{A}^V, X)$$

Let $x^1, \ldots x^n$ be local coordinates on M and let $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ be the components of g in this coordinate system. Let the Christoffel symbols Γ_{ij}^k be given as usual by $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x^k}$. Then by a well known formula

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{\ell=1}^{n} g^{k\ell}\Big(\frac{\partial g_{\ell j}}{\partial x^i} + \frac{\partial g_{\ell i}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\ell}\Big)\,. \tag{10.2}$$

(where $[g^{ij}]$ is the inverse of the matrix $[g_{ij}]$). If the vector field V is locally given by $V = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}$ and the components $(\mathcal{A}^V)_i^j$ and $(\nabla_V \mathcal{A}^V)_i^j$ are given by

$$(\mathcal{A}^V)\frac{\partial}{\partial x^i} = \sum_{j=1}^n (\mathcal{A}^V)_i^j \frac{\partial}{\partial x^j}\,, \quad (\nabla_V \mathcal{A}^V)\frac{\partial}{\partial x^i} = \sum_{j=1}^n (\nabla_V \mathcal{A}^V)_i^j \frac{\partial}{\partial x^j}$$

then a little calculation shows that

$$(\mathcal{A}^V)_i^j = \frac{\partial v^j}{\partial x^i} + \sum_{k=1}^n v^k \Gamma_{ik}^j\,. \tag{10.3}$$

$$(\nabla_V \mathcal{A}^V)_i^j = \sum_{k=1}^n v^k \frac{\partial a_i^j}{\partial x^k} + \sum_{k,\ell=1}^n (a_i^\ell v^k \Gamma_{k\ell}^j - a_\ell^j v^k \Gamma_{ki}^\ell)\,, \tag{10.4}$$

where $a_i^j = (\mathcal{A}^V)_i^j$. Indeed,

$$\begin{aligned}
(\nabla_V \mathcal{A}^V)\frac{\partial}{\partial x^i} &= \nabla_V\big(\mathcal{A}^V(\frac{\partial}{\partial x^i})\big) - \mathcal{A}^V(\nabla_V \frac{\partial}{\partial x^i}) \\
&= \nabla_{\sum_{k=1}^n v_k \frac{\partial}{\partial x^k}}\Big(\sum_{j=1}^n a_i^j \frac{\partial}{\partial x^j}\Big) - \nabla_{\nabla_{\sum_{k=1}^n v_k \frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}} V\Big) \\
&= \sum_{j,k=1}^n v^k \frac{\partial a_i^j}{\partial x^k}\frac{\partial}{\partial x^j} + \sum_{k,\ell=1}^n (a_i^j v^k \Gamma_{kj}^\ell \frac{\partial}{\partial x^\ell} - v^k \Gamma_{ki}^\ell \nabla_{\frac{\partial}{\partial x^\ell}} V)\,, \\
&= \sum_{j=1}^n \Big(\sum_{k=1}^n v^k \frac{\partial a_i^j}{\partial x^k} + \sum_{k,\ell=1}^n (a_i^\ell v^k \Gamma_{k\ell}^j - a_\ell^j v^k \Gamma_{ki}^\ell)\Big)\frac{\partial}{\partial x^j}\,,
\end{aligned}$$

Putting (10.2) into (10.3) and (10.4) and the result of that into (10.1) gives $\mathcal{E}_p(V, u, g)$ as a rational function of the g_{ij} and their first two derivatives. Thus $\mathcal{E}_p(V, u, g)$ is clearly a continuous function of g in the strong C^2 topology. Analogously, we can show the function

$$(u, g) \mapsto \frac{1}{p}\sum_{i=1}^\ell \frac{d^2}{dt^2}\Big|_{t=0} ||\big(u \circ \phi_t^{\mathrm{v}_i^T}\big) * X||^p \tag{10.5}$$

is a continuous function of g in the strong topology. This completes the proof.

□

There is an analog of a neighborhood of g_0 in the C^2 topology in unstable rectifiable currents (cf. [24, Theorem 2.1]). Proceed in the same spirit, by a continuous function of metric g in the strong topology as in the proof of Theorem 10.1, we obtain

Theorem 10.2. *Let $M_1^{m_1}, \cdots, M_\ell^{m_\ell}$ be X-SSU manifolds, where X- is one of the following: $\Phi_{(1)}$-, Φ_S-, $\Phi_{S,p}$-, $\Phi_{(2)}$-, $\Phi_{(3)}$-. Then (i) The product manifold $M = M_1^{m_1} \times \cdots \times M_\ell^{m_\ell}$ is an X-SSU manifold of dimension $m = m_1 + \cdots + m_\ell$. Hence, M is X-SU, i.e. M is the corresponding $\Phi_{(1)}$-SU, Φ_S-SU, $\Phi_{S,p}$-SU, $\Phi_{(2)}$-SU, or $\Phi_{(3)}$-SU. (ii) There is a neighborhood of the product metric g_0 of the product manifold $M = M_1^{m_1} \times \cdots \times M_\ell^{m_\ell}$ in the C^2 topology (if M is not compact we must use the strong C^2 topology) such that for every g in this neighborhood the Riemannian manifold (M, g) is an X-SSU manifold of dimension $m = m_1 + \cdots + m_\ell$. Hence, (M, g) is X-SU.*

Remark 10.3. *The extrinsic average variational method in the calculus variations also gives*

(i) *the first nonexistence theorem of stable Yang-Mills fields on product manifolds* ([41]).
(ii) *the first classification of stable rectifialbe currents on product manifolds* ([48]).
(iii) *the first nonexistence theorem of stable nonconstant harmonic maps into product manifolds*([40]).

11. Liouville type theorems for stable harmonic maps into SSU manifolds

We extended our study on the nonexistence of stable harmonic maps between compact manifolds ([23]) to that between complete, non-compact ones. Thus, employing the extrinsic average variational method in the calculus of variations, S.W. Wei established the first Liouvile-type theorem of stable harmonic maps into SSU manifolds.

We recall a manifold M is said to be *parabolic* if M admits no nonconstant positive superharmonic function. Whereas a complete noncompact manifold with quadratic volume growth is parabolic, Wei-Li-Wu constructed examples of p-parabolic manifolds with exponential volume growth (cf. [54]).

Theorem 11.1 ([43])**.** *Every smooth, stable harmonic map $u : M \to N$ from a parabolic manifold M into any* SSU*-manifold N is constant.*

Using the extrinsic average variational method in the calculus of variations, S.W. Wei and C.M. Yau ([56]) extend and generalize the above results to Liouville Theorems for stable p-harmonic maps into p-SSU manifolds

In contrast to vanishing theorems for differential forms with values in vector bundles ([11, 10]), Liouville theorems for $\Phi_{(1)}$-harmonic maps, (resp. $\Phi_{(3)}$-harmonic maps, and $\Phi_{S,p}$-harmonic maps), by using techniques of $\Phi_{(1)}$-stress-energy tensor, $\Phi_{(1)}$-conservation law, monotonicity formula for $\Phi_{(1)}$-energy $\big($resp. $\Phi_{(3)}$-stress-energy tensor, $\Phi_{(3)}$-conservation law, monotonicity formula for $\Phi_{(3)}$-energy, and $\Phi_{S,p}$-stress-energy tensor, $\Phi_{S,p}$-conservation law, and monotonicity formula for $\Phi_{S,p}$-eneergy$\big)$ are derived in ([26, 14, 16]).

12. Regularity of Energy-Minimizing Maps and SSU-index

We recall a map $\overline{u} : \mathbb{R}^{j+1} \to N$ is said to be a **p-enegy-minimizing tangent map** if $\overline{u}$ is p-energy minimizing on every compact subset of $\mathbb{R}^{j+1}$ and is a homogeneous extension of $u : S^j \to N$ of degree-zero, i.e., $\overline{u}(x) = u(\frac{x}{|x|})$ for every $0 \neq x \in \mathbb{R}^{j+1}$

Theorem 12.1 ([20], Hardt-Lin, Theorem 4.5, p.573)**.** *Suppose ℓ is the largest integer such that any p-energy minimizing tangent map from the unit ball in $\mathbb{R}^j$ into N is a constant map for each $j = 1, \cdots, \ell$. Then the interior singular set of any p-energy minimizing tangent map $u \in L^p_1(\Omega, N)$ is empty in case $n < \ell + 1$, is a discrete set in case $n = \ell + 1$, and has Hausdorff dimension $n - \ell - 1$ in case $n \geq \ell + 1$ (Where Ω is a C^2 bounded open subset of $\mathbb{R}^n$ with the Euclidean metric).*

For a given SSU manifold, S.W. Wei found the first SSU-index w which serves as an indication of the regularity of energy minimizing harmonic maps into SSU manifolds.

In applying Theorem 12.1, we find SSU-index w is a number between $0 < w < 1$, the *higher* the index w is (or the *closer w is to 1*), the *higher dimension* ℓ of the domain of trivial minimizing tangent map $\overline{u}$ is (or the *easier* Liouville theorem for minimizing tangent map $\overline{u}$ to hold)), and hence the *smaller* the size of the Hausdorff dimension of the singular set $= n - \ell - 1$ is (or the *smoother* of $L^{1,2}$- energy minimizing map into SSU-manifold is, in terms of SSU-index (cf. [43]).

This result is extended to the regularity of p-energy minimizing $L^{1,p}$ map into p-SSU manifolds in terms of p-SSU-index by Wei-Yau. ([56])

On the other hand, the following regularity theorem is achieved.

Theorem 12.2 ([56] Theorem 1.4)**.** *Every p-energy minimizing $L^{1,p}$- map into a manifold N with Riem$^N \leq 0$ or into a domain of a convex function is $C^{1,\alpha}$.*

Proof. By the first variation formula for p-energy E_p formula ([?] p.249), any p-energy minimizing tangent map $\overline{u}$ from the unit ball in $\mathbb{R}^j$ into a N is a constant map, for every integer j. The singular set of u is empty, due to p-energy minimizing tangent map $\overline{u}$ from the unit ball in $\mathbb{R}^j$ into a nonpositively curved N is a constant map, for every integer j. In view of Theorem 12.1, $n < \ell + 1$, the singular set of $L^{1,p}$ map u is empty. It folllows from the regularity theorem u is $C^{1,\alpha}$.([20, 31]). □

13. Dirichlet boundary value problems

Regularity Theorem 12.2 of p-energy minimizing $L^{1,p}$ maps is used to solve Dirichlet boundary value problem for p-harmonic maps into manifolds with nonpositive sectional curvature ([45]), generalizing the case $p = 2$ for harmonic maps due to R. S. Hamilton ([17]).

S.W. Wei also proves the uniqueness of solutions of Dirichlet boundary value problem for p-harmonic maps into [51]), manifolds with nonpositive sectional curvature ([45]). The case $p = 2$ is due to P. Hartman ([18]), where the heat flow method is used.

Solutions of some Dirichlet Boundery Value Problem for differential 1-form can be found in (e.g., [27, 11]).

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Department of Mathematics, University of Oklahoma, Norman, Oklahoma 73019-0315, U.S.A.
Email address: wwei@ou.edu