A NOTE ON BERTRAND CURVES AND SURFACES AND THEIR APPLICATIONS

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Abstract. A Bertrand curve is a space curve having its principal normals as principal normals for another space curve. Interesting well-known geometric properties are recalled and a less known property is proven. One gives a way to construct a Bertrand curve and an example.

As a natural generalization, Bertrand surfaces are defined and a geometric property is given. Applications of Bertrand curves and surfaces in engineering are mentioned.

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1. Introduction

In this section we will shortly recall the definition of a parametrized space curve, the Frenet frame associated to a space curve and Frenet formulae. We will give the definition of Bertrand curves and some of its well-known geometrical properties.

Let $I \subset \mathbf{R}$ be a real interval, $\mathbf{E_3}$ the 3-dimensional Euclidean space and

$$c : I \to \mathbf{E_3},\ c(s) = (x(s), y(s), z(s)),$$

be a canonical parametrized space curve, i.e.,

$$\|\dot{c}(s)\| = 1,\ \forall\ s \in I,$$

where the dot denotes the first derivative.

The parameter s for which $\|\dot{c}(s)\| = 1$ is called the *canonical parameter*. From the theory of curves one knows that any regular space curve admits a canonical parameter.

Denote by $\{t,\ n,\ b\}$ the Frenet frame associated to the curve c (see Figure 1).

We recall the following notations: t is the *unit tangent vector*, n is the *unit principal normal vector*, b is the *unit binormal vector*. Also, (n, b) is the *normal plane*, orthogonal to the tangent vector, (t, b) is the *rectifying plane*, orthogonal to the principal normal vector and (t, n) is the *osculating plane*, orthogonal to the binormal vector.

The existence of a Frenet frame at a point of a space curve is assured by the condition that the vectors $\dot{c}(s)$ and $\ddot{c}(s)$ to be linearly independent, where double dots mean the second derivative.

The *curvature* of a curve (with respect to the canonical parameter) is denoted by $K(s)$ and its *torsion* is denoted by $\tau(s)$ and they are given by

$$K(s) = \|\ddot{c}(s)\|,$$

$$\tau(s) = \frac{\det(\dot{c}(s), \ddot{c}(s),\ \dddot{c}(s))}{\|\ddot{c}(s)\|^2}.$$

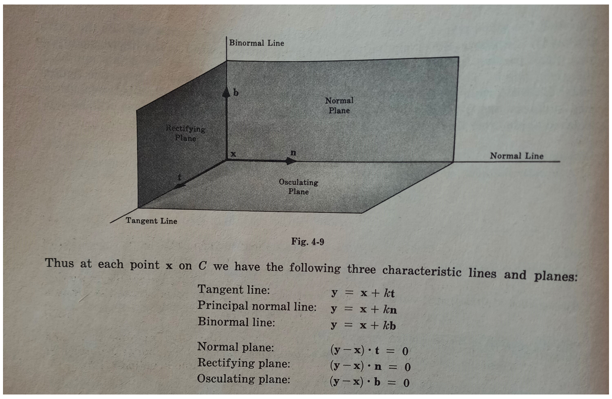

Fig. 4-9

Thus at each point **x** on C we have the following three characteristic lines and planes:

Tangent line:	$\mathbf{y} = \mathbf{x} + k\mathbf{t}$
Principal normal line:	$\mathbf{y} = \mathbf{x} + k\mathbf{n}$
Binormal line:	$\mathbf{y} = \mathbf{x} + k\mathbf{b}$
Normal plane:	$(\mathbf{y}-\mathbf{x}) \cdot \mathbf{t} = 0$
Rectifying plane:	$(\mathbf{y}-\mathbf{x}) \cdot \mathbf{n} = 0$
Osculating plane:	$(\mathbf{y}-\mathbf{x}) \cdot \mathbf{b} = 0$

FIGURE 1. Frenet frame ([2])

The *Frenet formulae* with respect to the canonical parameter are:

$$\dot{t}(s) = K(s)n(s),$$

$$\dot{n}(s) = -\ K(s)t(s) + \tau(s)b(s),$$

$$\dot{b}(s) = -\tau(s)n(s),$$

or, in an equivalent way, by using matrices:

$$\begin{pmatrix} \dot{t}(s) \\ \dot{n}(s) \\ \dot{b}(s) \end{pmatrix} = \begin{pmatrix} 0 & K(s) & 0 \\ -K(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix}.$$

Remark 1.1. The unit normal principal vector $n(s)$ is always collinear with $\ddot{c}(s)$, for s canonical parameter, because, by definition, $n(s) = \frac{\ddot{c}(s)}{||\ddot{c}(s)||}$.

This property holds also for an arbitrary parameter t for which $||\dot{c}(t)||$ is constant.

A *Bertrand curve* is a curve $c : I \to \mathbf{E_3}$ for which its principal normals are principal normals for another curve $\overline{c}$.

The nontrivial case is that of space curves and we will consider Bertrand (space) curves in this note.

The curves c and $\overline{c}$ are called *Bertrand associated curves* and the points in which the principal normals coincide are *associated points* or *corresponding points* (see Figure 2).

These curves have interesting geometric properties; some of them are recalled below:

Proposition 1.2. ([2], [3])

1) *The distance between two corresponding points on two associated Bertrand curves, measured on the common principal normal, is constant.*
2) *The angle between the tangents at two Bertrand curves in two corresponding points is constant.*
3) *If $K(s)$ and $\tau(s)$ are the curvature, respectively the torsion of a Bertrand curve $c : I \to \mathbf{E_3}$ at an arbitrary point $c(s)$, then there exist 3 real constants $a_1, a_2, a_3 \in \mathbf{R}$, $a_3 \neq 0$, such that*

$$a_1K(s) + a_2\tau(s) + a_3 = 0, \quad \forall\ s \in I.$$

Remark 1.3. A similar relation holds for the associated curve $\overline{c}$: $\exists\ \overline{a}_1, \overline{a}_2, \overline{a}_3\ \in \mathbf{R}, \overline{a}_3 \neq 0$, such that

$$\overline{a}_1\overline{K}(\overline{s}) + \overline{a}_2\overline{\tau}(\overline{s}) + \overline{a}_3 = 0, \quad \forall\ \overline{s} \in \overline{I}.$$

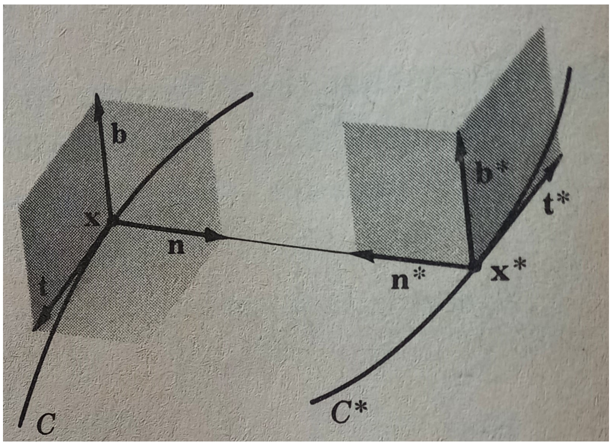

FIGURE 2. Two associated Bertrand curves ([2])

Proposition 1.4. [2] *If for a space curve $c : I \to \mathbf{E_3}$ (of zero torsion) there exist at least two curves c^* si c^{**} which are Bertrand associated, then the curve c is a circular helix. The converse statement also holds.*

2. Torsions of two Bertrand associated curves

This section contains the proof of a less-known property of associated Bertrand curves.
More precisely, in [2] the following problem is proposed:

Prove that the product of the torsions of two Bertrand associated curves is constant.

We propose the following:

Proof.
From item 1) of Proposition 1.2. we have

$$\overline{c}\,(\overline{s}) = c(s) + an(s),$$

because the distance between two corresponding points on two associated Bertrand curves is constant, a.
From the previous relation and from the Frenet formulae it follows that:

$$\frac{d\overline{c}\,(\overline{s})}{ds} = \frac{dc(s)}{ds} + a\frac{dn(s)}{ds} = \; t(s) + a\dot{n}(s) =$$

$$t(s) + a\left(-K(s)t(s) + \tau(s)b(s)\right) = \left(1 - aK(s)\right)t(s) + a\tau(s)b(s).$$

We have

$$\overline{t}\,(\overline{s}) = \frac{d\overline{c}\,(\overline{s})}{ds}\frac{ds}{d\overline{s}} = \left[\left(1 - aK(s)\right)t(s) + a\tau(s)b(s)\right]\frac{ds}{d\overline{s}}.$$

On the other hand, we can write

$$\overline{t}\,(\overline{s}) = (\cos\theta)t(s) + (\sin\theta)b(s),$$

where θ is the constant angle of the tangents (see 2) from Proposition 1.2); then

$$\cos\theta = \left(1 - aK(s)\right)\frac{ds}{d\overline{s}},$$

$$\sin\theta = a\tau(s)\frac{ds}{d\overline{s}}.$$

We shall have

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1 - aK(s)}{a\tau(s)}.$$

On the other hand,

$$\begin{aligned}\bar{b}\left(\bar{s}\right) = \bar{t}\left(\bar{s}\right) \times \overline{n}\left(\bar{s}\right) = \\ = (\cos\theta t(s) + \sin\theta b(s)) \times (\pm n(s)) = \\ = \pm\cos\theta t(s) \times n(s) \pm \sin\theta b(s) \times n(s) = \\ = \ \pm\left[\cos\theta b(s) - \sin\theta t(s)\right].\end{aligned}$$

Then

$$\begin{aligned}\frac{d\bar{b}\left(\bar{s}\right)}{d\bar{s}} &= \pm\left[\cos\theta\dot{b}(s) - \sin\theta\dot{t}(s)\right]\frac{ds}{d\bar{s}} = \\ &= \pm\left[-\cos\theta\tau(s)n(s) - \sin\theta K(s)n(s)\right]\frac{ds}{d\bar{s}} \\ &= \left[-\cos\theta\tau(s)\overline{n}\left(\bar{s}\right) - \sin\theta K(s)\overline{n}\left(\bar{s}\right)\right]\frac{ds}{d\bar{s}}.\end{aligned}$$

It follows that

$$\overline{\tau}\left(\bar{s}\right)\overline{n}\left(\bar{s}\right) = (\cos\theta\tau(s) + \sin\theta K(s))\overline{n}\left(\bar{s}\right)\frac{ds}{d\bar{s}},$$

and then

$$\begin{aligned}\overline{\tau}\left(\bar{s}\right) &= \left[\cos\theta\tau(s) + \sin\theta K(s)\right]\frac{ds}{d\bar{s}} = \sin\theta[\cot\theta\tau(s) + K(s)]\frac{ds}{d\bar{s}} \\ &= \sin\theta\left[\frac{1 - aK(s)}{a\tau(s)}\tau(s) + K(s)\right]\frac{ds}{d\bar{s}} = \frac{\sin\theta}{a}\frac{ds}{d\bar{s}}.\end{aligned}$$

Similarly,

$$\tau(s) = \frac{\sin\theta}{a}\frac{d\bar{s}}{ds}.$$

Then *the product of torsions of two Bertrand curves is constant*, more precisely

$$\tau(s)\overline{\tau}\left(\bar{s}\right) = \frac{\sin^2\theta}{a^2}.$$

3. Construction of a Bertrand curve. Example

Let $c : I \to \mathbf{E_3}$ be defined by:

$$c(t) = \alpha\int g(t)dt + \beta\int g(t) \times \dot{g}(t)dt,$$

where $g : I \to \mathbf{E_3}$ is a vector-valued smooth function, with $\|g(t)\| = \|\dot{g}(t)\| = 1$, and α, β are real constants.

We shall prove that c is a Bertrand curve.

More precisely, this problem (proposed in [2]) represents a way to construct Bertrand curves.

Starting from the definition of c, by calculating the first derivative we find $\dot{c}(t) = \alpha g(t) + \beta g(t) \times \dot{g}(t)$, where $\alpha,\ \beta \in \mathbf{R}$; it follows that $\|\dot{c}(t)\| = \sqrt{\alpha^2 + \beta^2}$ is constant.

From the second derivative we obtain

$$\begin{aligned}\ddot{c}(t) = \alpha\dot{g}(t) + \beta[g(t) \times \dot{g}(t)]' = \\ = \alpha\dot{g}(t) + \beta\dot{g}(t) \times \dot{g}(t) + \beta g(t) \times \ddot{g}(t) = \alpha\dot{g}(t) + \beta g(t) \times \ddot{g}(t).\end{aligned}$$

It follows that the principal normal vector $n(t)$, being collinear with $\ddot{c}(t)$ (according to Remark 1.1) will be written as a linear combination of the vectors $\dot{g}(t)$ and $g(t) \times \ddot{g}(t)$.

On the other hand, $\{g(t), \dot{g}(t), g(t) \times \dot{g}(t)\}$ is an orthonormal frame of unit vectors (this follows immediately from the definition of the vector function g).

We have $< g(t) \times \ddot{g}(t), g(t) > = (g(t), \ddot{g}(t), g(t)) = 0$ (mixed product) and

$$< g(t) \times \ddot{g}(t), g(t) \times \dot{g}(t) > = \frac{1}{2} < g(t) \times \dot{g}(t), g(t) \times \dot{g}(t) >' = 0.$$

It follows that $g(t) \times \ddot{g}(t)$ is collinear with $\dot{g}(t)$, and then we can write $g(t) \times \ddot{g}(t) = \gamma \dot{g}(t)$.

Then $\ddot{c}(t) = \alpha \dot{g}(t) + \beta\gamma \dot{g}(t) = (\alpha + \beta\gamma)\dot{g}(t)$, and so

$$n(t) = \pm \dot{g}(t).$$

By defining the curve $c^*(t) = \int g(t)dt$, where t is the canonical parameter for $\int g(t)dt$ (because $\|g(t)\| = \left\|(\int g(t)dt)'\right\| = 1$), then t it is also canonical parameter for c^*, it follows that the principal normal vector $n^*(t)$ of the curve $c^*(t)$, is collinear with $\ddot{c}^*(t)$, and then it is collinear with $\dot{g}(t)$.

Then, we proved that c and c^* are associated Bertrand curves, with the same principal normals of direction $\dot{g}(t)$.

We construct the following

Example. Let $g(t) = \left(\frac{1}{\sqrt{2}} \sin t, \frac{1}{\sqrt{2}} \sin t, \cos t\right)$, from where $\|g(t)\| = 1$.

We have

$$\dot{g}(t) = \left(\frac{1}{\sqrt{2}} \cos t, \frac{1}{\sqrt{2}} \cos t, \ -\sin t\right)$$

and $\|\dot{g}(t)\| = 1$. Then

$$g(t) \times \dot{g}(t) = \begin{vmatrix} i & j & k \\ \frac{1}{\sqrt{2}} \sin t & \frac{1}{\sqrt{2}} \sin t & \cos t \\ \frac{1}{\sqrt{2}} \cos t & \frac{1}{\sqrt{2}} \cos t & -\sin t \end{vmatrix} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right).$$

It follows that

$$c(t) = \alpha \int g(t)dt + \frac{\beta}{\sqrt{2}} \int (-1, 1, 0)dt =$$

$$= \left(-\frac{\alpha}{\sqrt{2}} \cos t - \frac{\beta}{\sqrt{2}} t, \ -\frac{\alpha}{\sqrt{2}} \cos t + \frac{\beta}{\sqrt{2}} t, \alpha \sin t \right)$$

is a Bertrand curve.

The associated Bertrand curve is

$$c^*(t) = \int g(t)dt = \left(-\frac{1}{\sqrt{2}} \cos t, \ -\frac{1}{\sqrt{2}} \cos t, \sin t\right).$$

By calculations, one proves that the principal normals of directions $n(t)$ and $n^*(t)$ of the curves $c(t)$ and $c^*(t)$, respectively, are:

$$\frac{x + \frac{\alpha}{\sqrt{2}} \cos t + \frac{\beta}{\sqrt{2}} t}{-\frac{\alpha^3}{\sqrt{2}} \cos t} = \frac{y + \frac{\alpha}{\sqrt{2}} \cos t - \frac{\beta}{\sqrt{2}} t}{-\frac{\alpha^3}{\sqrt{2}} \cos t} = \frac{z - \alpha \sin t}{\alpha^3 \sin t},$$

$$\frac{x + \frac{1}{\sqrt{2}} \cos t}{-\frac{1}{\sqrt{2}} \cos t} = \frac{y + \frac{1}{\sqrt{2}} \cos t}{-\frac{1}{\sqrt{2}} \cos t} = \frac{z - \sin t}{\sin t}.$$

Obviously, the principal normal of the curve c coincides with the principal normal of c^* (both have the director vector $\left(-\frac{1}{\sqrt{2}} \cos t, \ -\frac{1}{\sqrt{2}} \cos t, \sin t\right)$, and then c and c^* are Bertrand associated curves.

In Figure 3 we illustrated this example for $\alpha = 3$ şi $\beta = 5$ (by using https://www.math3d.org/tnb).

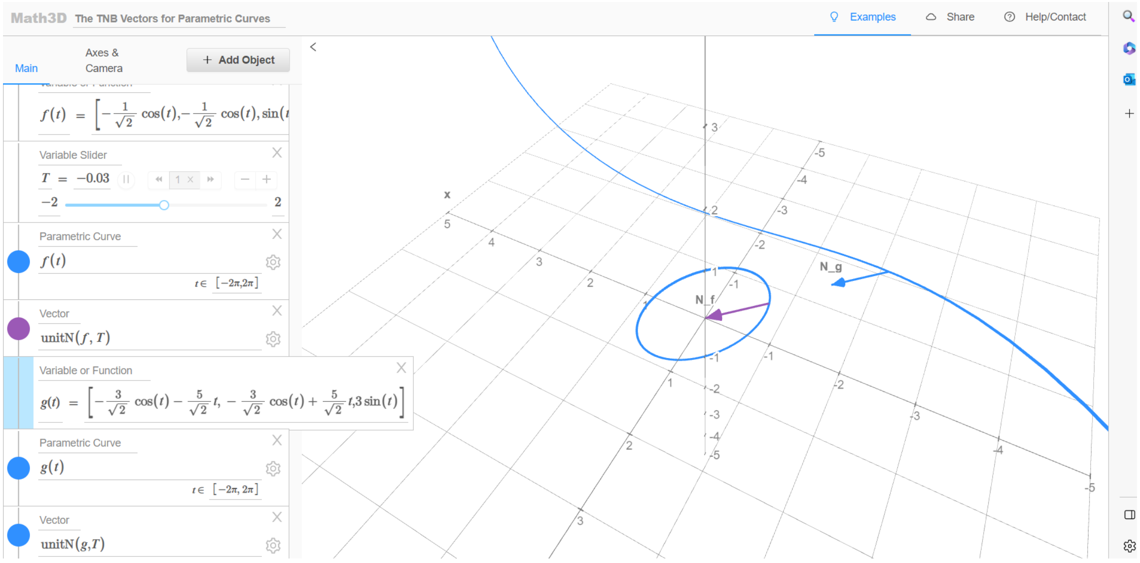

FIGURE 3. Example for $\alpha = 3$ şi $\beta = 5$

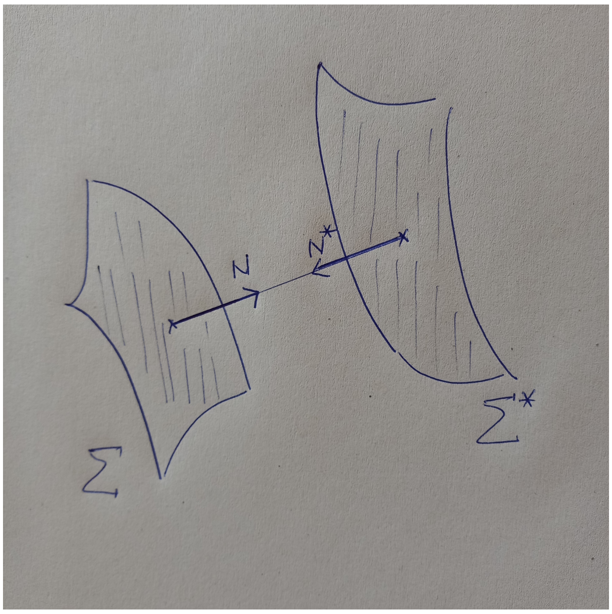

FIGURE 4. Associated Bertrand surfaces

4. BERTRAND SURFACES

A natural generalization of the notion of a Bertrand curve is that of a *Bertrand surface.*

We define two *Bertrand associated surfaces* as two surfaces which (at corresponding points) have common normals (see Figure 4).

Let Σ be the surface defined by

$$\Sigma : r(u, v) = (x(u, v), y(u, v), z(u, v)) .$$

FIGURE 5. https://www.gwstoolgroup.com/the-ins-and-outs-of-ball-nose-end-mills/

We denote by $N(u,v)$ the normal to the surface Σ at the (regular) point $(x(u,v), y(u.v), z(u,v))$ (i.e. $r_u = \frac{\partial r}{\partial u}$ and $r_v = \frac{\partial r}{\partial v}$ are linearly independent vectors at that point).

We define the surface Σ^* by $r^*(u^*,v^*) = r(u,v) + a(u,v)N(u,v)$, where $a(u,v)$ represents the distance between the corresponding points, measured on the common normal.

We have

$$\frac{\partial r^*}{\partial u}(u^*,v^*) = \frac{\partial r}{\partial u}(u,v) + \frac{\partial a}{\partial u}(u,v)N(u,v) + a(u,v)\frac{\partial N}{\partial u}(u,v).$$

The left hand side of the equality is tangent to Σ^*, and then is tangent to Σ.

The first term of the right hand side is tangent to Σ.

The last term of the right hand side is tangent to Σ.

It follows that $\frac{\partial a}{\partial u}(u,v) = 0$, because the right hand side of the equality cannot have a normal component to Σ.

In a similar way, one can proves that $\frac{\partial a}{\partial v}(u,v) = 0$.

Then it follows that $a(u,v) = a$ (constant).

We proved the following

Proposision 4.1. *The distance between the corresponding points, measured on the common normal of a two Bertrand surfaces is constant.*

Remark 4.2. The Proposition 4.1. corresponds to the item 1 of Proposition 1.2. for Bertrand curves.

Proposition 4.3. *The angle between the tangent planes to the Bertrand surfaces Σ and Σ^* at corresponding points is zero.*

The proof is obvious, because the two tangent planes are normal to the same line.

Remark 4.4. The Proposition 4.3. corresponds to the item 2 of Proposition 1.2. for Bertrand curves.

5. Applications

The Bertrand curves are well-known in geometry. Still, their applications were not intensively studied.

In [4] applications of Bertrand curves in CADCAM (Computer-Aided Design Computer-Aided Manufacturing) are described, more precisely in programming of cutter motions, i.e., cutting movements of different objects.

In the same paper [4] we also found the notion of Bertrand surfaces, but they were constructed conversely, i.e., starting from a practical application, called the ball-end cutter with nose radius (see Figure 5), but the mathematical definition which arises from this practical application coincides with our definition from the previous section and there geometrical properties were not investigated.

For applications of Bertrand surfaces in engineering we can also refer to [1].

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