COMPUTING THE CLOSENESS CENTRALITIES OF A TYPE OF THE PROBABILISTIC NEURAL NETWORKS

TUFAN TURACI

Abstract. The study of communication networks is an important area of different researches. The vulnerability of a network computation the reliability of the network to deterioration of transaction after the disturbance of certain centers or communication links. Recently, different centrality parameters have been proposed and studied for communication networks, also they play important role in the areas of network vulnerability. The position of a node in a network can be determined using the these measures. In this paper, the closeness, the vertex residual closeness and normalized vertex residual closeness of a type of the probabilistic neural networks which has been recently defined by in [14, 18] have been computed. In here, the probabilistic neural networks are consisted of three layers of nodes, also they are denoted by $PNN(n, k, m)$, where n, k and m are positive integers.

Mathematics Subject Classification (2010): 05C12, 68M10, 68R10.
Key words: Graph theory, network design, communications, probabilistic neural networks, closeness centrality, residual closeness, normalized vertex residual closeness.

Article history:
Received: October 15, 2020
Received in revised form: October 6, 2021
Accepted: October 9, 2021

1. Introduction

The study of networks has been become a very important area of different research such as computer science, mathematics, informatics, chemistry, social sciences, and other applied sciences [21, 22]. Furthermore, networks can be seen in many different area, settings and applications, also they define a wide range of systems in community and nature including some examples such as the metabolic networks, the neural networks, the Internet, the electric power grids, supply chains and so on. The vulnerability and reliability of networks are of major importance to designers of networks. There are a lot of vulnerability parameters have derived for networks. These values show that the reluctance of the network after the deterioration of some vertices which represents centers or edges which represents line between any two centers until a communication breakdown [5]. Graph theory techniques should be used to determine vulnerability values of a networks. Moreover, it is most strong mathematical tools and a network has been represented by a graph [5].

Let $G = (V(G), E(G))$ be a simple undirected graph of order n and size m. Now, we give some standard definitions for graphs. They have been used throughout this paper. For any vertex $v \in V(G)$, the *open neighborhood* of v is $N_G(v) = \{u \in V(G)|uv \in E(G)\}$ and *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of vertex v denoted by $deg_G(v)$ is the size of its open neighborhood in the graph G. The *maximum degree* of G denoted by $\Delta(G)$ is defined as $\max\{\deg_G(v)\,|v \in V(G)\}$. Similarly, the *minimum degree* of G denoted by $\delta(G)$ is defined as $\min\{\deg_G(v)\,|v \in V(G)\}$. Let $u, v \in V(G)$ be two vertices of

G. The *distance* between the vertices u and v in G denoted by $d_G(u, v)$ and it is defined that the length of a shortest path between them [15].

Centrality parameters are important in the areas of analysis of network. Numerous centrality parameters were proposed and analyzed for networks. These concept determines the importance of position of a node in a network. For example, graph theoretical parameters such as the edge betweenness, the average vertex and edge betweenness, the normalized average vertex and the edge betweenness [16, 19, 23, 24, 25], the closeness, the vertex residual closeness, the normalized vertex residual closeness [6, 7, 8] were derived and proposed recently.

A neural network which modeled a computer system is defined the nervous system and nerve tissue. The neural networks are used many different applied fields. For example, artificial intelligence, localization, image processing, neurochemistry, intrusion detection system, medicine and so on [18]. Furthermore, they are used some problems such as to medicine such as for detecting resistivity of antibiotics in [11] , to identify the damage localization for bridges in [13], diagnosing hepatitis in [17] and to email security enhancement [26]. Furthermore, several interesting results can be found in [10, 12, 20, 29].

In this paper we want to compute the closeness, the vertex residual closeness(VRC) and normalized vertex residual closeness(NVRC) of a type of the probabilistic neural networks, also is denoted by $PNN(n, k, m)$. In Section 2, some definitions and some basic results are given for closeness, VRC and NVRC, respectively. In Section 3, the closeness, the VRC and NVRC of $PNN(n, k, m)$ are determined. Finally, we present our conclusions in Section 4.

2. Closeness centaralities and basic results

The concept of closeness, vertex residual closeness and normalized vertex residual closeness have been derived and proposed on 2006 by Dangalchev in [6], then same author has studied VRC and NVRC of some special graphs in [7, 8]. Furthermore, they have been further studied by Aytac et al. [1, 2, 3, 4, 30, 31], Turaci et al. [27, 28] and Yigit et al. [32]. The aim of residual closeness is to measure the reliability of networks when removal of the each vertex from the graph G. In a network, closeness is a useful parameter that predicts how fast the flow of information would be owing to a given vertex to other vertex [6]. The closeness of a graph G is defined as: $C(G) = \sum\limits_{v_i \in V(G)} C(v_i)$, where $C(v_i)$ is the closeness of a vertex v_i, and it is defined as: $C(v_i) = \sum\limits_{v_i \neq v_j} \frac{1}{2^{d_G(v_i, v_j)}}$ [6]. Let $d_{v_k}(v_i, v_j)$ be the distance between vertices v_i and v_j in the graph G, received from the original graph where all links of vertex v_k are deleted. Then the closeness after removing vertex v_k is defined as: $C_{v_k} = \sum\limits_{v_i} \sum\limits_{v_i \neq v_j} \frac{1}{2^{d_{v_k}(v_i, v_j)}}$. The vertex residual closeness (VRC) of the graph G is defined as: $R(G) = \min\limits_{v_k}\{C_{v_k}\}$ [6]. The normalized vertex residual closeness ($NVRC$) of the graph G is defined as dividing the residual closeness by the closeness $C(G)$: $R'(G) = R(G)/C(G)$ [6].

Theorem 2.1. [6] *The VRC and NVRC of*
(a) *If* $G = K_n$, *then* $R(G) = ((n-1)(n-2))/2$ *and* $R'(G) = (n-2)/n$.
(b) *If* $G = S_n$, *then* $R(G) = 0$ *and* $R'(G) = 0$.

Theorem 2.2. [6] *For a graph* G, $0 \leq R'(G) < 1$.

Theorem 2.3. [6] *Let* G *be a graph of order* n. *If* H *is a proper subgraph of* G, *then* $R(H) < R(G)$.

Theorem 2.4. [1, 30] *The vertex residual closeness (VRC) of*
(a) *the cycle graph* C_n *with* n *vertices is* $R(C_n) = 2n - 6 + 1/2^{n-3}$;
(b) *the wheel graph* W_n *with* $n+1$ *vertices is*

$$R(W_n) = \begin{cases} n(\sum_{i=1}^{\lfloor n/2 \rfloor} 1/2^{(i-1)}) & , \text{ if } n \text{ is odd;} \\ n((\sum_{i=1}^{(n/2)-1} 1/2^{(i-1)}) + 1/2^{(n/2)}) & , \text{ if } n \text{ is even.} \end{cases}$$

3. Calculation of closeness centralities of probabilistic neural networks

In this section, fistly the definition of the probabilistic neural networks. Then, we compute the closeness, VRC and NVRC of a type of probabilistic neural networks.

Definition 3.1. [14, 18] *A probabilistic neural network includes three layers of nodes. They are input layer, hidden layer and output layer, where the number of output layer is equal the number of classes of hidden layers. Furthermore, each nodes in input layer is connected to each nodes of hidden layers and each nodes in output layer is connected to each nodes in the distinct classes of hidden layers. Let n be number of nodes in input layers, let k be number of nodes in output layers, and let m be the number of nodes of each classes in hidden layers. So, a probabilistic neural network denoted by $PNN(n, k, m)$, where n, k and m are positive integers. Clearly, $|V(PNN(n, k, m))| = n + k(m + 1)$. In Figure 1, the probabilistic neural network is shown for $n = 5$, $k = 3$ and $m = 2$.*

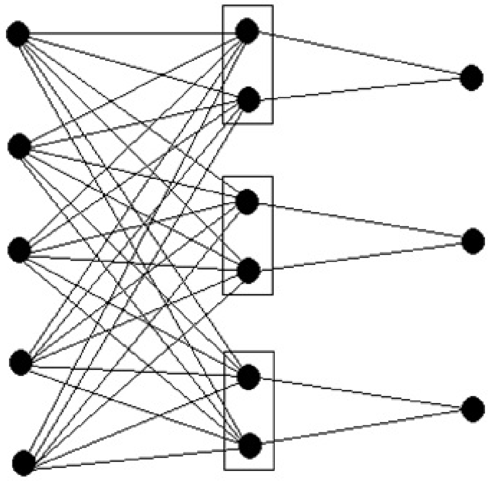

Figure 1. The probabilistic neural network $P(5, 3, 2)$.

In order to make the proof of the given theorems understandable, we use some notations are as follows: Let $G \cong P(n, k, m)$. The vertices of G will be as $V(G) = V_1 \cup V_2 \cup V_3$, where:

$V_1 = \{v \in V(G)|deg_G(v) = km\}$;
$V_2 = \{v^* \in V(G)|deg_G(v^*) = n + 1\}$;
$V_3 = \{v' \in V(G)|deg_G(v') = m\}$.

In [14], the Hosoya polynomial $H(G, x)$ of the graph $G \cong PNN(n, k, m)$ was obtained. Using the Hosoya polynomial $H(G, x)$, the following Lemma 3.2 can be obtained.

Lemma 3.2. *Let G be a probabilistic neural network $PNN(n, k, m)$ of order $n + k(m + 1)$, where $n, k, m \geq 2$. The closeness of G is*

$$C(G) = \frac{nk}{2}(2m + 1) + \frac{km}{4}(km + k + 2) + \frac{n}{4}(n - 1) + \frac{k}{16}(k - 1).$$

Proof. We have three cases depending on the vertices of G.

Case 1. For any vertex of $v_i \in V_1$ in G, we have $|N_G(v_i)| = km$. There are $(k+n-1)$- paths of length 2 by the structure of G. Thus,

$$(n)C(v_i) = (n)\Big((km)(2^{-1}) + (k+n-1)(2^{-2})\Big). \tag{3.1}$$

Case 2. For any vertex of $v_i^* \in V_2$ in G, we have $d_G(v_i^*, v_i) = 1$ for all vertices of $v_i \in V_1$. Furthermore, we have $d_G(v_i^*, v_i') = 1$ for only one vertex of V_3. Then, there are $(km-1)$- paths of length 2 and $(k-1)$- paths of length 3. So, we get:

$$(km)C(v_i^*) = (km)\Big((n+1)(2^{-1}) + (km-1)(2^{-2}) + (k-1)(2^{-3})\Big). \tag{3.2}$$

Case 3. Let $v_i' \in V_3$. Then, we have $|N_G(v_i')| = m$. Let $v_i \in V_1$. So we get $d_G(v_i', v_i) = 2$ for each vertex of $v_i \in V_1$. Furthermore, there are $(km-m)$- paths of length 3 and $(k-1)$- paths of length 4. So, we have the following for each vertex of V_3.

$$(k)C(v_i') = (k)\Big((m)(2^{-1}) + (n)(2^{-2}) + (km-m)(2^{-3}) + (k-1)(2^{-4})\Big). \tag{3.3}$$

By summing the Equations (3.1), (3.2) and (3.3), then we obtain the following result:

$$C(G) = \frac{nk}{2}(2m+1) + \frac{km}{4}(km+k+2) + \frac{n}{4}(n-1) + \frac{k}{16}(k-1).$$

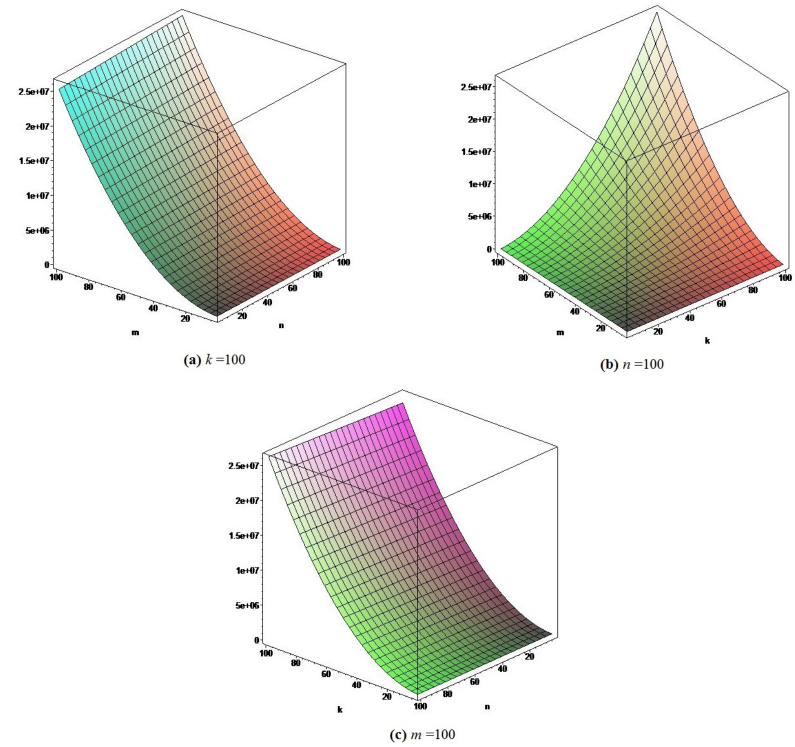

FIGURE 2. Computer-based comparative graph of the Closeness $(C(G))$ for $G \cong PNN(n,k,m)$.

□

Theorem 3.3. *Let G be a probabilistic neural network $PNN(n,k,m)$ of order $n+k(m+1)$, where $n,k,m \geq 2$. If $n \leq m$ or $n \leq k$, then the VRC of G is*

$$R(G) = \frac{1}{2} + \frac{k}{2}(2nm+n-m) + \frac{k^2m}{4}(m+1) + \frac{n}{4}(n-3) + \frac{k}{16}(k-9).$$

Proof. We have three cases depending on the vertices of G.

Case 1. Removing a vertex $v_i \in V_1$ in the graph G. We have three sub cases depending on the vertices of the survival sub graph $G \setminus \{v_i\}$.

Subcase 1.1. For a vertex $v_j \in V_1 \setminus \{v_i\}$ in the graph $G \setminus \{v_i\}$. The vertex v_j is adjacent to each vertex of V_2. So, $d_{v_i}(v_j, v_i^*) = 1$ for all $v_i^* \in V_2$. Then, it is clear that the distance from the vertex v_j to remaining $(n+k-2)$-vertices is two. Thus,

$$\begin{aligned}(n-1)C_{v_i}(v_j) &= (n-1)\Big((km)(2^{-1}) + (n+k-2)(2^{-2})\Big)\\ &= \frac{(n-1)(2km+n+k-2)}{4}. \qquad (3.4)\end{aligned}$$

Subcase 1.2. For a vertex $v_i^* \in V_2$ in the graph $G \setminus \{v_i\}$. The vertex v_i^* is adjacent to each vertex of $V_1 \setminus \{v_i\}$ and a vertex $v_i' \in V_3$. So, $d_{v_i}(v_i^*, v_j) = 1$ for all $v_j \in V_1 \setminus \{v_i\}$ and $d_{v_i}(v_i^*, v_i') = 1$. Furthermore, there are $(km-1)$-paths of length 2 and $(k-1)$- paths of length 3. So, we get:

$$\begin{aligned}(km)C_{v_i}(v_i^*) &= (km)\Big((n)(2^{-1}) + (km-1)(2^{-2}) + (k-1)(2^{-3})\Big)\\ &= \frac{(km)(2km+4n+k-3)}{8}. \qquad (3.5)\end{aligned}$$

Subcase 1.3. For a vertex $v_i' \in V_3$ in the graph $G \setminus \{v_i\}$. The vertex v_i' is adjacent to m-vertices in V_2. Furthermore, there are $(n-1)$- paths of length 2, $(km-m)$- paths of length 3 and $(k-1)$- paths of length 4. So, we get:

$$\begin{aligned}(k)C_{v_i}(v_i') &= (k)\Big((m)(2^{-1}) + (n-1)(2^{-2}) + (km-m)(2^{-3}) + (k-1)(2^{-4})\Big)\\ &= \frac{(k)(2km+6m+4n+k-5)}{16}. \qquad (3.6)\end{aligned}$$

By summing Equations (3.4), (3.5) and (3.6), then we obtain the following result:

$$C_{v_i} = \frac{1}{2} + \frac{k}{2}(2nm+n-m) + \frac{k^2m}{4}(m+1) + \frac{n}{4}(n-3) + \frac{k}{16}(k-9).$$

Case 2. Removing a vertex $v_i^* \in V_2$ in the graph G. We have three sub cases depending on the vertices of the survival sub graph $G \setminus \{v_i^*\}$.

Subcase 2.1. For a vertex $v_i \in V_1$ in the graph $G \setminus \{v_i^*\}$. The vertex v_i is adjacent to each vertex of $V_2 \setminus \{v_i^*\}$. So, $d_{v_i^*}(v_i, v_j^*) = 1$ for all $v_j^* \in V_2 \setminus \{v_i^*\}$. Then, it is clear that the distance from the vertex v_i to remaining $(n+k-1)$-vertices is two. Thus,

$$\begin{aligned}(n)C_{v_i^*}(v_i) &= (n)\Big((km-1)(2^{-1}) + (n+k-1)(2^{-2})\Big)\\ &= \frac{(n)(2km+n+k-3)}{4}. \qquad (3.7)\end{aligned}$$

Subcase 2.2. For a vertex $v_j^* \in V_2 \setminus \{v_i^*\}$ in the graph $G \setminus \{v_i^*\}$. The vertex v_j^* is adjacent to each vertex of V_1 and a vertex $v_i' \in V_3$. So, $d_{v_i^*}(v_j^*, v_i) = 1$ for all $v_i \in V_1$ and $d_{v_i^*}(v_j^*, v_i') = 1$. Furthermore, there are $(km-2)$- paths of length 2 and $(k-1)$- paths of length 3. So, we get:

$$(km-1)C_{v_i^*}(v_j^*) = (km-1)\Big((n+1)(2^{-1}) + (km-2)(2^{-2}) + (k-1)(2^{-3})\Big)$$

$$= \frac{(km-1)(2km+4n+k-1)}{8}. \tag{3.8}$$

Subcase 2.3. For a vertex $v_i' \in V_3$ in the graph $G \setminus \{v_i^*\}$. The vertex v_i' is adjacent to $(m-1)$-vertices in V_2, if the vertex v_i' is adjacent to the vertex v_i^* in the graph G. Furthermore, there are n-paths of length 2, $(km-m)$- paths of length 3 and $(k-1)$- paths of length 4. So we get the following $Sum1$.

$$Sum1 = (m-1)(2^{-1}) + (n)(2^{-2}) + (km-m)(2^{-3}) + (k-1)(2^{-4})$$

$$= \frac{2km+4n+6m+k-9}{16}. \tag{3.9}$$

Otherwise, that is the vertex v_i' is not adjacent to the vertex v_i^* in the graph G, then the vertex v_i' is adjacent to m- vertices in V_2. Furthermore, there are n- paths of length 2, $(km-m-1)$- paths of length 3 and $(k-1)$- paths of length 4. So we get the following $Sum2$.

$$Sum2 = (k-1)\Big((m)(2^{-1}) + (n)(2^{-2}) + (km-m-1)(2^{-3}) + (k-1)(2^{-4})\Big)$$

$$= \frac{(k-1)(2km+4n+6m+k-3)}{16}. \tag{3.10}$$

By the Equations (3.9) and (3.10), we get the following:

$$C_{v_i^*}(v_i') = Sum1 + Sum2$$

$$= \frac{2km+4n+6m+k-9}{16} + \frac{(k-1)(2km+4n+6m+k-3)}{16}$$

$$= \frac{2k^2m+k^2+6km+4kn-3k-6}{16}. \tag{3.11}$$

By summing the Equations (3.7), (3.8) and (3.11), then we obtain:

$$C_{v_i^*} = -\frac{1}{4} + \frac{nk}{2}(2m+1) + \frac{k^2m}{4}(m+1) + \frac{n}{4}(n-5) + \frac{k}{16}(k-5).$$

Case 3. Removing a vertex $v_i' \in V_3$ in the graph G. We have three sub cases depending on the vertices of the survival sub graph $G \setminus \{v_i'\}$.

Subcase 3.1. For a vertex $v_i \in V_1$ in the graph $G \setminus \{v_i'\}$. The vertex v_i is adjacent to each vertex of V_2. So, $d_{v_i'}(v_i, v_i^*) = 1$ for all $v_i^* \in V_2$. Then, it is clear that the distance from the vertex v_i to remaining $(n+k-2)$- vertices is two. Thus,

$$(n)C_{v_i'}(v_i) = (n)\Big((km)(2^{-1}) + (n+k-2)(2^{-2})\Big)$$

$$= \frac{(n)(2km+n+k-2)}{4}. \tag{3.12}$$

Subcase 3.2. For a vertex $v_i^* \in V_2$ in the graph $G \setminus \{v_i'\}$. The vertex v_i' is adjacent to n- vertices of V_1, if the vertex v_i^* is adjacent to the vertex v_i' in the graph G. Furthermore, there are $(km-1)$- paths of length 2 and $(k-1)$- paths of length 3. So we get the following $Sum1$.

$$Sum1 = (m)\Big((n)(2^{-1}) + (km-1)(2^{-2}) + (k-1)(2^{-3})\Big)$$

$$= \frac{(m)(2km+4n+k-3)}{8}. \tag{3.13}$$

Otherwise, that is the vertex v_i^* is not adjacent to the vertex v_i' in the graph G, then the vertex v_i^* is adjacent to $(n+1)$- vertices in V_1 and V_3. Furthermore, there are $(km-1)$- paths of length 2 and $(k-2)$- paths of length 3. So we get the following $Sum2$.

$$Sum2 = (km-m)\Big((n+1)(2^{-1}) + (km-1)(2^{-2}) + (k-2)(2^{-3})\Big)$$

$$= \frac{(km-m)(2km+4n+k)}{8}. \tag{3.14}$$

By the Equations (3.13) and (3.14), we get the following:

$$C_{v_i'}(v_i^*) = Sum1 + Sum2$$

$$= \frac{(m)(2km+4n+k-3)}{8} + \frac{(km-m)(2km+4n+k)}{8}$$

$$= \frac{4nkm + 2k^2m^2 + k^2m - 3m}{8}. \tag{3.15}$$

Subcase 3.3. For a vertex $v_j' \in V_3 \setminus \{v_i'\}$ in the graph $G \setminus \{v_i'\}$. The vertex v_i' is adjacent to m- vertices in V_1. Furthermore, there are n- paths of length 2, $(km-m)$- paths of length 3 and $(k-2)$- paths of length 4. Thus, we get:

$$(k-1)C_{v_i'}(v_j') = (k-1)\Big((m)(2^{-1}) + (n)(2^{-2}) + (km-m)(2^{-3}) + (k-2)(2^{-4})\Big)$$

$$= \frac{(k-1)(2km+4n+6m+k-2)}{16}. \tag{3.16}$$

By summing the Equations (3.12), (3.15) and (3.16), then we obtain:

$$C_{v_i'} = \frac{1}{8} + \frac{k}{4}(4nm+2n+m) + \frac{k^2m}{4}(m+1) + \frac{n}{4}(n-3) + \frac{k}{16}(k-3) - \frac{3m}{4}.$$

By the definition of the VRC of the graph G, we have $R(G) = min\Big\{C_{v_i}, C_{v_i^*}, C_{v_i'}\Big\}$. Furthermore, we know that $n \leq m$ or $n \leq k$ for $n, k, m \geq 2$. As a result, we obtain:

$$R(G) = \frac{1}{2} + \frac{k}{2}(2nm + n - m) + \frac{k^2 m}{4}(m+1) + \frac{n}{4}(n-3) + \frac{k}{16}(k-9).$$

□

In Figures 3,4,5 and 6, three-dimensional graphics of VRC for "$n \leq k \leq m$", "$n \leq m \leq k$", "$k \leq n \leq m$" and "$m \leq n \leq k$" are shown.

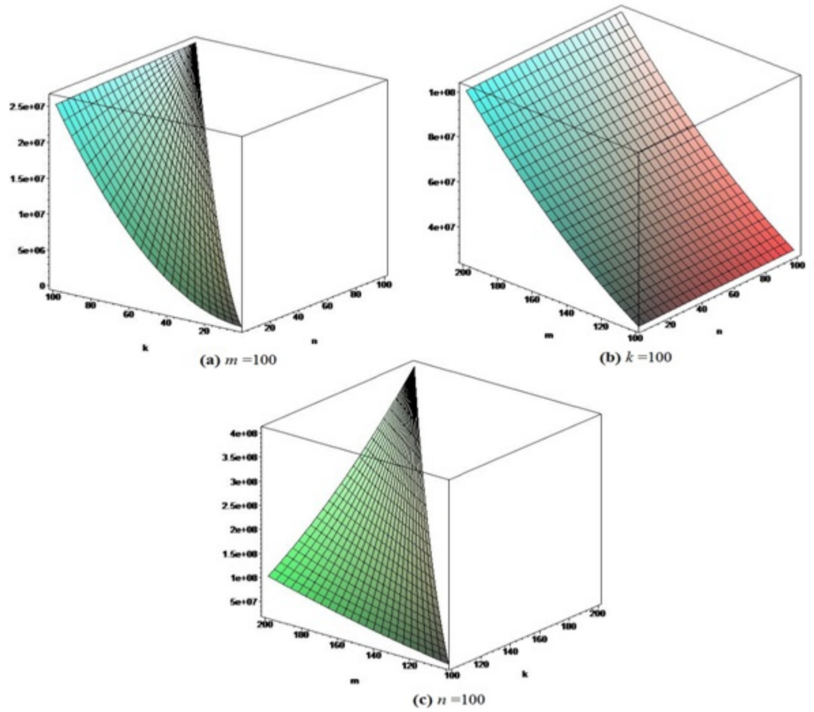

FIGURE 3. Computer-based comparative graph of the VRC $(R(G))$ of $G \cong PNN(n, k, m)$ for $n \leq k \leq m$.

Theorem 3.4. *Let G be a probabilistic neural network $PNN(n, k, m)$ of order $n + k(m+1)$, where $n, k, m \geq 2$. If $n \leq m$ or $n \leq k$, then the NVRC of G is*

$$R'(G) = \frac{\frac{1}{2} + \frac{k}{2}(2nm + n - m) + \frac{k^2 m}{4}(m+1) + \frac{n}{4}(n-3) + \frac{k}{16}(k-9)}{\frac{nk}{2}(2m+1) + \frac{km}{4}(km + k + 2) + \frac{n}{4}(n-1) + \frac{k}{16}(k-1)}.$$

Proof. By the definition of NVRC, the proof is clear.

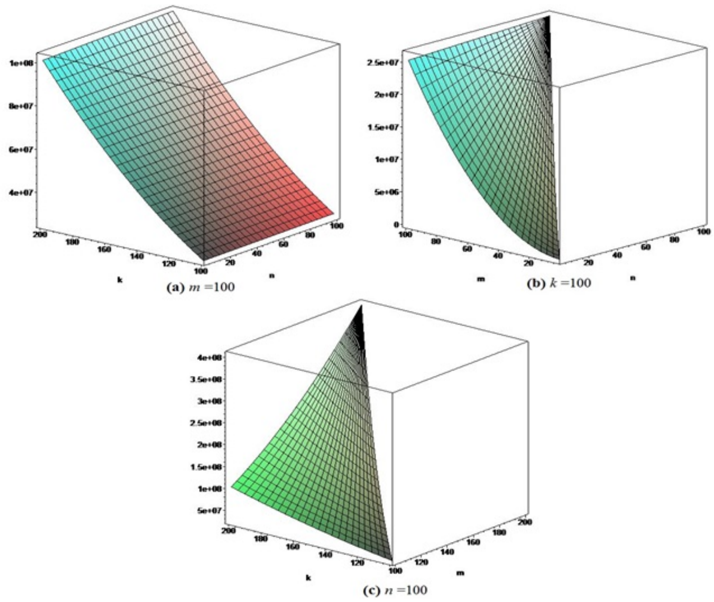

FIGURE 4. Computer-based comparative graph of the VRC $(R(G))$ of $G \cong PNN(n,k,m)$ for $n \leq m \leq k$.

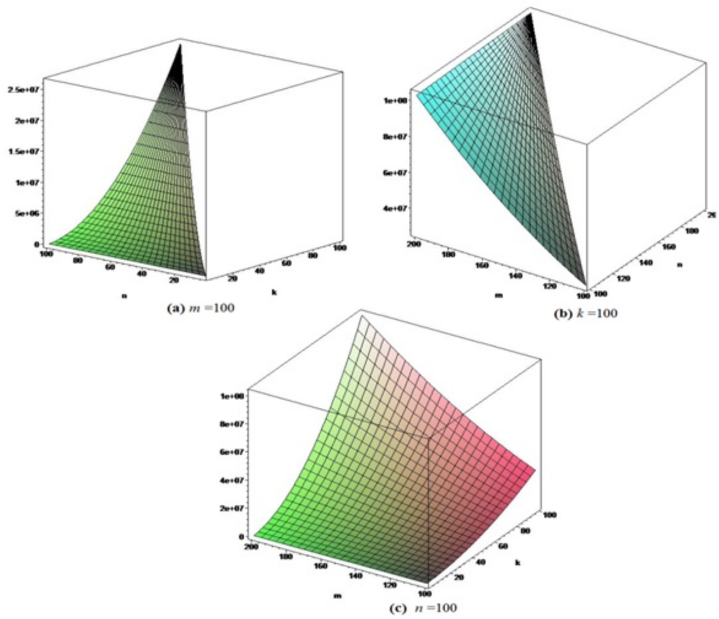

FIGURE 5. Computer-based comparative graph of the VRC $(R(G))$ of $G \cong PNN(n,k,m)$ for $k \leq n \leq m$.

□

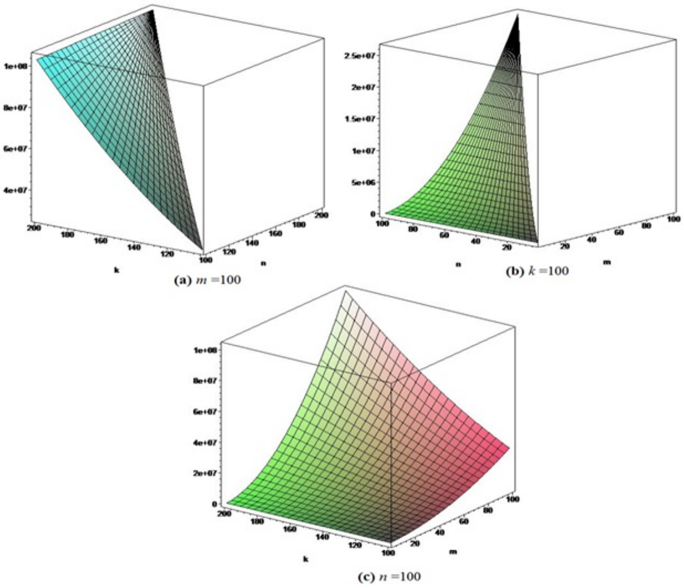

FIGURE 6. Computer-based comparative graph of the VRC ($R(G)$) of $G \cong PNN(n, k, m)$ for $m \leq n \leq k$.

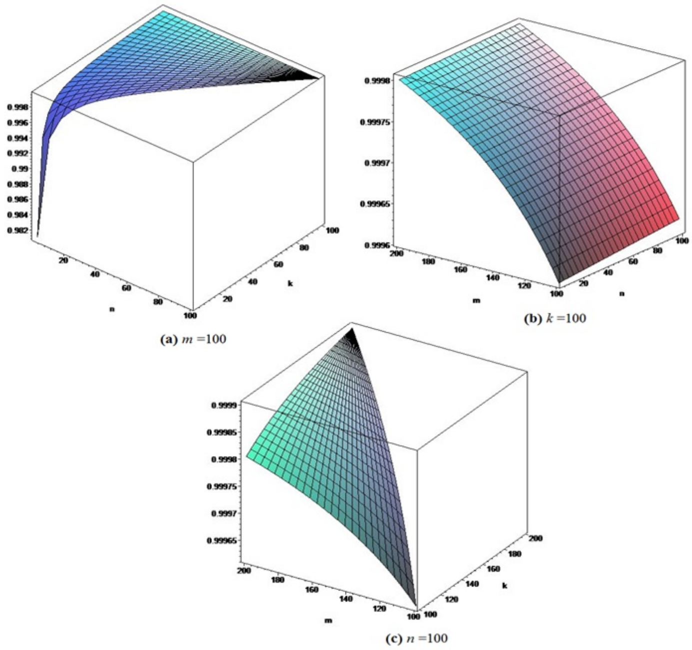

FIGURE 7. Computer-based comparative graph of the NVRC ($R'(G)$) of $G \cong PNN(n, k, m)$ for $n \leq k \leq m$.

In Figure 7, three-dimensional graphs of NVRC for $n \leq k \leq m$ is shown. Remaining conditions, that is three-dimensional graphics of NVRC of any graph G for "$n \leq m \leq k$", "$k \leq n \leq m$" and "$m \leq n \leq k$", can be seen by readers. So, we omit them. It is clear that $0 \leq R'(PNN(n,k,m)) \leq 1$ has been obtained for all cases.

4. Conclusion

Communication systems are often subjected to failures and attacks. We have investigated the residual closeness of networks as a measure of network analysis. Calculation of residual closeness for simple graph types is important because we get which vertices in the network are responsible for fast communication flow. Thus, vertices giving the residual closeness of a graph are important and fast in distributing information through the network. In this paper we have studied the closeness centralities for the probabilistic neural networks $PNN(n,k,m)$. In order to help us to know the properties of closeness centralities, which we had already computed, we plotted the three-dimensional graphics of $C(PNN(n,k,m))$, $R(PNN(n,k,m))$ and $R'(PNN(n,k,m))$ with the help of space cartesian coordinate system.

References

[1] A. Aytaç and Z.N. Odabaş, *Residual Closeness of Wheels and Related Networks*, International Journal of Foundations of Computer Science **22(5)** (2011), 1229–1240.

[2] A. Aytaç and Z.N. Odabaş Berberler, *Robustness of Regular Caterpillars*, International Journal of Foundations of Computer Science **28(7)** (2017), 835-841.

[3] A. Aytaç and Z.N. Odabaş Berberler, *Network robustness and residual closeness*, RAIRO-Oper. Res. **52(3)** (2018), 839-847.

[4] A. Aytaç and Z.N. Odabaş Berberler, *Residual Closeness for Helm and Sunflowar Graphs*, TWMS J. App. Eng. Math. **7(2)** (2017), 209-220.

[5] C.A.Barefoot, B.Entringer and H.Swart, *Vulnerability in graphs-a comparative survey*, J. Combin. Math. Combin. Comput. **1** (1987), 13-22.

[6] Ch. Dangalchev, *Residual Closeness in Networks*, Physica A **365** (2006), 556-564.

[7] Ch. Dangalchev, *Residual Closeness and Generalized Closeness*, International Journal of Foundations of Computer Science **22(8)** (2011), 1938–1948.

[8] Ch. Dangalchev, *Residual Closeness of Generalized Thorn Graphs*, Fundamenta Informaticae **162(1)** (2018), 1-15.

[9] D.F. Specht, *Probabilistic neural networks*, Neural networks **3(1)** (1990), 109-118.

[10] E. Holmes, J.K Nicholson, J. K. and G. Tranter, *Metabonomic characterization of genetic variations in toxicological and metabolic responses using probabilistic neural networks*, Chemical research in toxicology **14(2)** (2001), 182-191.

[11] F. Budak and E.D. Ubeyli, *Detection of resistivity for antibiotics by probabilistic neural networks*, Journal of medical systems **35(1)** (2011), 87-91.

[12] I.B. Standal, J. Rainuzzo, D.E. Axelson, S. Valdersnes, K. Julshamn and M. Aursand, *Classification of geographical origin by PNN analysis of fatty acid data and level of contaminants in oils from Peruvian anchovy*, Journal of the American Oil Chemists Society **89(7)** (2012), 1173-1182.

[13] J.J.Lee and C.B. Yun, *Damage localization for bridges using probabilistic neural networks*, KSCE Journal of civil engineering **11(2)** (2007), 111-120.

[14] J. B. Liu, J. Zhao, S. Wang, M. Javaid and J. Cao, *On the Topological Properties of the Certain Neural Networks*, Journal of Artificial Intelligence and Soft Computing Research **8(4)** (2018), 257-268.

[15] J. A. Bondy and U. S. R. Murty, *Graph theory with applications*, American Elsevier Publishing Co., Inc., New York, 1976.

[16] L. Freeman, *The development of social network analysis: A study in the sociology of science*, , Empirical Press Vancouver, 2004.

[17] M.S. Bascil and H. Oztekin, *A study on hepatitis disease diagnosis using probabilistic neural network*, Journal of medical systems **36(3)** (2012), 1603-1606.

[18] M. Javaid and J. Cao, *Computing topological indices of probabilistic neural network*, Neural Computing and Applications **30(12)** (2018), 3869-3876.

[19] M.E.J. Newman, *The structure and function of complex networks*, SIAM Review **45** (2003), 167-256.

[20] P.A. Kowalski and P. Kulczycki, *Interval probabilistic neural network*, Neural Computing and Applications **28(4)** (2017), 817-834.

[21] R. Albert and A.-L. Barabási, *Statistical mechanics of complex networks*, Reviews of Modern Physics **74** (2002), 47-97.

[22] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez and D.-U. Hwang, *Complex Networks: Structure and Dynamics*, Physics Reports **424** (2006), 175-308.

[23] S. Borgatti, *Centrality and network flow*, Social Networks **27(1)** (2005), 55-71.

[24] S.N. Dorogovtsev and J.F.F. Mendes, *Evolution of networks*, Adv. Phys. **51)** (2002), 1079-1187.

[25] S.Wasserman and K. Faust, *Social Network Analysis: Methods and Applications (Structural Analysis in the Social Sciences)*, first ed. Cambridge University Press, Cambridge, UK., 1994.

[26] T.P. Tran, T.T.S. Nguyen, P. Tsai and X. Kong, *BSPNN: boosted subspace probabilistic neural network for email security*, Artificial Intelligence Review **35(4)** (2011), 369-382.

[27] T. Turaci and M. Okten, *Vulnerability Of Mycielski Graphs Via Residual Closeness*, Ars Combinatoria **118** (2015), 419-427.

[28] T. Turaci and V. Aytac, *Residual Closeness of Splitting Networks*, Ars Combinatoria **(130)** (2017), 17-27.

[29] Y. Wang, T. Adali, S.Y. Kung and Z. Szabo, *Quantification and segmentation of brain tissues from MR images: a probabilistic neural network approach*, IEEE transactions on image processing **7(8)** (1998), 1165-1181.

[30] Z.N. Odabas and A. Aytac, *Residual closeness in cycles and related networks*, Fundamenta Informaticae **124(3)** (2013), 297-307.

[31] V. Aytac and T. Turaci, *Closeness Centrality in Some Splitting Networks*, Computer Science Journal of Moldova **26(3)** (2018), 251-269.

[32] E. Yigit and Z.N. Berberler, *Link failure in wheel type networks*, International Journal of Modern Physics C **30(9)** (2019), 1950072.

Faculty of Engineering, Department of Computer Engineering, Pamukkale University, 20160 Denizli, Turkey

Email address: `tturaci@pau.edu.tr`

SOME SOLITON TYPES ON RIEMANNIAN MANIFOLDS

HALİL İBRAHİM YOLDAŞ

ABSTRACT. In this paper we concentrate on almost Ricci and almost quasi Yamabe solitons on Riemannian manifolds equipped with some special vector fields such as projective, affine conformal and give some notable characterizations which classify such manifolds. Also, we provide some necessary conditions for which affine conformal and projective vector field are affine. Furthermore, we give some necessary and sufficient conditions for a Riemmannian manifold M to be Ricci symmetric.

Mathematics Subject Classification (2010): Almost Ricci soliton, Ricci soliton, almost quasi Yamabe soliton, Yamabe soliton, Ricci symmetric manifold.
Key words: 35Q51, 53C44, 53B20.

Article history:
Received: October 25, 2020
Received in revised form: October 8, 2021
Accepted: October 10, 2021

1. INTRODUCTION

The interest in the study of the problems on Ricci and Yamabe flows, which are evolution equations for Riemannian metrics, has increased in the last years. After the concepts of Yamabe and Ricci flow were defined by Hamilton, geometric flows have started to gain more importance in the studies of differential geometry of manifolds. According to Hamilton, on a Riemannian manifold M the definitions of *Ricci* and *Yamabe flow* are given by, respectively ([11], [12]):

$$\frac{\partial}{\partial t}g(t) = -2S(g(t))$$

and

$$\frac{\partial}{\partial t}g(t) = -2r(t)g(t).$$

Here, S, r and g are the Ricci tensor, the scalar curvature and the metric tensor of M, respectively. In a manifold of dimension $n = 2$, these geometric flows are equivalent. However, if $n > 2$ there does not exist such an equivalence between them. While the Yamabe flow preserves the conformal class of metrics, the Ricci flow does not in general.

A Riemannian manifold (M, g) with a vector field $V \in \Gamma(TM)$ is named as *a Ricci soliton* if it satisfies [12]

$$(\pounds_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \quad \lambda \in \mathbb{R} \tag{1.1}$$

or is named as *a Yamabe soliton* if it satisfies [12]

$$\frac{1}{2}(\pounds_V g)(X, Y) = (r - \lambda)g(X, Y), \quad \lambda \in \mathbb{R} \tag{1.2}$$

where $\pounds_V$ denotes the Lie-derivative with respect to V, which is called the potential vector field. The Ricci solitons (or Yamabe solitons) can be categorized as *shrinking, steady* or *expanding* depending on $\lambda < 0, \lambda = 0$ or $\lambda > 0$, respectively.

Yamabe and Ricci solitons are special solutions of Hamiton's Yamabe and Ricci flow, respectively. Inspired by Hamilton's work, many mathematicians have worked comprehensively the varied generalizations of such solitons in recent years. For example, authors in [18] introduced the almost Ricci solitons as a more general case of Ricci solitons by setting λ as a function in (1.1). Similarly, Barbosa et al. defined the notion of almost Yamabe soliton in [1]. Then, Blaga defined a new kind of Yamabe solitons called almost quasi Yamabe solitons in [2]. According to Blaga, a Riemannian manifold (M, g) is named as *an almost quasi Yamabe soliton* if there exist some smooth functions λ, μ and a vector field V satisfying

$$\frac{1}{2}(\pounds_V g)(X,Y) = (r-\lambda)g(X,Y) + \mu V^\star(X)V^\star(Y), \tag{1.3}$$

such that $V^\star$ is the dual 1-form of V, namely $V^\star(X) = g(X, V)$. An almost quasi Yamabe soliton which satisfies (1.3) is denoted by (V, λ, μ). If variable λ in (1.3) is a constant on (M, g), then eq. (1.3) becomes *a quasi Yamabe soliton* [7]. If μ vanishes identically in (1.3), then eq. (1.3) defines *an almost Yamabe soliton* [1]. For the latest studies as regards Ricci and Yamabe solitons, we refer to ([8], [14], [15], [17], [18], [20], [24]) and references therein.

On the other hand, in manifold theory, the specialists used many methods to classify manifolds. One of these methods is by using the vector fields, both diferential in geometry and physics. They are useful tools to characterize the most geometric structures of the related topic of the manifolds. In addition to this, they arise in many fields and play a notable role in the works of Riemannian geometry. Hence, the manifolds endowed with special geometric vector fields have been considered by several authors such as [3]-[7], [9], [13], [22] and [23].

Motivated by these circumstances, we deal with almost Ricci and almost quasi Yamabe solitons on Riemannian manifolds equipped with some special vector fields, which prove to be rich in geometrical structures. The present paper is organized in the following way: in section 2, we give some essential definitions, notations and formulas; in section 3, we give our main results that we obtain in this work.

2. Preliminaries

In this section, we recall some fundamental definitions and formulae which are going to be used.

Let (M, g) be a Riemannian manifold and let S be the Ricci tensor of (M, g). The manifold is named *Ricci recurrent* if it satisfies [16]

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z), \tag{2.1}$$

such that

$$(\nabla_X S)(Y,Z) = \nabla_X S(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z),$$

for any $X, Y, Z \in \Gamma(TM)$, where A is a non-zero 1-form. If $A = 0$ in (2.1), then (M, g) is called a *Ricci symmetric manifold.* Also, if the Ricci tensor S of (M, g) satisfies

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0$$

then, it is called a *cyclic Ricci tensor.*

On the other hand, a vector field V on a Riemannian manifold (M, g) is said to be *affine conformal* if it satisfies [10]

$$(\pounds_V \nabla)(X,Y) = X(\rho)Y + Y(\rho)X - g(X,Y)D\rho, \tag{2.2}$$

or is said to be *projective* if it satisfies [19]

$$(\pounds_V \nabla)(X,Y) = p(X)Y + p(Y)X, \tag{2.3}$$

where $\pounds_V$ is the Lie-derivative with respect to V, p is an exact 1-form, ρ is a smooth function on M and $D\rho$ is the gradient of ρ. If variable ρ in (2.2) is a constant, then the vector field V is called *affine.* Also, if the 1-form p vanishes identically in (2.3), then V is called an *affine vector field.*

3. Main Results

In this section, we deal with almost quasi Yamabe and almost Ricci solitons on Riemannian manifolds endowed with some special vector fields such as affine conformal, projective vector field and obtain results which classify such manifolds.

Assumption: Let (M, g) be a Riemannian manifold. Throughout this paper, unless otherwise stated, we assume that the potential vector field V is an unit non-parallel vector field with respect to the Levi-Civita connection ∇ on the Riemannian manifold (M, g).

3.1. **Almost quasi Yamabe solitons.** Let (V, λ, μ) be an almost quasi Yamabe soliton on a Riemannian manifold (M, g). Then, from (1.3) we have

$$g(\nabla_Y V, Z) + g(Y, \nabla_Z V) = 2(r - \lambda)g(Y, Z) + 2\mu V^\star(Y)V^\star(Z), \tag{3.1}$$

for any $Y, Z \in \Gamma(TM)$. Substituting Z for V in (3.1) and using $g(V, V) = 1$, we get

$$g(Y, \nabla_V V) = 2(r - \lambda + \mu)V^\star(Y). \tag{3.2}$$

Setting $Y = V$ in (3.2), one gives

$$r - \lambda + \mu = 0. \tag{3.3}$$

Making use of (3.1) and (3.3), the equation (1.3) takes the form

$$(\pounds_V g)(Y, Z) = -2\mu g(Y, Z) + 2\mu V^\star(Y)V^\star(Z). \tag{3.4}$$

On the other hand, from Yano [21] we know that we have the following commutation formula

$$\begin{aligned}(\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[V,X]} g)(Y, Z) &= -g((\pounds_V \nabla)(X, Y), Z) \\ &\quad -g((\pounds_V \nabla)(X, Z), Y),\end{aligned} \tag{3.5}$$

for any $X, Y, Z \in \Gamma(TM)$. Since the Riemannian metric g is parallel, i.e. $\nabla g = 0$, the equation (3.5) reduces to

$$(\nabla_X \pounds_V g)(Y, Z) = g((\pounds_V \nabla)(X, Y), Z) + g((\pounds_V \nabla)(X, Z), Y). \tag{3.6}$$

Also, it follows from (2.2) that we obtain

$$g((\pounds_V \nabla)(X, Y), Z) + g((\pounds_V \nabla)(X, Z), Y) = 2X(\rho)g(Y, Z). \tag{3.7}$$

In view of (3.4) and after some straightforward calculations, we infer

$$\begin{aligned}(\nabla_X \pounds_V g)(Y, Z) &= -2X(\mu)g(Y, Z) + 2X(\mu)V^\star(Y)V^\star(Z) \\ &\quad +2\mu(V^\star(Y)g(Z, \nabla_X V) + V^\star(Z)g(Y, \nabla_X V)).\end{aligned} \tag{3.8}$$

By combining (3.6), (3.7) and (3.8), one immediately has

$$\begin{aligned}2X(\rho)g(Y, Z) &= -2X(\mu)g(Y, Z) + 2X(\mu)V^\star(Y)V^\star(Z) \\ &\quad +2\mu(V^\star(Y)g(Z, \nabla_X V) + V^\star(Z)g(Y, \nabla_X V)).\end{aligned} \tag{3.9}$$

Taking V instead of Z in (3.9) yields

$$X(\rho)V^\star(Y) = \mu g(Y, \nabla_X V). \tag{3.10}$$

Again, taking V instead of Y in (3.10), we have $X(\rho) = 0$. This means that ρ is a constant. In this case, the vector field V is affine on M and the equation (3.10) becomes

$$\mu g(Y, \nabla_X V) = 0.$$

which implies either $\mu = 0$ or $\nabla_X V = 0$. Because V is a non-parallel vector field, we have $\mu = 0$.

Thus, this leads to the following.

Theorem 3.1. *Let (M, g) be a Riemannian manifold admitting an almost quasi Yamabe soliton (V, λ, μ) such that the vector field V is affine conformal. Then V is an affine vector field on M and the almost quasi Yamabe soliton reduces to the almost Yamabe soliton.*

Using the equality (3.9), we can state the following:

Corollary 3.2. *Let (M, g) be a Riemannian manifold endowed with an affine conformal vector field V. Then we have the followings:*
i) If (M, g) admits a quasi Yamabe soliton (V, λ, μ), then the vector field V is affine on M and (M, g) has constant scalar curvature.
ii) If (M, g) admits an almost Yamabe soliton (V, λ), then the vector field V is affine on M.
iii) If (M, g) admits a Yamabe soliton (V, λ), then the vector field V is affine on M and (M, g) has constant scalar curvature.

The next theorem presents a characterization of Riemannian manifolds which admit almost quasi Yamabe solitons.

Theorem 3.3. *Let (M, g) be a Riemannian manifold admitting an almost quasi Yamabe soliton (V, λ, μ) such that the vector field V is projective. Then V is an affine vector field on M and the almost quasi Yamabe soliton reduces to the almost Yamabe soliton.*

Proof. Since V is a projective vector field on M, we write

$$(\pounds_V \nabla)(X, Y) = p(X)Y + p(Y)X, \tag{3.11}$$

for any $X, Y \in \Gamma(TM)$. Using the almost quasi Yamabe soliton equation (1.3) in the commutation formula (3.6) and also using (3.11), we obtain

$$\begin{aligned} &-2X(\mu)g(Y, Z) + 2X(\mu)V^\star(Y)V^\star(Z) + 2\mu V^\star(Y)g(Z, \nabla_X V) \\ &+2\mu V^\star(Z)g(Y, \nabla_X V) = 2p(X)g(Y, Z) + p(Y)g(X, Z) + p(Z)g(X, Y), \end{aligned} \tag{3.12}$$

for any $X, Y, Z \in \Gamma(TM)$. Plugging $Y = V$ in (3.12) and using $g(V, V) = 1$, we get

$$2\mu g(Z, \nabla_X V) = 2p(X)V^\star(Z) + p(V)g(X, Z) + p(Z)V^\star(X). \tag{3.13}$$

Putting $Z = V$ in (3.13), one has

$$p(X) + p(V)V^\star(X) = 0. \tag{3.14}$$

Again, putting $X = V$ in (3.2) provides $p(V) = 0$. Using this fact in (3.14), we have $p(X) = 0$. Then, V is an affine vector field and the equation (3.13) reduces to

$$\mu g(Z, \nabla_X V) = 0. \tag{3.15}$$

Because V is a non-parallel vector field, the equation (3.15) gives the conclusion $\mu = 0$. Therefore, we get the requested result. □

Using the equality (3.12), we can state the following:

Corollary 3.4. *Let (M, g) be a Riemannian manifold endowed with a projective vector field V. Then we have the followings:*
i) If (M, g) admits a quasi Yamabe soliton (V, λ, μ), then the vector field V is affine on M and (M, g) has constant scalar curvature.
ii) If (M, g) admits an almost Yamabe soliton (V, λ), then the vector field V is affine on M.
iii) If (M, g) admits a Yamabe soliton (V, λ), then the vector field V is affine on M and (M, g) has constant scalar curvature.

3.2. **Almost Ricci solitons.** Let (V, λ) be an almost Ricci soliton on a Riemannian manifold (M, g). Then, by definition of the almost Ricci soliton we write

$$(\pounds_V g)(Y, Z) = -2S(Y, Z) - 2\lambda g(Y, Z), \tag{3.16}$$

for any $Y, Z \in \Gamma(TM)$. Taking covariant derivative of (3.16) with respect to X, we get

$$\begin{aligned} \nabla_X(\pounds_V g)(Y, Z) &= -2(\nabla_X S(Y, Z) + X(\lambda)g(Y, Z) \\ &\quad +\lambda g(\nabla_X Y, Z) + \lambda g(Y, \nabla_X Z)), \end{aligned} \tag{3.17}$$

for any $X \in \Gamma(TM)$. Also, making use of (3.16) we have

$$(\pounds_V g)(\nabla_X Y, Z) = -2(S(\nabla_X Y, Z) + \lambda g(\nabla_X Y, Z)) \tag{3.18}$$

and

$$(\pounds_V g)(Y, \nabla_X Z) = -2(S(Y, \nabla_X Z) + \lambda g(Y, \nabla_X Z)). \tag{3.19}$$

By virtue of the equalities (3.17), (3.18) and (3.19), we obtain

$$(\nabla_X \pounds_V g)(Y, Z) = -2((\nabla_X S)(Y, Z) + X(\lambda)g(Y, Z)). \tag{3.20}$$

Because the vector field V is affine conformal on M, with the help of (3.6), (3.7) and (3.20), we find that

$$(\nabla_X S)(Y, Z) = X(\lambda - \rho)g(Y, Z). \tag{3.21}$$

Now, if (M, g) is a Ricci symmetric manifold, then we arrive at

$$X(\lambda - \rho)g(Y, Z) = 0. \tag{3.22}$$

Taking $Y = Z = V$ in (3.22) gives $X(\lambda - \rho) = 0$. This implies that $\lambda - \rho$ is a constant. Conversely, if $\lambda - \rho$ is a constant, then from (3.21) it follows that (M, g) is a Ricci symetric manifold.

Therefore, we are in a position to state the following:

Theorem 3.5. *Let (M, g) be a Riemannian manifold admitting an almost Ricci soliton (V, λ) such that the vector field V is affine conformal. Then (M, g) is a Ricci symmetric manifold if and only if $\lambda - \rho$ is a constant, where ρ is defined by (2.2).*

As an immediate consequence of the Theorem 3.5, we can give the following corollary:

Corollary 3.6. *Let (M, g) be a Riemannian manifold admitting a Ricci soliton (V, λ) such that the vector field V is affine conformal. Then (M, g) is a Ricci symmetric manifold if and only if V is a affine vector field on M.*

Using the equality (3.11), we have the following:

Theorem 3.7. *Let (M, g) be a Riemannian manifold admitting an almost Ricci soliton (V, λ) such that the vector field V is affine conformal. Then (M, g) has cyclic Ricci tensor if and only if $\lambda - \rho$ is a constant, where ρ is defined by (2.2).*

Proof. It follows from (3.11) that we get

$$\begin{aligned} X(\lambda - \rho)g(Y, Z) + Y(\lambda - \rho)g(X, Z) + Z(\lambda - \rho)g(X, Y) \\ = (\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) \end{aligned} \tag{3.23}$$

for any $X, Y, Z \in \Gamma(TM)$. If (M, g) has cyclic Ricci tensor, then (3.23) turns into

$$X(\lambda - \rho)g(Y, Z) + Y(\lambda - \rho)g(X, Z) + Z(\lambda - \rho)g(X, Y) = 0. \tag{3.24}$$

Also, taking V in place of $Y$and Z in (3.24) provides

$$X(\lambda - \rho) + 2V(\lambda - \rho)g(X, V) = 0. \tag{3.25}$$

Replacing X by V in (3.25) we obtain

$$V(\lambda - \rho) = 0. \tag{3.26}$$

If we use (3.26) in (3.25), then we have $X(\lambda - \rho) = 0$, which implies that $\lambda - \rho$ is a constant.
The converse of the theorem is clear. Therefore, the proof is completed. □

The proof of the following corollary easily follows from Theorem 3.7.

Corollary 3.8. *Let (M, g) be a Riemannian manifold admitting a Ricci soliton (V, λ) such that the vector field V is affine conformal. Then (M, g) has cyclic Ricci tensor if and only if V is an affine vector field on M.*

Next theorem gives a classification a result on an almost Ricci soliton and its potential vector field.

Theorem 3.9. *Let (M,g) be a Ricci symmetric Riemannian manifold admitting an almost Ricci soliton (V,λ) such that the vector field V is projective. Then the almost Ricci soliton reduces to Ricci soliton if and only if V is an affine vector field on M.*

Proof. Owing to being V projective vector field on M, from (2.3) we have

$$\begin{aligned} 2p(X)g(Y,Z) + p(Y)g(X,Z) + p(Z)g(X,Y) \\ = g((\pounds_V\nabla)(X,Y),Z) + g((\pounds_V\nabla)(X,Z),Y), \end{aligned} \tag{3.27}$$

for any $X,Y,Z \in \Gamma(TM)$. Also, keeping in mind (3.6) and (3.20) we get

$$\begin{aligned} -2((\nabla_X S)(Y,Z) + X(\lambda)g(Y,Z)) &= 2p(X)g(Y,Z) + p(Y)g(X,Z) \\ &\quad + p(Z)g(X,Y). \end{aligned} \tag{3.28}$$

Since M is a Ricci symmetric Riemannian manifold, the eq. (3.28) transforms into

$$-2X(\lambda)g(Y,Z) = 2p(X)g(Y,Z) + p(Y)g(X,Z) + p(Z)g(X,Y). \tag{3.29}$$

Putting $Y = Z = V$ in (3.29), we derive

$$X(\lambda) = -p(X) - p(V)g(X,V). \tag{3.30}$$

After, inserting (3.30) in (3.29), one has

$$2p(V)g(X,V)g(Y,Z) = p(Y)g(X,Z) + p(Z)g(X,Y). \tag{3.31}$$

If we put $X = Z = V$ in (3.31), we arrive at

$$p(Y) = p(V)g(V,Y). \tag{3.32}$$

Hence, from (3.30) and (3.32) we obtain

$$X(\lambda) = -2p(X),$$

which completes the proof. □

As a result of the Theorem 3.9, we can give following.

Corollary 3.10. *Let (M,g) be a Ricci symmetric Riemannian manifold admitting a Ricci soliton (V,λ) such that the vector field V is projective. Then V is an affine vector field on M.*

Example 3.11. [3] One considers the three-dimensional manifold

$$M = \{(x,y,z) \in \mathbb{R}^3, z > 0\},$$

where (x,y,z) are the Cartesian coordinates in $\mathbb{R}^3$. Take

$$g := \frac{1}{z^2}\{dx \otimes dx + dy \otimes dy + dz \otimes dz\},$$

$$V^\star := -\frac{1}{z}dz, \qquad V := -z\frac{\partial}{\partial z}.$$

Let e_1, e_2 and e_3 be the linearly independent vector fields in $\mathbb{R}^3$ as follows:

$$e_1 = z\frac{\partial}{\partial x}, \quad e_2 = z\frac{\partial}{\partial y} \quad \text{and} \quad e_3 = -z\frac{\partial}{\partial z}.$$

Then, we have

$$V^\star(e_1) = 0, \quad V^\star(e_2) = 0, \quad V^\star(e_3) = 1,$$

$$[e_1,e_2] = 0, \quad [e_2,e_3] = e_2, \quad [e_3,e_1] = -e_1.$$

On the other hand, using Koszul's formula for the Riemannian metric g, we get:

$$\nabla_{e_1}e_2 = \nabla_{e_2}e_1 = \nabla_{e_3}e_1 = \nabla_{e_3}e_2 = \nabla_{e_3}e_3 = 0,$$
$$\nabla_{e_1}e_1 = -e_3, \quad \nabla_{e_1}e_3 = e_1, \quad \nabla_{e_2}e_2 = -e_3, \quad \nabla_{e_2}e_3 = e_2.$$

Also, with the help of the above equations we find that

$$R(e_1, e_2)e_2 = -e_1, \quad R(e_2, e_1)e_1 = -e_2,$$
$$R(e_3, e_1)e_1 = -e_3, \quad R(e_2, e_3)e_3 = -e_2,$$
$$R(e_1, e_3)e_3 = -e_1, \quad R(e_3, e_2)e_2 = -e_3.$$

In view of the expressions of the curvature tensors, one has

$$S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = -2$$

and hence

$$r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) = -6.$$

Writing the quasi Yamabe soliton equation in (e_i, e_i) we obtain:

$$\frac{1}{2}(\pounds_V g)(e_i, e_i) = (-6 - \lambda)g(e_i, e_i) + \mu V^\star(e_i)V^\star(e_i),$$

for all $i = 1, 2, 3$. Hence, we get $\lambda = -7$ and $\mu = -1$. Then, (V, λ, μ) defines a quasi Yamabe soliton.

Example 3.12. [3] Let $M = \mathbb{R}^3$ and (x, y, z) be the standard coordinates in $\mathbb{R}^3$. Let g be the Lorentzian metric defined by:

$$g := e^{-2z}dx \otimes dx + e^{2x-2z}dy \otimes dy - dz \otimes dz.$$

Let e_1, e_2 and e_3 be the linearly independent vector fields in $\mathbb{R}^3$ as follows:

$$e_1 = e^z \frac{\partial}{\partial x}, \quad e_2 = e^{z-x}\frac{\partial}{\partial y} \quad and \quad e_3 = \frac{\partial}{\partial z}.$$

Also, consider the 1-form $V^\star$ and the potential vector field V:

$$V^\star := dz, \quad V := \frac{\partial}{\partial z}.$$

Therefore, we find:

$$\nabla_{e_1}e_2 = \nabla_{e_3}e_1 = \nabla_{e_3}e_2 = \nabla_{e_3}e_3 = 0,$$
$$\nabla_{e_1}e_1 = -e_3, \quad \nabla_{e_2}e_3 = -e_2, \quad \nabla_{e_1}e_3 = -e_1,$$
$$\nabla_{e_2}e_1 = e^z e_2, \quad \nabla_{e_2}e_2 = -e^z e_1 - e_3.$$

Using the above equations, one has

$$R(e_1, e_2)e_2 = (1 - e^{2z})e_1, \quad R(e_2, e_1)e_1 = (1 - e^{2z})e_2,$$
$$R(e_1, e_3)e_3 = -e_1, \quad R(e_3, e_1)e_1 = e_3,$$
$$R(e_2, e_3)e_3 = -e_2, \quad R(e_3, e_2)e_2 = e_3.$$

Using the expressions of the curvature tensors, we have

$$S(e_1, e_1) = -e^{2z}, \quad S(e_2, e_2) = -e^{2z}, \quad S(e_3, e_3) = -2$$

and

$$r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) = -2 - 2e^{2z}.$$

In this case, it follows from the eq. (1.3) that (V, λ, μ) defines an almost quasi Yamabe soliton with $\lambda = 1 - 2e^{2z}$ and $\mu = -1$.

References

[1] E. Barbosa and E. Ribeiro, *On conformal solutions of the Yamabe flow*, Arch. Math. **101** (2013), 79–89.

[2] A. M. Blaga, *A note on warped product almost quasi Yamabe solitons*, Filomat **33(7)** (2019), 2009-2016.

[3] A. M. Blaga and C. Özgür, *Almost η-Ricci and almost η-Yamabe solitons with torse-forming vector field*, Quaest. Math. **2020** (2020), 1-21.

[4] A. M. Blaga and M. Crasmareanu, *Torse-forming η-Ricci solitons in almost paracontact η-Einstein geometry*, Filomat **31(2)** (2017), 499-504.

[5] B.-Y. Chen, *Classification of torqued vector fields and its applications to Ricci solitons*, Kragujevac J. Math. **41(2)** (2017), 239-250.

[6] B.-Y. Chen and S. Deshmukh, *Ricci solitons and concurrent vector fields*, Balkan J. Geom. Appl. **20(1)** (2015), 14-25.

[7] B.-Y. Chen and S. Desmukh, *Yamabe and quasi-Yamabe solitons on Euclidean submanifolds*, Mediterr. J. Math. **15(5)** (2018), Article 194.

[8] J. T. Cho and M. Kimura, *Ricci solitons and real hypersurfaces in a complex space form*, Tohoku Math. J. **61(2)** (2009), 205-212.

[9] M. Crasmareanu, *Parallel tensors and Ricci solitons in $N(k)$-quasi Einstein manifolds*, Indian J. Pure Appl. Math. **43** (2012), 359-369.

[10] K. L. Duggal, *Affine conformal vector fields in semi-Riemannian manifolds*, Acta Applicandae Mathematicae **23** (1991), 275-294.

[11] R. S. Hamilton, *The Ricci flow on surfaces, Mathematics and general relativity (Santa Cruz, CA, 1986)*, Contemp. Math. A.M.S. **71** (1988), 237-262.

[12] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Diff. Geom. **17** (1982), 255-306.

[13] P. Majhi and G. Ghosh, *Concircular vector field in (k,μ)-contact metric manifolds*, Int. Electron. J. Geom. **11(1)** (2018), 52-56.

[14] Ş. E. Meriç and E. Kılıç, *Riemannian submersions whose total manifolds admit a Ricci soliton*, Int. J. Geom. Methods Mod. Phys. **16(12)** (2019), 1950196.

[15] D. M. Naik and V. Venkatesha, *η-Ricci solitons and almost η-Ricci solitons on para-Sasakian manifolds*, Int. J. Geom. Methods Mod. Phys. **16(9)** (2019), 1950134.

[16] E. M. Patterson, *Some theorems on Ricci-recurrent spaces*, J. London. Math. Soc. **27** (1952), 287-295.

[17] D. S. Patra, *Ricci solitons and paracontact geometry*, Mediterr. J. Math. **16** (2019), Article 137.

[18] S. Pigola, M. Rigoli, M. Rimoldi and A. Setti, *Ricci almost solitons*, Ann. Sc. Norm. Pisa C1. Sci. **10(4)** (2019), 757-799.

[19] A. Romero and M. Sanchez, *Projective vector fields on Lorentzian manifolds*, Geometriae Dedicata **93** (2002), 95–105.

[20] R. Sharma, *On certain results on K-contact and (k,μ)-contact manifolds*, J. Geom. **89(1-2)**, (2008), 138-147.

[21] K. Yano, *Integral formulas in Riemannian geometry*, Marcel Dekker, New York, 1970.

[22] K. Yano, *On torse-forming direction in a Riemannian Space*, Proc. Imp. Acad. Tokyo **20** (1944), 340-345.

[23] H. İ. Yoldaş, *Certain results on $N(k)$-contact metric manifolds and torse-forming vector fields*, J. Math. Ext. **15(3)** (2021), 1-16.

[24] H. İ. Yoldaş, Ş. E. Meriç and E. Yaşar, *On submanifolds of Kenmotsu manifold with torqued vector field*, Hacet. J. Math. Stat. **49 (2)** (2020), 843-853.

Department of Mathematics, Mersin University, Mersin, Turkey
Email address: hibrahimyoldas@gmail.com

ZAGREB DOMINATION IN GRAPHS

B. CHALUVARAJU[1], S. A. DIWAKAR[1] AND SUNILKUMAR M. HOSAMANI[2]

Abstract. A set $D \subseteq V$ is a dominating set of a graph G if every vertex in $V - D$ is adjacent to one or more vertices in D. The first Zagreb index $M_1(G)$ is the sum of the square of the degree of each vertex in G. In this paper, these two classical concepts are combined and initiated the novel Zagreb-domination parameters in terms of first perfect/strong/weak Zagreb domination in graphs. Here, we investigate several properties and bounds of these newly introduced Zagreb-domination parameters. Further, we examine the applicability of the Zagreb domination in the QSPR-analysis of octane isomers.

Mathematics Subject Classification (2020): 05C05, 05C012, 05C35, 05C69.
Key words: Zagreb index, Dominating set, First perfect Zagreb dominating set, First strong Zagreb dominating set, First weak Zagreb dominating set.

Article history:
Received: August 6, 2021
Received in revised form: November 10, 2021
Accepted: November 12, 2021

1. Introduction

All graphs considered here are finite, undirected and connected with no loops and multiple edges. As usual $p = |V| = |V(G)|$ and $q = |E| = |E(G)|$ denote the number of vertices and edges of G, respectively. In general $\langle X \rangle$ to denote the subgraph induced by the set of vertices X. $N(v) = \{d_G(v) : v \in V\}$ and $N[v] = N(v) \cup \{v\}$ denote the open and closed neighborhoods of a vertex v, respectively. For a set $S \subseteq V$, its open neighborhood is the set $N(S) = \bigcup_{v \in S} N(v)$ and its closed neighborhood is the set $N[S] = N(S) \cup S$. The private neighborhood $PN(v, S)$ of $v \in S$ is defined by $PN(v, S) = N(v) - N(S - \{v\})$. For graph-theoretic terminology and notation not defined here we follow [9].

The concept of domination has existed and has been studied for a long time. Books on domination [10, 11, 12] have stimulated sufficient inspiration, leading to the expansive growth of this field. A set $D \subseteq V$ is a dominating set of G if every vertex in $V - D$ is adjacent to one or more vertices in D. The minimum cardinality taken over all dominating sets in G is called domination number and is denoted by $\gamma(G)$. A dominating set D with minimum cardinality is called γ-set of G. Further, a minimal dominating set is a dominating set in a graph that is not a proper subset of any other dominating set. For more details on domination and its related parameters, we refer to [1, 3, 4, 14].

Gutman and Trinajstić [8] have put forward a degree based topological indices viz., Zagreb indices for analyzing the structure-dependency of total $\pi-$electron energy, these invariants called the first Zagreb index and second Zagreb index are defined as

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2 = \sum_{uv \in E(G)} \left[d_G(u) + d_G(v)\right]$$
$$M_2(G) = \sum_{uv \in E(G)} d_G(u)\, d_G(v).$$

For a comprehensive study of Zagreb type indices and their history, applications, and mathematical properties, see [2, 5, 6, 7, 8] and the references cited therein.

2. First Perfect Zagreb Domination

Let $G = (V, E)$ be a graph. A subset $D \subseteq V$ of vertex set of G is said to be first perfect Zagreb dominating (FPZD) set if

(i) for every $v \in D$ there exist $u \in V - D$ such that $uv \in E(G)$.
(ii) $\sum_{v\in D} d_G(v)^2 = \sum_{u\in V-D} d_G(u)^2$.

The minimum cardinality among all FPZD-sets of the graph G are called the first perfect Zagreb domination number $\gamma_{pz}(G)$. Further, an FPZD-set D is a minimal FPZD-set if no proper subset of D is an FPZD-set. The FPZD-set D with minimum cardinality is called γ_{pz}-set of a graph G.

Now, we illustrate some examples, which yield $\gamma_{pz}(G)$ when G is an odd order.

Example 1. Consider the 5-vertex graph obtained from two triangles, by identifying a vertex of one triangle with a vertex of another triangle. This graph has one vertex of degree 4 and four vertices of degree 2. The vertex of degree 4 forms an FPZD-set.

Example 2. Consider the 15-vertex graph G obtained from one C_5 and five triangles by identifying all the vertices of a C_5 with the vertex of triangles.

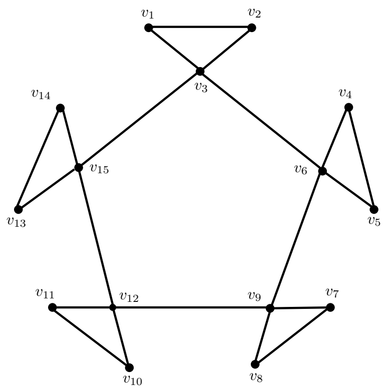

Figure 1. 15-vertex graph

This graph has $p = 15$, $q = 20$ and five vertices of degree 4 and the remaining vertices of degree 2. Here, $D = \{v_3, v_4, v_9, v_{12}, v_{13}, v_{14}\}$ is the FPZD-set of G. Therefore $\sum_{v\in D} d_G(v)^2 = \sum_{u\in V-D} d_G(u)^2 = 60$.

To prove our next result we make use of the following result.

Tutte's Theorem [16]**.** Let G be a graph. Then G has a perfect matching if and only if $O(G-S) \le |S|$ for every subset of $S \subseteq V$, where $O(G-S)$ is the number of odd components that gets generated if S is removed from G.

Theorem 2.1. *For any connected (p, q)-graph satisfying FPZD-set,*

$$p - q \le \gamma_{pz}(G) \le \frac{p}{2}.$$

Further, the lower bound attained if and only if the graph $G \cong K_2$ and the upper bound is attained if and only if $G \ncong T$ (except path $P_{2n}; n \ge 1$) is a perfect matching (i.e., every vertex of the graph is incident to exactly one edge of the matching).

Proof. Let D be a FPZD-set of a connected graph G. Since every vertex in $V - D$ has a common neighborhood with at least one vertex of D, then $q \ge |V - D|$. Hence lower bound follows.

Since every connected non-trivial graph G has a FPZD-set D whose complement $V-D$ is also a FPZD-set. Hence $|D|+|V-D|=p$ and $D^{'} \subseteq V-D$. Thus $\gamma_{pz}(G) \leq min\{|D|,|D^{'}|\} \leq \frac{p}{2}$.
Now we prove the second part. On contrary, suppose $\gamma_{pz}(G) \neq p-q$. Then there exist atleast three vertices u, v and w such that u is not adjacent to v and w is adjacent to both u and v. This implies that $w \in D$ is a FPZD-set and $u, v \in V-D$ of G. This implies that $\sum_{w\in D} d_G(w)^2 > \sum_{x=u,v\in V-D} d_G(x)^2$, which is a contradiction. The converse part is obvious. Hence $\gamma_{pz}(G) = p-q$ follows.

Now, for equality of an upper bound, let $V_i; 1 \leq i \leq k$ be a vertex set partition of a non-trivial connected graph G with even order. If each induced subgraph $\langle V_i \rangle$ is regular, then $\sum\limits_{u\in D} deg(u)^2 = \sum\limits_{v\in V-D} deg(v)^2$. Therefore, D contains $\frac{|V_i|}{2}$ vertices for every i. Let $M=\{v_1v_2, v_3v_4, v_5v_6, \cdots, v_{p-1}v_p\}$ be a perfect matching in G. If each vertex set partition of V_i is the union k-pairs of vertices of same degree. Therefore $\gamma_{pz}(G) = \frac{p}{2}$.

Conversely, suppose $\gamma_{pz}(G) \neq \frac{p}{2}$. We consider the following cases
Case 1. If $|D| < \frac{p}{2}$ with even order, then $|V-D| \geq \frac{p}{2}+1$. By condition (ii) of the definition of FPZD-set, we have $\sum_{v\in D} d_G(v)^2 = \frac{pr^2}{2} = \sum_{u\in V-D} d_G(u)^2$. Since $|V-D| \geq \frac{p}{2}+1$. Hence $\sum_{u\in V-D} d_G(u)^2 \geq \frac{pr^2}{2}+r^2$, which is a contradiction.
Case 2. If $|D| > \frac{p}{2}$, then by the similar arguments as in Case 1, we have $\sum_{v\in D} d_G(v)^2 \geq \frac{pr^2}{2}+r^2$, which is again a contradiction.

In general, if $\gamma_{pz}(G) = \frac{p}{2}$ yields, then the $V(G)$ is necessarily even. Since the vertex set of G should be partitioned into the union of k- pairs of vertices of same degree. Hence it is easy to see that $O(G-S) \leq |S|$ for every subset of $S \subseteq V$. By Tutte's theorem, G must has a 1-factor or a perfect matching. Thus the result follows. □

Theorem 2.2. *Let G be a connected graph satisfying FPZD-set D. If D is a minimal FPZD-set, then $V-D$ is also a FPZD-set of G.*

Proof. Let D be a minimal FPZD-set of G. Suppose $V-D$ is not an FPZD-set. Then there exists a vertex u such that u is not dominated by any vertex in $V-D$. Since G, a non-trivial connected graph satisfies FPZD-set, u is dominated by at least one vertex in $D-\{u\}$. Thus $D-\{u\}$ is a FPZD-set, a contradiction. Hence $V-D$ is an FPZD-set of a graph G. □

To prove our next result, we make use of the following definition:

Let G and H be two graphs. Then the corona of G and H is the graph $G \circ H$ which is the disjoint union of G and $|V(G)|$ copies of H and every vertex v of G is adjacent to every vertex in the corresponding copy of H.

Corollary 2.3. *Let G be a non-trivial connected graph satisfying FPZD-set. Then $\gamma(G) = \gamma_{pz}(G) = \frac{p}{2}$ if and only if $G \cong \{C_4$ or $H \circ K_1\}$ for any connected graph H with even order.*

Proof. Let G be a graph isomorphic with C_4 or $H \circ K_1$ for any connected graph H with even order satisfying FPZD-set D. Since each vertex v of D covers all the edges due to r-regular graph G. There fore $\sum_{v\in D} d_G(v)^2 = \sum_{u\in V-D} d_G(u)^2$ and $\gamma_{pz}(G) = \gamma(G) = \frac{p}{2}$.

Conversely, let $\gamma(G) = \gamma_{pz}(G)$. Since the graph is satisfying perfect matching and r-regular. The result follows. □

Theorem 2.4. *A dominating set D of a graph G is minimal FPZD-set if and only if it satisfies the following conditions,*

(i) $PN(v, D) \neq \emptyset$ *for every* $v \in D$
(ii) $\sum_{v\in D} d_G(v)^2 = \sum_{u\in V-D} d_G(u)^2$.

Proof. Let D be a minimal FPZD-set. Then every vertex $v \in D$, $D-\{v\}$ not a FPZD-set, there exists a vertex $u \in V-(D-\{v\})$. Therefore $u \in PN(v, D)$. Hence for every vertex $v \in D$ has at least one neighbor. Thus $PN(v, D) \neq \emptyset$. Also, $\sum_{v\in D} d_G(v)^2 = \sum_{u\in V-D} d_G(u)^2$.

Conversely, suppose $PN(v, D) \neq \emptyset$ and $\sum_{v\in D} d_G(v)^2 = \sum_{u\in V-D} d_G(u)^2$. Now we have to prove that D is a minimal FPZD-set. Assume D is not a minimal FPZD-set which implies that there exists a vertex $v \in D$ such that $D - \{v\}$ a dominating set. Then v is adjacent to at least one vertex in $D - \{v\}$ and also every vertex in $V - D$ is adjacent to at least one in $D - \{v\}$. Therefore, neither (i) nor (ii) holds, which is a contradiction. □

Theorem 2.5. *Let G be any connected graph having minimum FPZD-set D. Then G is a minimal FPZD-set.*

Proof. Let D be any FPZD-set. If for each vertex $v \in D$, then there exist $\sum_{v\in D} d_G(v)^2 = \sum_{u\in V-D} d_G(u)^2$ such that $uv \in E(G)$. Hence D is a minimal FPZD-set. □

Remark 2.6. *Converse of the Theorem 2.5 need not be true. For example, consider a graph G having vertex set $\{v_1, v_2, \ldots, v_{19}\}$. Here, $D_1 = \{v_1, v_3, v_4, v_6, v_{11}, v_{13}, v_{16}\}$, $D_2 = \{v_2, v_5, v_7, v_8, v_9, v_{10}, v_{12}, v_{14}, v_{15}, v_{17}, v_{18}, v_{19}\}$ are the FPZD-sets of G. Since D_1 is a minimum FPZD-set, which is a minimal and D_2 is minimal FPZD-set, which is not minimum.*

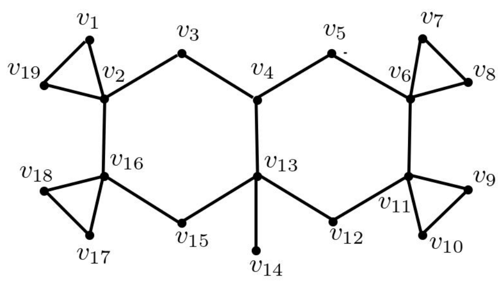

FIGURE 2. A graph G with 19 vertices

Theorem 2.7. *For any connected graph G satisfying FPZD-set,*

$$\gamma(G) \leq \gamma_{pz}(G) \leq p - \gamma(G).$$

Proof. Since every FPZD-set is a dominating set. Hence, $\gamma(G) \leq \gamma_{pz}(G)$.

Let D be an FPZD-set of a graph G. Suppose $V - D$ is not a dominating set of G. Then there exists a vertex v in D such that v is not adjacent to any of the vertex in $V - D$. Hence $D - \{v\}$ is an FPZD-set of G, which is a contradiction to the minimality of D. In continuation, $V - D$ is a dominating set of G, we have $|V - D| \geq \gamma(G)$ implies $\gamma(G) + \gamma_{pz}(G) \leq p$. Thus the upper bound follows. □

We showed that there are G graphs for which the $\gamma_{pz}(G) = \gamma(G)$ equality is maintained. However, the difference between $\gamma_{pz}(G)$ and $\gamma(G)$ can be arbitrarily large.

Theorem 2.8. *The difference $\gamma_{pz}(G) - \gamma(G)$ can be arbitrarily large.*

Proof. Consider a complete bipartite graph $K_{m,n}$ with $m = n$ and $\gamma(K_{m,n}) = 2$. $\gamma_{pz}(K_{m,n}) = \frac{p}{2}$. Hence $\gamma_{pz}(G) - \gamma(G) = \frac{p-4}{2}$. □

To prove next result, we use certain definitions that are:

1. The n-pan graph with $p = n+1$ is the graph obtained by joining a cycle graph C_n to a singleton graph K_1 with a cut vertex.
2. The fish graph is the graph obtained by taking one copy of a complete graph K_3 with an odd cycle $C_{2n+1}; n \geq 1$ of atmost one vertex in common and is denoted by $(2n + 1)$-fish graph.

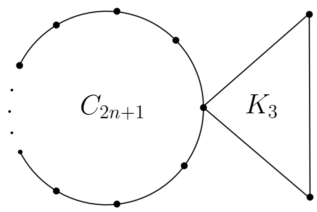

FIGURE 3. C_{2n+1} Fish graph

Theorem 2.9. *For any connected graph G satisfying a FPZD-set,*

(i) $\gamma_{pz}(G) = 1$ *if and only if G is P_2 or 3-pan graph or 3-fish graph.*
(ii) $\gamma_{pz}(G) = 2$ *if and only if G is P_4 or C_4 or 5-fish graph or 5-pan graph.*
(iii) $\gamma_{pz}(G) = p - 1$ *if and only if $G \cong P_2$.*
(iv) $\gamma_{pz}(G) = p - 2$ *if and only if $G \cong P_4$ or C_4 or K_4.*
(v) $\gamma_{pz}(G) = \gamma_{pz}(G - v)$ *if and only if G is a $(2n + 1)$-fish graph.*

Proof.
(i) Let D be a FPZD-set of a connected graph and $\gamma_{pz}(G) = 1$. Then every vertex $v \in D$, there exists a vertex $u \in V - (D - \{v\})$. Therefore u is a open neighborhood of v with respect to D. Hence for every vertex v in D has at least one neighbor. Hence G is P_2 or 3-pan graph or 3-fish graph.

Conversely, suppose G is P_2 or 3-pan graph or 3-fish graph. By Theorem 2.2, D is a dominating set. We have to prove that $\gamma_{pz}(G) = 1$. Suppose D is not a minimal FPZD-set. Then there exists at least one vertex v in D such that $D - \{v\}$ is a dominating set. If $D - \{v\}$ dominates $V - (D - \{v\})$, then at least one vertex $D - \{v\}$ is adjacent to v. This contradicts that D is a FPZD-set. Hence $\gamma_{pz}(G) = 1$.
(ii) By (i), the result (ii) follows.
(iii) Let G be a connected graph with even order p. Since $\gamma_{pz}(G) \leq \frac{p}{2}$, we have $\gamma(G) = \gamma_{pz}(G)$ with the graph $G \cong P_2$. Therefore, $\gamma(G) = p - 1 = \frac{p}{2}$ for $p = 2$. Thus the result follows.

The converse is obvious.
(iv) By (iii), the result (iv) follows.
(v) Let $v \in V(G)$ and $d_G(v) = \Delta(G)$. Let D be the minimal FPZD-set of $G - v$ with minimum cardinality. So, $\gamma_{pz}(G) = |D|$. Since $d_G(v) = \Delta(G)$, v is in G and v is a cut vertex. By the definition of FPZD-set, every FPZD-set of G contains at least one $\Delta(G)$, which implies $\gamma_{pz}(G) = \gamma_{pz}(G - v)$.

The converse is obvious. □

Theorem 2.10. *For any connected graph G with even order,*

$$\left\lfloor \frac{\alpha(G)}{2} \right\rfloor \leq \gamma_{pz}(G) \leq \alpha(G),$$

where $\alpha(G)$ is a vertex covering number of a graph G.

Proof. Let G be any connected graph, which satisfies an FPZD-set. Then at least one of $\Delta(G)$ cover the edges of maximum degree; at least one of $\delta(G)$ cover the remaining edges of G and also, if $\delta(G) = \Delta(G)$, then $\alpha(G)$ covers all the edges of G. Hence the lower bound follows.

Let D be a FPZD-set of a connected graph G and let $u \in V - D$, which implies $D \subseteq N(u)$. Since $\delta(G) \geq 1$, the vertex u is adjacent to atleast one vertex of D, is a vertex covering set of G. Hence $\gamma_{pz}(G) \leq \alpha(G)$. □

Theorem 2.11. *Let G be a connected graph with even order. Then*

$$\left\lceil \frac{diam(G)}{2} \right\rceil \leq \gamma_{pz}(G) \leq p - 1 - \left\lfloor \frac{diam(G) - 1}{2} \right\rfloor.$$

Proof. Let G be a connected graph satisfying FPZD-set. Then by the definition of diameter of a graph G. The lower bound folows.

To prove the upper bound. Let D be any connected graph contains a diametral path P_p. Then by Theorem 2.1, we obtain

$$\gamma_{pz}(G) \quad \leq \quad (p - diam(G) - 1) + \left\lfloor \frac{diam(G) - 1}{2} \right\rfloor = p - 1 - \left\lfloor \frac{diam(G) - 1}{2} \right\rfloor.$$

□

Theorem 2.12. *For any connected graph G with even order p has $\delta(G) \geq 2$ and $g(G) \geq 3$,*

$$\left\lfloor \frac{g(G)}{2} \right\rfloor \leq \gamma_{pz}(G) \leq p - \left\lfloor \frac{g(G)}{2} \right\rfloor + 1,$$

where $g(G)$ is a girth (i.e., the length of its shortest cycle) of a graph G.

Proof. The lower bound is straight forward. Now, to prove the upper bound, let G be a connected graph with even order p with $\delta(G) \geq 2$ and $g(G) \geq 3$. Form $G^{'}$ by removing the cycle C_p of shortest length from G. Suppose an arbitrary vertex $v \in V(G^{'})$, then v has at least two neighbors say u and w. If $u, w \in C_p$ and $d(u, w) \geq 2$ then by replacing the path from u and w on C_p with the path uvw which reduces the girth of G, a contradiction. If $d(u, w) \leq 1$, then u, w, v are on a cycle C_p in G, which is a contradiction to the hypothesis that $g(G) \geq 3$. Hence no vertex in $G^{'}$ has two or more neighbors on C_p. Therefore, $\gamma_{pz}(G) \leq p - \lfloor \frac{g(G)}{2} \rfloor + 1$. □

Theorem 2.13. [7] *For any (p, q)-simple graph G with $p \geq 3$ vertices,*

$$M_1(G) \quad \leq \quad p(\Delta(G))^2.$$

Equality holds if and only if G is r-regular graph.

Clearly, a graph G having no isolates with even order is $\gamma_{pz}(G) \leq M_1(G)$. Now, we obtain sharp upper bound for $\gamma_{pz}(G)$ in terms of order, maximum degree and the first Zagreb index of G.

Theorem 2.14. *For any (p, q)-simple graph G with $p \geq 3$,*

$$\gamma_{pz}(G) \leq \frac{p^2(\Delta(G))^2}{2M_1(G)}.$$

Equality holds if and only if G is a regular graph.

Proof. Let G be a graph of order p with maximum degree $\Delta(G)$ and if $\frac{p^2(\Delta(G))^2}{2M_1(G)} \geq \gamma_{pz}(G)$. Then $p^2(\Delta(G))^2 \geq 2M_1(G)\gamma_{pz}(G)$ But $\gamma_{pz}(G) \leq \frac{p}{2}$. We have $p(\Delta(G))^2 \geq M_1(G)$ That is $M_1(G) \leq p(\Delta(G))^2$. Thus, the result follows from Theorem 2.13. □

3. First strong Zagreb domination

A dominating set $D \subseteq V$ of a graph $G = (V, E)$ is a first strong Zagreb dominating (FSZD) set if

(i) for every $v \in D$ there exist $u \in V - D$ such that $uv \in E(G)$.
(ii) $\sum_{v \in D} d_G(v)^2 > \sum_{u \in V-D} d_G(u)^2$.

The minimum cardinality among all the FSZD-sets of a graph G is called the first strong Zagreb domination number $\gamma_{sz}(G)$. The FSZD-set D with minimum cardinality is called γ_{sz}-set of G.

Here, we start with a couple of Observation and Proposition, which are straightforward, hence we omit the proofs.

Observation 3.1. *Let G be any totally disconnected graph. Then the FSZD-set of G does not exist.*

Proposition 3.2. *Let G be some standard class of graphs with $p \geq 3$ vertices. Then* *(i)* $\gamma_{sz}(P_p) = \begin{cases} \lfloor \frac{p}{2} \rfloor & \textit{if } p \textit{ is odd;} \\ \frac{p}{2} & \textit{if } p \textit{ is even.} \end{cases}$

(ii) $\gamma_{sz}(C_p) = \begin{cases} \lceil \frac{p}{2} \rceil & \textit{if } p \textit{ is odd;} \\ \lceil \frac{p}{2} \rceil + 1 & \textit{if } p \textit{ is even.} \end{cases}$

(iii) $\gamma_{sz}(K_p) = \begin{cases} \lceil \frac{p}{2} \rceil & \textit{if } p \textit{ is odd;} \\ \frac{p}{2} + 1 & \textit{if } p \textit{ is even.} \end{cases}$

(iv) $\gamma_{sz}(K_{1,p-1}) = 1$.

Theorem 3.3. *Let G be a graph. Then $\gamma_{sz}(G) = 1$ if and only if there exist only cut vertex v in G with degree $p-1$.*

Proof. Let $D = \{v\}$ be a γ_{sz}-set of a graph G. Then $\{v\}$ has the maximum degree $\Delta(G)$ dominates all the vertices. This implies that $\sum_{v \in D} d_G(v)^2 > \sum_{u \in V-D} d_G(u)^2$. Hence $\{v\}$ is a cutvertex of G with degree $p-1$.

Conversely, if v is a cut vertex of a graph G with degree $p-1$ since v is the only one cut vertex, it dominates all the remaining vertices of G. Thus $\gamma_{sz}(G) = 1$. □

Theorem 3.4. *If an FSZD-set is a dominating set, then there exists two verticess $u, w \in V - D$ such that $d(u, w) \geq 1$.*

Proof. Suppose there exists two vertex $u, w \in V - D$ such that $d(u, w) - 0$. Then $\langle V - D \rangle$ is disconnected, which is a contradiction to the given hypothesis. Hence the result follows. □

Theorem 3.5. *If a graph G satisfying an FSZD-set D, then D is always a minimal dominating set.*

Proof. Since D is a minimal dominating set of G. Then for every vertex $v \in D$, $D - \{v\}$ is a FSZD-set of G. This implies that $\sum_{v \in D} d_G(v)^2 \ngtr \sum_{u \in V-D} d_G(u)^2$. Hence every FSZD-set D is always a minimal dominating set of G. □

Remark 3.6. *Every FSZD-set D of a graph G is a dominating set but the converse need not be true. For example, Figure 3 having vertex set $\{v_1, v_2, \ldots, v_{19}\}$. Here, $D_1 = \{v_1, v_2, v_3, v_4, v_6, v_{11}, v_{13}, v_{16}\}$ is the FSZD-set of G. Since D_1 is a minimum FSZD-set, and also, a dominating set. Further, $D_2 = \{v_2, v_6, v_{11}, v_{13}, v_{16}\}$ which is a dominating set but not the FSZD-set of G.*

Theorem 3.7. *If G is a graph with $\Delta(G) \geq 2$, then γ_{sz}-set always depends on support vertex or $\Delta(G) - 1$ and $\Delta(G)$, and both.*

Proof. Let D be a FSZD-set of G. Since $\Delta(G) - 1$ and $\Delta(G)$ dominates maximum number of vertices and support vertex dominates vertex of degree one, which automaticaaly redirect $\sum_{v \in D} d_G(v)^2 > \sum_{u \in V-D} d_G(u)^2$. Hence the result follows. □

Theorem 3.8. *A tree T with more than one vertex of degree one is adjacent to a support vertex if and only if every γ-set is a γ_{sz}-set of T.*

Proof. Let D be a γ_{sz}-set of T. If a support vertex v adjacent to more than one vertex of degree one, then v must be in D. Hence D is a γ_{sz}-set of T.

Conversely, suppose every γ-set of T is a γ_{sz}-set of T. We have to prove that a tree T with more than one vertex of degree one is adjacent to a support vertex. Since a dominating set of T depends mainly on support vertex, which implies to dominate vertex of degree one. Therefore, all support vertices dominate the remaining vertices of T. Hence the proof. □

Theorem 3.9. *If T is a tree of order $p \geq 3$ vertices,*

$$1 \leq \gamma_{sz}(T) \leq \frac{p+1}{2}.$$

Further, the lower bound attains if and only if T is isomorphic with star.

Proof. Let T be a tree of order $p \geq 3$ vertices. Then T has at least one support vertex and any FSZD-set must contain all the support vertices. Thus the lower bound follows. By Proposition 3.2, the FSZD-set is adjacent to atmost $\frac{p}{2}+1$ vertices. This implies that $2\gamma_{sz}(T) \leq q+2$ with $q = p-1$, the upper bound follows.

Now we prove the next part. On contrary, if T is a star of order $p \geq 3$ vertices implies $\gamma_{sz}(T) = 1$, let D be a γ-set of T. Hence D must contain all support vertices. Since T has atleast one support vertex, the induced subgraph $\langle D \rangle$ in T is a star with the same support vertex as T. Let t denote the center of the star. If $p \geq \Delta(T)+1$, then there exist two vertices $u, w \in V - D$ such that $uw \in E$. Since u and w must be dominated by D, both u and w are adjacent to t. It follows that T has a cycle, which is a contradiction. Therefore, $\gamma_{sz}(T) = 1$ and T is a star. □

4. First weak Zagreb domination

A dominating set $D \subseteq V$ of a graph $G = (V, E)$ is a first weak Zagreb dominating (FWZD) set if

(i) for every $v \in D$ there exist $u \in V - D$ such that $uv \in E(G)$.

(ii) $\sum_{v \in D} d_G(v)^2 < \sum_{u \in V-D} d_G(u)^2$.

The minimum cardinality among all the FWZD-sets of a graph G is called the first weak Zagreb domination number $\gamma_{wz}(G)$. The FWZD-set D with minimum cardinality is called γ_{wz}-set of G.

Observation 4.1. *For any totally disconnected graph, the FWZD set does not exist.*

Observation 4.2. *Let G be a graph with the vertex of degree one. Then at least one vertex of degree one belongs to γ_{wz}-set of G.*

Proposition 4.3. *Let G be some standard class of graphs. Then*

(i) $\gamma_{wz}(P_p) = \begin{cases} 2 & \text{if } p = 3; \\ \lceil \frac{p}{3} \rceil & \text{if } p \geq 4. \end{cases}$

(ii) $\gamma_{wz}(C_p) = \begin{cases} 1 & \text{if } p = 3; \\ \text{does not exist} & \text{if } p = 4; \\ \lceil \frac{p}{3} \rceil & \text{if } p \geq 5. \end{cases}$

(iii) $\gamma_{wz}(K_p) = 1$.

(iv) $\gamma_{wz}(K_{m,n}) = \begin{cases} 1 & \text{if } 1 = m \leq n; \\ \text{does not exist} & \text{if } m = n; \\ \lceil \frac{p}{3} \rceil & \text{if } 3 \leq m < n. \end{cases}$

Theorem 4.4. *For any graph G with $p \geq 3$ vertices with $diam(G) = 2$,*

$$2 \leq \gamma_{wz}(G) \leq p-1.$$

Proof. If G satisfies the hypothesis of the theorem, then clearly $\gamma_{wz}(G) \geq 2$.

For upper bound, let $\gamma_{wz}(G) \geq p-1$ and $diam(G) \neq 2$. If D is a minimal FWZD-set of G, then the following cases arise:

Case 1. If $diam(G) = 1$, then $G \cong K_p$, which implies $\gamma_{wz}(G) = 1$, which is a contradiction.

Case 2. If $diam(G) \geq p-1$, then clearly $\gamma_{wz}(G) \geq p$, which is a contradiction.

Hence $diam(G) = 2$ holds. □

Observation 4.5. *For any graph G has a FWZD-set D if and only if following conditions are satisfied:*

(i) $|V(G)| \geq 3$

(ii) D is totally disconnected.

Observation 4.6. *Let G be a connected graph. If D is a FWZD-set of G, then $V - D$ is a FSZD set of G and vice-versa.*

Observation 4.7. *If a FWZD-set D of a graph G is a minimal FWZD-set of each vertex $v \in D$, then v is an isolated vertex in $\langle D \rangle$ in G.*

Theorem 4.8. *If T is a tree of $p \geq 3$, then*

$$\gamma_{wz}(G) \geq \Delta(T).$$

Furthermore, the bound is attained if and only if T is a star.

Proof. Let T be a tree with $p \geq 3$ vertices. Since T has at least two vertices of degree one and any FWZD-set must contain all the vertices of degree one. Hence the result follows.

Now, we prove the second part. It T is a tree of order $p \geq 3$, then $\gamma_{wz}(G) = \Delta(T)$. Let D be a γ_{wz}-set of T. Since T containing all its vertices of degree one. T has atleast two vertices of degree one, the $\langle D \rangle$ of a tree T is a star with the same vertex of degree one as T. Therefore, $\gamma_{wz}(T) = 1$ and T is a star.

The converse is obvious. □

Theorem 4.9. *Let T be any tree with at least two vertices of degree one are adjacent to a support vertex. If D is a γ-set of T, then $V - D$ is a γ_{wz}-set of a tree T.*

Proof. Let D be a γ-set of T. A support vertex u adjacent to at least two vertices of degree one. Since $|D| = 1$, which implies its complement is a γ_{wz}-set of T. Thus the result follows. □

5. Applicability of the Zagreb domination in QSPR-Analysis

In this section we examine the applicability of the $\gamma_{pz}(G)$, $\gamma_{sz}(G)$ and $\gamma_{wz}(G)$ in QSPR-analysis of octane isomers. For this, we consider the physico-chemical properties [acentric factor(AF), standard enthalpy of vaporization (DHVAP), boiling points(BP), critical temperature (TC), critical pressure (PC), entropy(S), density (D), mean radius (R_m^2), heat of vaporization (H_v) and heat of formation (H_f)] of octane isomers. The values are compiled in Table 1.

Alkane	**AF**	**DHVAP**	**BP**	**TC**	**PC**	**S**	**D**	R_m^2	$-\Delta H_f$	$-\Delta H_v$
n-octane	0.39	9.9	125.7	296.2	24.64	111.67	0.70	2.04	208.6	41.49
2M	0.37	9.4	117.6	288.0	24.80	109.84	0.69	1.89	215.4	39.67
3M	0.37	9.5	118.9	292.0	25.60	111.26	0.70	1.79	212.5	39.83
4M	0.37	9.4	117.7	290.0	25.60	109.32	0.70	1.76	210.7	39.64
3E	0.36	9.4	118.5	292.0	25.74	109.43	0.71	1.76	210.7	39.64
2,2MM	0.33	8.9	106.8	279.0	25.6	103.42	0.69	1.67	224.6	37.28
2,3MM	0.34	9.272	115.6	293.0	26.6	108.0	0.71	1.64	213.8	38.78
2,4MM	0.34	9.0	109.4	282.0	25.80	106.98	0.70	1.61	219.2	37.76
2,5MM	0.35	9.051	109.1	279.0	25.0	105.72	0.69	1.64	222.5	37.85
3,3MM	0.32	8.9	112.0	290.8	27.2	104.74	0.71	1.73	220.0	37.53
3,4MM	0.34	9.3	117.7	298.0	27.4	106.59	0.72	1.52	212.8	38.97
2M3E	0.33	9.2	115.6	295.0	27.40	106.0	0.71	1.55	211.0	38.52
3M3E	0.3	9.0	118.3	305.0	28.9	101.48	0.72	1.52	214.8	37.99
2,2,3MMM	0.3	8.8	109.8	294.0	28.2	101.31	0.71	1.43	220.0	36.91
2,2,4MMM	0.30	8.4	99.24	271.1	25.50	104.0	0.69	1.4	224.0	35.14
2,3,3MMM	0.29	8.89	114.8	303.0	29.0	102.0	0.72	1.49	216.3	37.27
2,3,4MMM	0.31	9.0	113.5	295.0	27.6	102.39	0.71	1.36	217.3	37.75
2,2,3,3MMMM	0.25	8.41	106.5	270.8	24.5	93.0	0.82	1.46	225.6	42.9

Table 1- Physico-chemical properties of octane isomers.

where, M indicates methyl heptane, MM indictes dimethyl hexane, ME indictes methyl ethyl pentane, MMM indictes trimethyl pentane and MMMM indictes tetramethyl butane.

Now, we compute the respective first Zagreb index $M_1(G)$ and second Zagreb index $M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v)$, and the newly introduced Zagreb domination related parameters $\gamma_{pz}(G)$, $\gamma_{sz}(G)$ and $\gamma_{wz}(G)$ of octane isomers, which are listed in the following Table 2.

Alkane	$M_1(G)$	$M_2(G)$	$\gamma_{pz}(G)$	$\gamma_{sz}(G)$	$\gamma_{wz}(G)$
n-octane	26.00	24.00	4.00	4.00	4.00
2M	28.00	26.00	3.00	3.00	4.00
3M	28.00	27.00	3.00	3.00	3.00
4M	28.00	27.00	3.00	3.00	3.00
2,3MM	28.00	28.00	4.00	3.00	3.00
2,4MM	32.00	30.00	4.00	3.00	3.00
2,5MM	30.00	30.00	4.00	2.00	5.00
3,4MM	30.00	29.00	4.00	3.00	4.00
2,2MM	30.00	28.00	7.00	2.00	6.00
3,3MM	32.00	32.00	7.00	3.00	6.00
3E	30.00	31.00	5.00	4.00	3.00
2,2,3MMM	30.00	31.00	6.00	3.00	6.00
2,2,4MMM	32.00	34.00	6.00	2.00	6.00
2,3,3MMM	34.00	35.00	6.00	3.00	6.00
2,3,4MMM	34.00	32.00	0.00	3.00	6.00
2M3E	34.00	36.00	4.00	3.00	3.00
3M3E	32.00	33.00	7.00	4.00	6.00
2,2,3,3MMMM	38.00	40.00	4.00	2.00	6.00

Table 2- Octane isomers values for above mentioned parameters.

We frame the following linear regression model for our study:

$$PP = a\gamma_{pz}(G) + b\gamma_{sz}(G) + c\gamma_{wz}(G) + d.$$

where PP indicates the physical properties of octane isomers, and a, b, c, d are arbitrary constants.

Surprisingly, we could see that the predicting power of these newly introduced parameters is better than the first Zagreb index for the physical properties of octane isomers. For TC, the correlation coefficient value for M_1 is 0.342 where for model 1, r value is 0.589, For PC, the correlation coefficient value for M_1 is 0.294 where for model 1, r value is 0.470 and for heats of vaporization the correlation coefficient value for M_1 is 0.222 where for model 1, r value is 0.270. Thus, the Zagreb domination related parameters perform better than $M_1(G)$ in predicting PC, TC, and enthalpy of vaporization (HVAP) of octane isomers. The multiple range from 6.5 to 7.5 of $\gamma_{pz}(G)$, $\gamma_{sz}(G)$ and $\gamma_{wz}(G)$ are approximately same as $M_1(G)$ and $M_2(G)$. Further, many degree based molecular descriptiors correlate certain physico-chemical properties of octane isomers, for more details one can refer [13] and [15].

6. Conclusions

Being new Domination related parameters in terms of degree-based topological indices of a graph G, i.e., novel Zagreb domination-types of parameters are initiated. For the comparative advantages, applications, and mathematical point of view, many questions are suggested by this research, among them are the following.

1. Find the extremal values and extremal graphs of the Zagreb domination-types of parameters.
2. Characterize $\gamma_{pz}(G) = \gamma_{sz}(G)$.
3. Characterize $\gamma_{pz}(G) = \gamma_{wz}(G)$.
4. Characterize $\gamma_{pz}(G)$ interms of order and size, when G is odd order. (For illustrative examples, see Example 1 and Example 2).

7. Acknowledgements

Authors would like to thank the reviewers for their valuable comments and suggestions to improve the quality of the paper.

References

[1] B. D. Acharya, H. B. Walikar and E. Sampathkumar, *Recent developments in the theory of domination in graphs*, Mehta Research Institute, Allahabad, MRI Lecture Notes in Math. 1979.
[2] B. Chaluvaraju, H. S. Boregowda and I. N. Cangul, *Some inequalities for the first general Zagreb index of graphs and line graphs*, Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, Springer **91(1)**(2021), 79-88.
[3] B. Chaluvaraju, M. Chellali and K.A. Vidya, *Perfect k-domination in graphs*, Australasian Journal of Combinatorics **48**(2010), 175-184.
[4] B. Chaluvaraju and K.A. Vidya, *Generalized Perfect domination in graphs*, Journal of Combinatorial Optimization, Springer **27**(2014), 292-301.
[5] I. Gutman, *Degree-based topological indices*, Croat. Chem. Acta. **86**(2013), 351-361.
[6] I. Gutman, K. C. Das, *The first Zagreb indices 30 years after*, MATCH Commun. Math. Comput. Chem. **50** (2004), 83-92.
[7] I. Gutman, V.R.Kulli, B. Chaluvaraju and H.S. Boregowda, *On Banhatti and Zagreb Indices*, J. Int. Math. Virtual Inst. **7** (2017), 53-67.
[8] I. Gutman and N. Trinajstic, *Graph Theory and molecular orbitals, Total pi-electron energy of alternant hydrocarbons*, Chem. Phys. Lett. **17**(1972), 535-538.
[9] F. Harary, *Graph Theory*, Addison-Wesley, Reading Mass, 1969.
[10] T. W. Haynes, S. T. Hedetniemi, and M. A. Henning (Eds.,), *Topics in Domination in Graphs*, Springer International Publishing AG, 2020.
[11] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in graphs*, Marcel Dekker, New York, 1998.
[12] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Domination in graphs (Advanced Topics)*, Marcel Dekker, New York, 1998.
[13] M. Kuanar, S. K. Kuanar, B. K. Mishra and I. Gutman, *Correlation of line graph parameters with physicochemical properties of octane isomers*, Indian J. of Chem. **38A** (1999), 525-528.
[14] E. Sampathkumar and H. B. Walikar, *The connected domination number of a graph*, J. Math.Phys.Sci. **13(6)**(1979), 607-613.
[15] Sunilkumar M. Hosamani, *Correlation of domination parameters with physicochemical properties of octane isomers*, Appl. Math. Nonlinear Sci. **1(2)** (2016), 345-352.
[16] D. B. West, *Introduction to Graph Theory*, Prentice Hall, 2001.

(1) Department of Mathematics, Bangalore University, Jnana Bharathi Campus, Bengaluru-560 056, INDIA
Email address: bchaluvaraju/dsatumkur@gmail.com

(2) Department of Mathematics, Rani Channamma University, Belagavi - 591 156, INDIA
Email address: sunilkumar.rcu@gmail.com

WEAK FINITELY BI-CONJUGATIVE RELATIONS

DANIEL A. ROMANO

Abstract. The concept of weak finitely regular relations was introduced and applied to partially ordered sets by Luo and Xu in 2020. In this paper, the idea of weak finitely is extended to bi-conjugative relations and it is linked to finitely bi-conjugative relations, which were introduced and investigated by this author.

Mathematics Subject Classification (2010): 08A02, 06A11
Key words: Bi-conjugative relation, finitely bi-conjugative relation, weak finitely bi-conjugative relation.

Article history:
Received: September 2, 2019
Received in revised form: December 12, 2021
Accepted: December 20, 2021

1. Introduction and Preliminaries

For a set X, we call α a binary relation on X if $\alpha \subseteq X^2$. Let $\mathcal{B}(X)$ be denote the set of all binary relations on X. For $\alpha, \beta \in \mathcal{B}(X)$, define

$$\beta \circ \alpha = \{(x, z) \in X^2 : (\exists y \in X)((x, y) \in \alpha, (y, z) \in \beta)\}.$$

The relation $\beta \circ \alpha$ is called the composition of α and β. It is well known that $\mathcal{B}(X)$, with composition, is a monoid (semigroup with identity). Namely, $\triangle_X = \{(x, x) : x \in X\}$ is its identity element. For a binary relation α on a set X, define $\alpha^{-1} = \{(x, y) \in X^2 : (y, x) \in \alpha\}$ and $\alpha^c = X^2 \setminus \alpha$. Thus $(\alpha^c)^{-1} = (\alpha^{-1})^c$ holds.

Let A be a subset of X. For $\alpha \in \mathcal{B}(X)$, set

$$A\alpha = \{y \in X : (\exists a \in A)((a, y) \in \alpha)\}, \;\; \alpha A = \{x \in X : (\exists b \in A)((x, b) \in \alpha)\}.$$

It is easy to see that $A\alpha = \alpha^{-1}A$ holds. Specially, we put $a\alpha$ instead of $\{a\}\alpha$ and αb instead of $\alpha\{b\}$.

Notions and notations used in this article that are not previously defined may be find by the reader in articles [1, 3, 4, 7, 9].

1.1. **Bi-conjugative relations.** The regularity of binary relations were first characterized by Zareckiii [8]. Further criteria for regularity were given by Markowsky [2] and Schein [7]. In [9] Xu and Liu introduced the concepts of finitely regular relations and finite extensions of binary relations, and presented relational representations of hyper-continuous lattices. After that various similar types of binary relations were studied by some other mathematicians. The fundamental works of K. A. Zareckii, Markowsky and Schein and others on regular relations motivated several mathematicians to investigate similar classes of relations, obtained by putting α^{-1}, α^c or $(\alpha^c)^{-1}$ in place of one or both α's on the right side of the regularity equation

$$\alpha = \alpha \circ \beta \circ \alpha$$

(where β is some relation). The following class of elements in the semigroup $\mathcal{B}(X)$ have been investigated:

Definition 1.1 ([3], Definition 2.1)**.** The relation $\alpha \in \mathcal{B}(X)$ is called a *bi-conjugative* if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^{-1} \circ \beta \circ \alpha^{-1}.$$

The family of all bi-conjugative relations on set X is not empty [3]. In [3], the following proposition is proved:

Proposition 1.2 ([3], Theorem 2.3)**.** *For a binary relation α on a set X, the following conditions are equivalent:*
(1) *α is a bi-conjugative relation.*
(2) *For all $x, y \in X$, if $(x, y) \in \alpha$, there exist $v, u \in X$ such that:*
(a) *$(v, x) \in \alpha \wedge (y, u) \in \alpha$; and*
(b) *$(\forall s, t \in X)(((v, s) \in \alpha \wedge (t, u) \in \alpha) \implies (s, t) \in \alpha)$.*

In addition, it has been shown ([3], Corollary 2.4) that if $(L, \leqslant)$ is a poset, then $\nleqslant$ is not a bi-conjugative relation on L. .

1.2. **The concept of finite extension of bi-conjugative relations.** For any set X, let

$$X^{(<\omega)} = \{F \subseteq X : F \text{ is finite}\}.$$

For any positive integer m, we write $\overline{m} = \{1, 2, ..., m\}$.

Definition 1.3 ([9])**.** For a binary relation $\rho \subseteq X \times Y$, define a relation $\rho^{(<\omega)} \subseteq X^{(<\omega)} \times Y^{(<\omega)}$ by

$$(\forall (F, G) \in X^{(<\omega)} \times Y^{(<\omega)})((F, G) \in \rho^{(<\omega)} \iff G \subseteq F\rho).$$

$\rho^{(<\omega)}$ is called finite extension of ρ.

For illustration purposes, we will show definition of finite extension of bi-conjugative relations:

Definition 1.4 ([4])**.** A binary relation ρ on a set X is called finitely bi-conjugative if for all $(x, y) \in \rho$, there are $u \in X$ and $\{v_1, v_2, ..., v_k\} \in X^{(<\omega)}$, such that
(i) $(u, y) \in \rho^{-1} \wedge (\forall i \in \overline{k})((x, v_i) \in \rho^{-1})$, and
(ii) for all $\{s_1, s_2, ..., s_k\} \in X^{(<\omega)}$ and $t \in X$,
if $(u, t) \in \rho^{-1}$ and $(\forall i \in \overline{k})((s_i, v_i) \in \rho^{-1})$ then there is $j \in \overline{k}$ such that $(s_j, t) \in \rho$.

An important description of the finitely bi-conjugative relation is given in the following proposition:

Proposition 1.5 ([4])**.** *For a binary relation ρ on a set X, the following are equivalent:*
(i) *ρ is a finitely bi-conjugative relation on X; and*
(ii) *there is a binary relation $\delta \subseteq X^{(<\omega)} \times X^{(<\omega)}$ such that*

$$(\rho^{-1})^{(<\omega)} \circ \delta \circ (\rho^{-1})^{(<\omega)} = \rho^{(<\omega)}.$$

2. The concept of weak finitely bi-conjugative relations

In Definition 1.4, in statement (ii) the finite set $\{s_1, s_2, ..., s_m\}$, $(m \in \mathbb{N})$ appears as a very strong condition. In article [1], in a situation where regular relations is discussed, this requirement is weakened by taking $\{s\}$ instead of $\{s_1, s_2, ..., s_m\}$. We impose an analogous requirement on the finality of bi-conjugative relations.

Before that, we need the following definition

Definition 2.1 ([1], Definition 6)**.** For a binary relation $\rho \subseteq X \times Y$, define a binary relation $\rho^{(<\omega)} \subseteq X \times Y^{(<\omega)}$, called the *right finite extension* of ρ, by

$$\rho_r^{(<\omega)} = \{(x, G) \in X \times Y^{(<\omega)} : G \subseteq x\rho\}.$$

We can now introduce the concept of weak finitely bi-conjugative relations

Definition 2.2. A binary relation ρ on a set X is called *weak finitely bi-conjugative* if for all $(x,y)\in\rho$, there are $u\in X$ and $\{v_1,v_2,...,v_k\}\in X^{(<\omega)}$, such that

(i) $(u,y)\in\rho^{-1}\wedge(\forall i\in\overline{k})((x,v_i)\in\rho^{-1})$, and

(ii) for all $s\in X$ and $t\in X$,

if $(u,t)\in\rho^{-1}$ and $(\forall i\in\overline{k})((s,v_i)\in\rho^{-1})$ then $(s,t)\in\rho$.

3. The main result

In the following theorem, we give some characterizations of weak finality of bi-conjugative reactions.

Theorem 3.1. *Let $\rho\subseteq X\times X$ a binary relation. Then the following conditions are equivalent:*

(1) *ρ is weak finitely bi-conjugative;*

(2) *there is a relation $\delta\subseteq X^{(<\omega)}\times X^{(<\omega)}$ such that*

$$(\rho^{-1})^{(<\omega)}\circ\delta\circ(\rho_r^{-1})^{(<\omega)}=\rho_r^{(<\omega)};$$

(3) *for all $(x,G)\in\rho_r^{(<\omega)}$ there is $(U,V)\in X^{(<\omega)}\times X^{(<\omega)}$ such that*

(i) *$G\subseteq\rho U\wedge V\subseteq\rho x$, and*

(ii) *for any $(s,T)\in X\times X^{(<\omega)}$,*

if $V\subseteq\rho s$ and $T\subseteq\rho U$, then $(s,T)\in\rho_r^{(<\omega)}$.

Proof. (1) $\Longrightarrow$ (2). Define a relation $\delta\subseteq X^{(<\omega)}\times X^{(<\omega)}$ by $(G,F)\in\delta$ if and only if

$$(\forall(s,T)\in X\times X^{(<\omega)})((G\subseteq\rho s\wedge T\cap\rho F\neq\emptyset)\Longrightarrow T\cap s\rho\neq\emptyset).$$

For any $(h,W)\in X\times X^{(<\omega)}$, if $(h,W)\in(\rho^{-1})^{(<\omega)}\circ\delta\circ(\rho_r^{-1})^{(<\omega)}$, then there exists $(G,F)\in X^{(<\omega)}\times X^{(<\omega)}$ such that

$$(h,G)\in(\rho_r^{-1})^{(<\omega)}\wedge(G,F)\in\delta\wedge(F,W)\in(\rho^{-1})^{(<\omega)},$$

i.e., $G\subseteq h\rho^{-1}=\rho h$ and $W\subseteq F\rho^{-1}=\rho F$. Now we have to show that $W\subseteq h\rho$. For any $w\in W$, let $s=h$ and $T=\{w\}$. Then we have $G\subseteq\rho h=\rho s$ and $\emptyset\neq T\cap\rho F=\{w\}\cap\rho F=\{w\}$. From the definition of δ and $(G,F)\in\delta$, it follows $\emptyset\neq T\cap s\rho=\{w\}\cap h\rho$ what means that $w\in h\rho$. Therefore, $(h,W)\in\rho_r^{(<\omega)}$. We have proven inclusion $(\rho^{-1})^{(<\omega)}\circ\delta\circ\rho_r^{(<\omega)}\subseteq\rho_r^{(<\omega)}$.

For any $(h,W)\in X\times X^{(<\omega)}$, if $(h,W)\in\rho_r^{(<\omega)}$, then $W\subseteq h\rho$. This means for any $w\in W$ holds $(h,w)\in\rho$. Since ρ is weak finitely bi-conjugative, there are $u_w\in X$ and $V_w\in X^{(<\omega)}$ such that

(i) $(u_w,w)\in\rho^{-1}\wedge V_w\subseteq h\rho^{-1}=\rho h$, and

(ii) for all $s\in X$ and $t\in X$,

if $(u_w,t)\in\rho^{-1}$ and $V_w\subseteq s\rho^{-1}=\rho s$ then $(s,t)\in\rho$.

Let $G=\bigcup_{w\in W}V_w$, $F=\{u_w:w\in W\}$. Then $G\subseteq h\rho^{-1}$ and $W\subseteq F\rho^{-1}=\rho F$ since $w\in u_w\rho^{-1}=\rho u_w\subseteq\rho F$. Let $s\in X$ and $T\in X^{(<\omega)}$ be arbitrary elements such that $G\subseteq s\rho^{-1}$ and $T\cap\rho F\neq\emptyset$. Then there exist $u_{w_0}\in F$ and $t_0\in T$ such that $(u_{w_0},t_0)\in\rho^{-1}$. On the other hand, according to (ii), from $w\in W$ and $V_w\subseteq s\rho^{-1}$, follows $(s,t_0)\in\rho$. This means $T\cap s\rho\neq\emptyset$. By the definition of δ, we have $(G,F)\in\delta$. Hence $(h,W)\in(\rho^{-1})^{(<\omega)}\circ\delta\circ(\rho_r^{-1})^{(<\omega)}$. So, we have proven inclusion $\rho_r^{(<\omega)}\subseteq(\rho^{-1})^{(<\omega)}\circ\delta\circ(\rho_r^{-1})^{(<\omega)}$.

(2) $\Longrightarrow$ (3). This implication can be verified without difficulty.

(3) $\Longrightarrow$ (1). For any $(x,y)\in\rho$, there exist $V,U\in X^{(<\omega)}$ such that

(i) $y\in\rho U$ and $V\subseteq\rho x$; and

(ii) $(\forall(s,T)\in X\times X^{(<\omega)})((V\subseteq\rho s\wedge T\subseteq\rho U)\Longrightarrow(s,T)\in\rho_r^{(<\omega)})$.

Since $y\in\rho U$, there exists $u\in U$ such that $(u,y)\in\rho^{-1}$. Let $V=\{v_1,v_2,...,v_m\}$. Then $(u,y)\in\rho^{-1}$, $(\forall j\in\overline{m})((x,v_j)\in\rho^{-1}))$, i.e., the condition (i) in Definition 2.2 is satisfied. Now we check the condition (ii) in Definition 2.2. For any $(s,t)\in X\times X$, if $(u,t)\in\rho^{-1}$ and $(s,v_j)\in\rho^{-1}$ $(j=1,2,...,m)$, i.e., $V\subseteq\rho s$ and $\{t\}\subseteq u\rho^{-1}\subseteq\rho U$, then $t\in s\rho$ by the condition (ii). Thus ρ is weak finitely bi-conjugative. □

4. Conclusion

By applying the Luo and Xu idea, presented in the paper [1], to the bi-conjugative relation, in this text together with the previously published two papers ([5], [6]), the author seeks to see the peculiarity of being 'weak finitely relation' of some type.

Acknowledgement. The author thanks the reviewer for the suggestions that raised the consistency of the presented material in this paper.

References

[1] S. Luo and X. Xu, *Weak finitely regular relations and applications*, Semigroup forum, **100(2)**(2020), 505–512.
[2] G. Marcowsky, *Idempotents and product representations with applications weak finitely regular relations and applications to the semigroup of binary relations*, Semigroup forum, **5(1)**(1972), 95–119.
[3] D. A. Romano, *Bi-conjugative relations*, Rom. J. Math. Comp. Sci., **4(2)**(2014), 203–208.
[4] D. A. Romano, *Finitely bi-conjugative relations*, Rom. J. Math. Comp. Sci., **6(2)**(2016), 134–138.
[5] D. A. Romano, *Weak finitely dual quasi-conjugative relations*, Journal of Universal Mathematics, **3(1)**(2020), 28–32.
[6] D. A. Romano, *The concept of weak finality of conjugative relations*, MAT-KOL (Banja Luka), **26(1)**(2020), 17–23.
[7] B. M. Schein, *Regular elements of the semigroup of all binary relations*, Semigroup Forum, **13**(1976), 95-102.
[8] A. Zareckiĭ, *The semigroup of binary relations*, Mat. Sbornik, **61(3)**(1963), 291-305. (In Russian)
[9] X. Q. Xu and Y. M. Liu, *Relational representations of hypercontinuous lattices*, In: Zhang, G.Q., Lawson, J.D., Liu, Y.M., Luo, M.K. (Eds.) Domain Theory, Logic, and Computation (pp. 65-74). Kluwer Academic, Dordrecht, 2003.

International Mathematical Virtual Institute,
78000 Banja Luka, Kordunaška Streer 6, Bosnia and Herzegovina
E-mail address: bato49@hotmail.com, daniel.a.romano@hotmail.com

SOME NEW INEQUALITIES FOR (α, m_1, m_2)-GG CONVEX FUNCTIONS VIA GAMMA AND INCOMPLETE GAMMA FUNCTIONS

MAHIR KADAKAL

Abstract. In this study, we introduced the concept of (α, m_1, m_2)-GG convex functions; then we obtained Hermite-Hadamard type inequalities for these type of function classes via gamma and incomplete gamma functions together with Hölder, power-mean, Hölder-İşcan and improved power-mean integral inequalities.

Mathematics Subject Classification (2010): 26A51, 26D10
Key words: (α, m_1, m_2)-GG convex function, Hermite-Hadamard integral inequality.

Article history:
Received: March 21, 2020
Received in revised form: December 21, 2021
Accepted: December 27, 2021

1. Introduction

It is well known that convexity theory provides powerful principles and techniques to study a wide class of problems in both pure and applied mathematics. A convex function is a continuous function whose value at the midpoint of every interval in its domain does not exceed the arithmetic mean of its value at the ends of the considered interval. Readers can find more informations in the articles [1, 3, 7, 8, 10, 12, 13, 14, 17, 18, 21, 22, 23, 24, 25] and the references therein. One of the most important inequalities in convexity theory is Hermite-Hadamard integral inequality. [2, 4].This inequality gives us upper and lower bounds for the mean-value of a convex function.

Definition 1.1 ([20])**.** The GG-convex functions (called in what follows multiplicatively convex functions) are those functions $f : I \to J$ (acting on subintervals of $(0, \infty)$) such that

$$x, y \in I \text{ and } \lambda \in [0,1] \Longrightarrow f\left(x^{1-t}y^t\right) \leq f(x)^{1-\lambda} f(y)^{\lambda},$$

i.e., for which $\log f$ is convex.

Definition 1.2 ([15])**.** Let the function $f : [0, b] \to \mathbb{R}$ and $(m_1, m_2) \in [0,1]^2$. If

$$f\left(a^{m_1 t} b^{m_2(1-t)}\right) \leq [f(a)]^{m_1 t} [f(b)]^{m_2(1-t)}.$$

for all $[a, b] \in [0, b]$ and $t \in [0, 1]$, then the function f is said to be (m_1, m_2)-GG convex function, if the above inequality reversed, then the function f is said to be (m_1, m_2)-GG concave function.

Definition 1.3 ([11])**.** Let the function $f : [0, b] \to \mathbb{R}$ and $(\alpha, m_1, m_2) \in (0,1]^3$. If

$$f\left(a^{m_1 t} b^{m_2(1-t)}\right) \leq m_1 t^{\alpha} f(a) + m_2 (1 - t^{\alpha}) f(b).$$

for all $[a, b] \in [0, b]$ and $t \in [0, 1]$, then the function f is said to be (α, m_1, m_2)-geometric arithmetically convex function, if the inequality reversed, then the function f is said to be (α, m_1, m_2)-geometric arithmetically concave function.

An refinement of Hölder integral inequality better approach than Hölder's integral inequality can be given as follows:

Theorem 1.4 (Hölder-İşcan integral inequality [6])**.** *Let $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. If f and g are real functions defined on $[a,b]$ and if $|f|^p$, $|g|^q$ are integrable functions on interval $[a,b]$ then*

$$\begin{aligned}\int_a^b |f(x)g(x)|\,dx &\leq \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_a^b (x-a)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

An refinement of power-mean integral inequality better approach than power-mean inequality as a result of the Hölder-İşcan integral inequality can be given as follows:

Theorem 1.5 (Improved power-mean integral inequality [16])**.** *Let $q\geq 1$. If f and g are real functions defined on $[a,b]$ and if $|f|$, $|f|\,|g|^q$ are integrable functions on $[a,b]$ then*

$$\begin{aligned}\int_a^b |f(x)g(x)|\,dx &\leq \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b (b-x)\,|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_a^b (x-a)\,|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b (x-a)\,|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Definition 1.6 (Gamma function)**.** The classic gamma function is usually defined as

$$\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}dt.$$

Definition 1.7 (Upper incomplete gamma function)**.** The upper incomplete gamma function is defined as

$$\Gamma(s,x)=\int_x^\infty t^{s-1}e^{-t}dt.$$

Definition 1.8 (Lower incomplete gamma function)**.** The lower incomplete gamma function is defined as

$$\gamma(s,x)=\int_0^x t^{s-1}e^{-t}dt.$$

2. Main Results for (α,m_1,m_2)-GG convex functions

In this section, we give the concept of (α,m_1,m_2)-GG convex functions and some algebraic properties for them, as follows:

Definition 2.1. Let the function $f:[0,b]\to\mathbb{R}$ and $(\alpha,m_1,m_2)\in(0,1]^3$. If

$$f\left(a^{m_1t}b^{m_2(1-t)}\right)\leq[f(a)]^{m_1t^\alpha}[f(b)]^{m_2(1-t^\alpha)}. \tag{2.1}$$

for all $[a,b]\in[0,b]$ and $t\in[0,1]$, then the function f is said to be (α,m_1,m_2)-GG convex function, if the inequality (2.1) reversed, then the function f is said to be (α,m_1,m_2)-GG concave function.

Remark 2.2. When $m_1=m_2=1$ and $\alpha=1$, the (α,m_1,m_2)-GG convex (concave) function becomes a GG convex (concave) function in defined [20].

Remark 2.3. When $\alpha=1$, the (α,m_1,m_2)-GG convex (concave) function becomes a (m_1,m_2)-GG convex (concave) function in defined in [15].

Proposition 2.4. *Every nondecreasing (α, m_1, m_2)-GG convex function is (m_1, m_2)-GG convex function.*

Proof. If we substitute $[f(a)]^{m_1 t^\alpha} \le [f(a)]^{m_1 t}$ and $[f(b)]^{m_2(1-t^\alpha)} \le [f(b)]^{m_2(1-t)}$ in the equality (2.1), the proof is completed. □

The following Theorems can be proved similar to those in [11].

Theorem 2.5. *The function $f : (0, \infty) \to \mathbb{R}$ is (α, m_1, m_2)-GG convex function on the interval I if and only if $\ln \circ f : (0, \infty) \to \mathbb{R}$ is (α, m_1, m_2)-GA convex function on the interval I.*

Theorem 2.6. *Let $f, g : (0, \infty) \to \mathbb{R}$. If f and g are (α, m_1, m_2)-GG convex functions, then fg is an (α, m_1, m_2)-GG convex function.*

Theorem 2.7. *If $f : (0, \infty) \to (0, \infty)$ is a (m_1, m_2)-GG-convex and $g : (0, \infty) \to \mathbb{R}$ is a (α, m_1, m_2)-GG convex function and nondecreasing, then $g \circ f : (0, \infty) \to \mathbb{R}$ is a (α, m_1, m_2)-GG convex function.*

Theorem 2.8. *Let $\alpha, m_1, m_2 \in (0, 1], 0 < a^{m_1} < b^{m_2}$ and $f_\alpha : [a^{m_1}, b^{m_2}] \to \mathbb{R}$ be an arbitrary family of (α, m_1, m_2)-GG convex functions and let $f(x) = \sup_\alpha f_\alpha(x)$. If $J = \{u \in [a^{m_1}, b^{m_2}] : f(u) < \infty\}$ is nonempty, then J is an interval and f is an (α, m_1, m_2)-GG convex function on J.*

Theorem 2.9. *If the function $f : [a^{m_1}, b^{m_2}] \to \mathbb{R}$ is an (α, m_1, m_2)-GG convex function then f is bounded on the interval $[a^{m_1}, b^{m_2}]$.*

3. Hermite-Hadamard inequality for (α, m_1, m_2)-GG convex function

The goal of this section is to establish some inequalities of Hermite-Hadamard integral inequalities for (α, m_1, m_2)-GG convex functions. In this section, we will denote by $L[a, b]$ the space of (Lebesgue) integrable functions on the interval $[a, b]$.

Theorem 3.1. *Let $f : [a^{m_1}, b^{m_2}] \to \mathbb{R}$ be an (α, m_1, m_2)-GG convex function and let $[f(b)]^{m_2} > [f(a)]^{m_1}$. If $a < b$ and $f \in L[a, b]$, then the following Hermite-Hadamard type integral inequalities hold:*

$$
\begin{aligned}
f\left(\sqrt{a^{m_1} b^{m_2}}\right) &\le \frac{1}{\ln b^{m_2} - \ln a^{m_1}} \int_{a^{m_1}}^{b^{m_2}} \frac{1}{u} f\left(\sqrt{u}\right) f\left(\sqrt{\frac{a^{m_1} b^{m_2}}{u}}\right) du \\
&\le \frac{1}{\ln b^{m_2} - \ln a^{m_1}} \int_{a^{m_1}}^{b^{m_2}} \frac{f(u)}{u} du \\
&\le [f(b)]^{m_2} \frac{\Gamma\left(\frac{1}{\alpha}\right) - \Gamma\left(\frac{1}{\alpha}, \ln[f(b)]^{m_2} - \ln[f(a)]^{m_1}\right)}{\alpha\left[\ln[f(b)]^{m_2} - \ln[f(a)]^{m_1}\right]^{\frac{1}{\alpha}}} \le \frac{m_1 f(a) + \alpha m_2 f(b)}{\alpha + 1},
\end{aligned} \tag{3.1}
$$

where Γ is the Gamma function.

Proof. Firstly, the function f is (α, m_1, m_2)-GG convex function, we have

$$f(\sqrt{xy}) \le \sqrt{f(x) f(y)} \le \frac{f(x) + f(y)}{2}$$

for all $x, y \in I$. If we substitute $x = a^{m_1 t} b^{m_2(1-t)}$ and $y = a^{m_1(1-t)} b^{m_2 t}$ in the above inequalities for $t \in [0, 1]$, we can write as

$$
\begin{aligned}
f\left(\sqrt{a^{m_1} b^{m_2}}\right) &\le \sqrt{f\left(a^{m_1 t} b^{m_2(1-t)}\right) f\left(a^{m_1(1-t)} b^{m_2 t}\right)} \\
&\le \frac{f\left(a^{m_1 t} b^{m_2(1-t)}\right) + f\left(a^{m_1(1-t)} b^{m_2 t}\right)}{2}.
\end{aligned}
$$

Now, if we take integral in the last inequality with respect to $t \in [0,1]$, we deduce that

$$\begin{aligned} f\left(\sqrt{a^{m_1} b^{m_2}}\right) &\leq \int_0^1 \sqrt{f\left(a^{m_1 t} b^{m_2(1-t)}\right) f\left(a^{m_1(1-t)} b^{m_2 t}\right)} dt \\ &= \frac{1}{\ln b^{m_2} - \ln a^{m_1}} \int_{a^{m_1}}^{b^{m_2}} \frac{1}{u} f\left(\sqrt{u}\right) f\left(\sqrt{\frac{a^{m_1} b^{m_2}}{u}}\right) du \\ &\leq \frac{1}{\ln b^{m_2} - \ln a^{m_1}} \int_{a^{m_1}}^{b^{m_2}} \frac{f(u)}{u} du. \end{aligned}$$

Secondly, by using the property of the (α, m_1, m_2)-GG convex function of f, we can write

$$f\left(a^{m_1 t} b^{m_2(1-t)}\right) \leq [f(a)]^{m_1 t^\alpha} [f(b)]^{m_2(1-t^\alpha)} \leq m_1 t^\alpha f(a) + m_2 (1 - t^\alpha) f(b).$$

By taking integral, we obtain

$$\begin{aligned} &\frac{1}{\ln b^{m_2} - \ln a^{m_1}} \int_{a^{m_1}}^{b^{m_2}} \frac{f(u)}{u} du \\ = \; &[f(b)]^{m_2} \frac{\Gamma\left(\frac{1}{\alpha}\right) - \Gamma\left(\frac{1}{\alpha}, \ln [f(b)]^{m_2} - \ln [f(a)]^{m_1}\right)}{\alpha \left[\ln [f(b)]^{m_2} - \ln [f(a)]^{m_1}\right]^{\frac{1}{\alpha}}} \leq \frac{m_1 f(a) + \alpha m_2 f(b)}{\alpha + 1}. \end{aligned}$$

This completes the proof of theorem. □

Corollary 3.2. *If we take $m_1 = m_2 = 1$ and $\alpha = 1$ in the inequalities (3.1), then we have the following inequalities:*

$$\begin{aligned} f\left(\sqrt{ab}\right) &\leq \frac{1}{\ln b - \ln a} \int_a^b \frac{1}{u} f\left(\sqrt{u}\right) f\left(\sqrt{\frac{ab}{u}}\right) du \\ &\leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(u)}{u} du \leq \frac{f(b) - f(a)}{\ln f(b) - \ln f(a)} \leq \frac{f(a) + f(b)}{2}. \end{aligned}$$

This inequalities coincide with the inequalities [5].

Corollary 3.3. *If we take $\alpha = 1$ in the inequalities (3.1), then we have the following inequalities:*

$$\begin{aligned} f\left(\sqrt{a^{m_1} b^{m_2}}\right) &\leq \frac{1}{\ln b^{m_2} - \ln a^{m_1}} \int_{a^{m_1}}^{b^{m_2}} \frac{1}{u} f\left(\sqrt{u}\right) f\left(\sqrt{\frac{a^{m_1} b^{m_2}}{u}}\right) du \\ &\leq \frac{1}{\ln b^{m_2} - \ln a^{m_1}} \int_{a^{m_1}}^{b^{m_2}} \frac{f(u)}{u} du \\ &\leq \frac{[f(b)]^{m_2} - [f(a)]^{m_1}}{\ln [f(b)]^{m_2} - \ln [f(a)]^{m_1}} \leq \frac{m_1 f(a) + m_2 f(b)}{2}. \end{aligned}$$

This inequalities coincide with the inequalities [15].

4. Some new inequalities for (α, m_1, m_2)-GG convex functions

The aim of this section is to establish new estimates that refine Hermite-Hadamard integral inequality for functions whose first derivatives in absolute value is (α, m_1, m_2)-GG convex function via gamma and incomplete gamma functions together with Hölder, Hölder-İşcan, power-mean and improwed power-mean integral inequalities. In order to prove next theorems, we need the following identity for differentiable functions.

Lemma 4.1 ([9]). *Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be differentiable function and $a, b \in I$ with $a < b$. If $f' \in L([a,b])$, then*

$$\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx = \frac{\ln b - \ln a}{2}\int_0^1 a^{3(1-t)} b^{3t} f'\left(a^{1-t}b^t\right)dt.$$

Theorem 4.2. *Let $f : \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ be a differentiable function and $f' \in L[a,b]$ for $0 < a < b < \infty$ and let $\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2} > \left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1}$. If $|f'|$ is (α, m_1, m_2)-GG convex function on $\left[0, \max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$ for $[\alpha, m_1, m_2] \in (0,1]^2$, then we have*

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right| \le b^3 \left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1} \frac{\ln b - \ln a}{2} \times \frac{\Gamma\left(\frac{1}{\alpha}\right) - \Gamma\left(\frac{1}{\alpha}, \ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2} - \ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1}\right)}{\alpha\left[\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2} - \ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1}\right]^{\frac{1}{\alpha}}}. \tag{4.1}$$

Proof. By using Lemma 4.1, the inequality

$$\left|f'\left(a^{1-t}b^t\right)\right| = \left|f'\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)} f'\left(b^{\frac{1}{m_2}}\right)^{m_2 t}\right| \le \left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1(1-t^\alpha)} \left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2 t^\alpha},$$

and the property of $\left(\frac{b}{a}\right)^{3t} \le \left(\frac{b}{a}\right)^3$ for $t \in [0,1]$, we obtain

$$\begin{aligned}
&\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right| \\
\le\ & \frac{\ln b - \ln a}{2}\int_0^1 a^{3(1-t)} b^{3t}\left|f'\left(a^{1-t}b^t\right)\right| dt \\
\le\ & \frac{\ln b - \ln a}{2}\int_0^1 a^{3(1-t)} b^{3t}\left|f'\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)} f'\left(b^{\frac{1}{m_2}}\right)^{m_2 t}\right| dt \\
\le\ & \frac{\ln b - \ln a}{2}\int_0^1 a^3\left(\frac{b}{a}\right)^{3t}\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1(1-t^\alpha)}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2 t^\alpha} dt \\
\le\ & \frac{\ln b - \ln a}{2}\int_0^1 a^3\left(\frac{b}{a}\right)^{3}\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1(1-t^\alpha)}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2 t^\alpha} dt \\
=\ & b^3\frac{\ln b - \ln a}{2}\int_0^1 \left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1(1-t^\alpha)}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2 t^\alpha} dt \\
=\ & b^3\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1}\frac{\ln b - \ln a}{2}\frac{\Gamma\left(\frac{1}{\alpha}\right) - \Gamma\left(\frac{1}{\alpha}, \ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2} - \ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1}\right)}{\alpha\left[\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2} - \ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1}\right]^{\frac{1}{\alpha}}}
\end{aligned}$$

This completes the proof of theorem. □

Corollary 4.3. *Under the assumption of Theorem 4.2, If we take $m_1 = m_2 = 1$ and $\alpha = 1$ in the inequality (4.1), then we have*

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right| \le b^3\left|f'(a)\right|\frac{\ln b - \ln a}{2}\frac{1 - \Gamma\left(1, \ln\left|f'(b)\right| - \ln\left|f'(a)\right|\right)}{\ln\left|f'(b)\right| - \ln\left|f'(a)\right|}. \tag{4.2}$$

Theorem 4.4. *Let $f : \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ be a differentiable function and $f' \in L[a,b]$ for $0 < a < b < \infty$, $\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2} > \left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1}$ and assume that $q > 1$. If $|f'|^q$ is a (α, m_1, m_2)-GG convex function on*

$\left[0, \max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$ *for* $[\alpha, m_1, m_2] \in (0,1]^2$*, then we have*

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x) dx\right| \le \frac{\ln b - \ln a}{2} \left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1} \tag{4.3}$$

$$\times L^{\frac{1}{p}}\left(a^{3p}, b^{3p}\right) \left[\frac{\Gamma\left(\frac{1}{\alpha}\right) - \Gamma\left(\frac{1}{\alpha}, \ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2} - \ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right)}{\alpha\left[\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2} - \ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right]^{\frac{1}{\alpha}}}\right]^{\frac{1}{q}}$$

where Γ *is the gamma function and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By using Lemma 4.1, Hölder's integral inequality and the following inequality

$$\left|f'\left(a^{1-t} b^t\right)\right|^q = \left|f'\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)} f'\left(b^{\frac{1}{m_2}}\right)^{m_2 t}\right|^q \le \left(\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\right)^{m_1(1-t^\alpha)} \left(\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\right)^{m_2 t^\alpha},$$

we get

$$\begin{aligned}
&\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x) dx\right| \\
\le\; & \frac{\ln b - \ln a}{2} \int_0^1 a^{3(1-t)} b^{3t} \left|f'\left(a^{1-t} b^t\right)\right| dt \\
\le\; & \frac{\ln b - \ln a}{2} \left[\int_0^1 \left(a^{3(1-t)} b^{3t}\right)^p dt\right]^{\frac{1}{p}} \left[\int_0^1 \left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)} \left(b^{\frac{1}{m_2}}\right)^{m_2 t}\right)\right|^q dt\right]^{\frac{1}{q}} \\
\le\; & \frac{\ln b - \ln a}{2} \left[\int_0^1 \left(a^{3(1-t)} b^{3t}\right)^p dt\right]^{\frac{1}{p}} \left[\int_0^1 \left(\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\right)^{m_1(1-t^\alpha)} \left(\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\right)^{m_2 t^\alpha} dt\right]^{\frac{1}{q}} \\
=\; & \frac{\ln b - \ln a}{2} \left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1} L^{\frac{1}{p}}\left(a^{3p}, b^{3p}\right) \\
& \times \left[\frac{\Gamma\left(\frac{1}{\alpha}\right) - \Gamma\left(\frac{1}{\alpha}, \ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2} - \ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right)}{\alpha\left[\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2} - \ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right]^{\frac{1}{\alpha}}}\right]^{\frac{1}{q}},
\end{aligned}$$

where

$$\int_0^1 \left(a^{3(1-t)} b^{3t}\right)^p dt = L\left(a^{3p}, b^{3p}\right)$$

and

$$\begin{aligned}
&\int_0^1 \left(\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\right)^{m_1(1-t)} \left(\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\right)^{m_2 t} dt \\
=\; & \left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1} \frac{\Gamma\left(\frac{1}{\alpha}\right) - \Gamma\left(\frac{1}{\alpha}, \ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2} - \ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right)}{\alpha\left[\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2} - \ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right]^{\frac{1}{\alpha}}}.
\end{aligned}$$

This completes the proof of theorem. □

Corollary 4.5. *Under the assumption of Theorem 4.4, If we take* $m_1 = m_2 = 1$ *and* $\alpha = 1$ *in the inequality (4.3), then we have*

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x) dx\right| \le \left|f'(a)\right| \frac{\ln b - \ln a}{2} L^{\frac{1}{p}}\left(a^{3p}, b^{3p}\right) \left[\frac{1 - \Gamma\left(1, \ln\left|f'(b)\right|^q - \ln\left|f'(a)\right|^q\right)}{\ln\left|f'(b)\right|^q - \ln\left|f'(a)\right|^q}\right]^{\frac{1}{q}}.$$

Corollary 4.6. *Under the assumption of Theorem 4.4, If we take $\alpha = 1$ in the inequality (4.3), then we have*

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right| \leq \frac{\ln b - \ln a}{2}\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1}$$
$$\times L^{\frac{1}{p}}\left(a^{3p}, b^{3p}\right)\left[\frac{1 - \Gamma\left(1, \ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2} - \ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right)}{\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2} - \ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}}\right]^{\frac{1}{q}}$$

This inequality coincides with the inequalities [15].

Theorem 4.7. *Let $f : \mathbb{R}_0 = [0, \infty) \to \mathbb{R}$ be a differentiable function and $f' \in L[a, b]$ for $0 < a < b < \infty$, $\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2} > \left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1}$ and assume that $q \geq 1$. If $|f'|^q$ is a (α, m_1, m_2)-GG convex function on the interval $\left[0, \max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$, then we have*

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right| \leq b^{\frac{3}{q}}\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1}\frac{\ln b - \ln a}{2}L^{1-\frac{1}{q}}\left(a^3, b^3\right) \tag{4.4}$$
$$\times\left[\frac{\Gamma\left(\frac{1}{\alpha}\right) - \Gamma\left(\frac{1}{\alpha}, \ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2} - \ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right)}{\alpha\left[\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2} - \ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right]^{\frac{1}{\alpha}}}\right]^{\frac{1}{q}},$$

where L is the arithmetic mean.

Proof. By using Lemma 4.1, well known power-mean integral inequality, the property of the (m_1, m_2)-GG convex function of $|f'|^q$ and the property of $\left(\frac{b}{a}\right)^{3t} \leq \left(\frac{b}{a}\right)^3$ for $t \in [0, 1]$, we get

$$\begin{aligned}
&\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right| \\
\leq\ & \frac{\ln b - \ln a}{2}\int_0^1 a^{3(1-t)}b^{3t}\left|f'\left(a^{1-t}b^t\right)\right|dt \\
\leq\ & \frac{\ln b - \ln a}{2}\left(\int_0^1 a^{3(1-t)}b^{3t}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 a^{3(1-t)}b^{3t}\left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2 t}\right)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{\ln b - \ln a}{2}\left(\int_0^1 a^{3(1-t)}b^{3t}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 b^3\left(\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\right)^{m_1(1-t^\alpha)}\left(\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\right)^{m_2 t^\alpha}dt\right)^{\frac{1}{q}} \\
=\ & b^{\frac{3}{q}}\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1}\frac{\ln b - \ln a}{2}L^{1-\frac{1}{q}}\left(a^3, b^3\right) \\
&\times\left[\frac{\Gamma\left(\frac{1}{\alpha}\right) - \Gamma\left(\frac{1}{\alpha}, \ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2} - \ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right)}{\alpha\left[\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2} - \ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right]^{\frac{1}{\alpha}}}\right]^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof of theorem. □

Corollary 4.8. *Under the assumption of Theorem 4.7 with $q = 1$, we get the conclusion of Theorem 4.2 as follow:*

$$\begin{aligned}&\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right|\\ \leq\ & b^3\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1}\frac{\ln b-\ln a}{2}\left[\frac{\Gamma\left(\frac{1}{\alpha}\right)-\Gamma\left(\frac{1}{\alpha},\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1}\right)}{\alpha\left[\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1}\right]^{\frac{1}{\alpha}}}\right].\end{aligned}$$

This inequality coincides with the inequality (4.1).

Corollary 4.9. *Under the assumption of Theorem 4.7 with $\alpha = 1$, $q = 1$ and $m_1 = m_2 = 1$ in the inequality (4.4), we get the following inequality:*

$$\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right|\leq b^3\left|f'(a)\right|\frac{\ln b-\ln a}{2}\left[\frac{1-\Gamma\left(1,\ln\left|f'(b)\right|-\ln\left|f'(a)\right|\right)}{\ln\left|f'(b)\right|-\ln\left|f'(a)\right|}\right].$$

This inequality coincides with the inequality (4.2).

Theorem 4.10. *Let $f : \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ be a differentiable function and $f' \in L[a,b]$ for $0 < a < b < \infty$ and let $\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2} > \left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1}$. If $|f'|^q$ is (α, m_1, m_2)-GG convex function on $\left[0, \max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$ for $[\alpha, m_1, m_2] \in (0,1]^2$ and $q > 1$, then we have*

$$\begin{aligned}(4.5)\qquad &\left|\frac{b^2f(a)-a^2f(b)}{2}-\int_a^b xf(x)dx\right|\\ \leq\ &\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1}\frac{\ln b-\ln a}{2}\left[\frac{L\left(a^{3p},b^{3p}\right)-a^{3p}}{3\left(\ln b-\ln a\right)}\right]^{\frac{1}{p}}\\ &\times\left\{\frac{\Gamma\left(\frac{1}{\alpha}\right)-\Gamma\left(\frac{1}{\alpha},\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right)}{\alpha\left[\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right]^{\frac{1}{\alpha}}}\right.\\ &\left.+\frac{\Gamma\left(\frac{2}{\alpha},\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right)}{\alpha\left[\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right]^{\frac{2}{\alpha}}}\right\}^{\frac{1}{q}}\\ &+\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2}\frac{\ln b-\ln a}{2}\left[\frac{b^{3p}-L\left(a^{3p},b^{3p}\right)}{3\left(\ln b-\ln a\right)}\right]^{\frac{1}{p}}\\ &\times\left\{\frac{\Gamma\left(\frac{2}{\alpha}\right)-\Gamma\left(\frac{2}{\alpha},\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right)}{\alpha\left[\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right]^{\frac{2}{\alpha}}}\right\}^{\frac{1}{q}},\end{aligned}$$

where L is the logarithmic mean and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 4.1, Hölder-İşcan integral inequality and the (α, m_1, m_2)-GG convexity of the function $|f'|^q$ on $\left[0, \max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$, we get

$$\begin{aligned}
&\left|\frac{b^2 f(a)-a^2 f(b)}{2}-\int_a^b x f(x)dx\right| \\
\leq\ & \frac{\ln b-\ln a}{2}\left[\int_0^1 (1-t)\left(a^{3(1-t)}b^{3t}\right)^p dt\right]^{\frac{1}{p}}\left[\int_0^1 (1-t)\left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2 t}\right)\right|^q dt\right]^{\frac{1}{q}} \\
&+\frac{\ln b-\ln a}{2}\left[\int_0^1 t\left(a^{3(1-t)}b^{3t}\right)^p dt\right]^{\frac{1}{p}}\left[\int_0^1 t\left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2 t}\right)\right|^q dt\right]^{\frac{1}{q}} \\
\leq\ & \frac{\ln b-\ln a}{2}\left[\int_0^1 (1-t)a^{3p(1-t)}b^{3pt}dt\right]^{\frac{1}{p}}\left[\int_0^1 (1-t)\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1(1-t^\alpha)}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2 t^\alpha}dt\right]^{\frac{1}{q}} \\
&+\frac{\ln b-\ln a}{2}\left[\int_0^1 t a^{3p(1-t)}b^{3pt}dt\right]^{\frac{1}{p}}\left[\int_0^1 t\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1(1-t^\alpha)}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2 t^\alpha}dt\right]^{\frac{1}{q}} \\
=\ & \left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1}\frac{\ln b-\ln a}{2}\left[\frac{L\left(a^{3p},b^{3p}\right)-a^{3p}}{3(\ln b-\ln a)}\right]^{\frac{1}{p}} \\
&\times\left\{\frac{\Gamma\left(\frac{1}{\alpha}\right)-\Gamma\left(\frac{1}{\alpha},\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right)}{\alpha\left[\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right]^{\frac{1}{\alpha}}}+\frac{\Gamma\left(\frac{2}{\alpha},\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right)}{\alpha\left[\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right]^{\frac{2}{\alpha}}}\right\}^{\frac{1}{q}} \\
&+\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2}\frac{\ln b-\ln a}{2}\left[\frac{b^{3p}-L\left(a^{3p},b^{3p}\right)}{3(\ln b-\ln a)}\right]^{\frac{1}{p}}\left\{\frac{\Gamma\left(\frac{2}{\alpha}\right)-\Gamma\left(\frac{2}{\alpha},\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right)}{\alpha\left[\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right]^{\frac{2}{\alpha}}}\right\}^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof of theorem. □

Corollary 4.11. *Under the assumption of Theorem 4.10 with $m_1 = m_2 = 1$ and $\alpha = 1$ in the inequality (4.5), we get*

$$\begin{aligned}
(4.6)\qquad &\left|\frac{b^2 f(a)-a^2 f(b)}{2}-\int_a^b x f(x)dx\right| \leq \left|f'(a)\right|\frac{\ln b-\ln a}{2}\left[\frac{L\left(a^{3p},b^{3p}\right)-a^{3p}}{3(\ln b-\ln a)}\right]^{\frac{1}{p}} \\
&\times\left\{\frac{1-\Gamma\left(1,\ln\left|f'(b)\right|-\ln\left|f'(a)\right|\right)}{\ln\left|f'(b)\right|-\ln\left|f'(a)\right|}+\frac{\Gamma\left(2,\ln\left|f'(b)\right|-\ln\left|f'(a)\right|\right)}{\left[\ln\left|f'(b)\right|-\ln\left|f'(a)\right|\right]^2}\right\}^{\frac{1}{q}} \\
&+\left|f'(b)\right|\frac{\ln b-\ln a}{2}\left[\frac{b^{3p}-L\left(a^{3p},b^{3p}\right)}{3(\ln b-\ln a)}\right]^{\frac{1}{p}}\left\{\frac{1-\Gamma\left(2,\ln\left|f'(b)\right|-\ln\left|f'(a)\right|\right)}{\left[\ln\left|f'(b)\right|-\ln\left|f'(a)\right|\right]^2}\right\}^{\frac{1}{q}},
\end{aligned}$$

Theorem 4.12. *Let $f : \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ be a differentiable function and $f' \in L[a,b]$ for $0 < a < b < \infty$ and let $\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2} > \left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1}$. If $|f'|^q$ is (α, m_1, m_2)-GG convex function on $\left[0, \max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$ for $[\alpha, m_1, m_2] \in (0,1]^2$ and $q \geq 1$, then we have*

$$(4.7)\qquad \left|\frac{b^2 f(a)-a^2 f(b)}{2}-\int_a^b x f(x)dx\right| \leq b^{\frac{3}{q}}\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1}\frac{\ln b-\ln a}{2}\left[\frac{L\left(a^3,b^3\right)-a^3}{3(\ln b-\ln a)}\right]^{1-\frac{1}{q}}$$

$$\times\left\{\frac{\Gamma\left(\frac{1}{\alpha}\right)-\Gamma\left(\frac{1}{\alpha},\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right)}{\alpha\left[\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right]^{\frac{1}{\alpha}}}+\frac{\Gamma\left(\frac{2}{\alpha},\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right)}{\alpha\left[\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right]^{\frac{2}{\alpha}}}\right\}^{\frac{1}{q}}$$

$$+b^{\frac{3}{q}}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2}\frac{\ln b-\ln a}{2}\left[\frac{b^3-L\left(a^3,b^3\right)}{3\left(\ln b-\ln a\right)}\right]^{1-\frac{1}{q}}\left\{\frac{\Gamma\left(\frac{2}{\alpha}\right)-\Gamma\left(\frac{2}{\alpha},\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right)}{\alpha\left[\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right]^{\frac{2}{\alpha}}}\right\}^{\frac{1}{q}},$$

where L is the logarithmic mean.

Proof. By using Lemma 4.1, improwed power-mean integral inequality, the property of (α, m_1, m_2)-GG convexity of $|f'|^q$ on $\left[0, \max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$ and the property of $\left(\frac{b}{a}\right)^{3t} \leq \left(\frac{b}{a}\right)^3$ for $t \in [0,1]$ we get

$$\begin{aligned}
&\left|\frac{b^2 f(a)-a^2 f(b)}{2}-\int_a^b x f(x)dx\right| \\
\leq\;& \frac{\ln b-\ln a}{2}\left[\int_0^1(1-t)\left(a^{3(1-t)}b^{3t}\right)dt\right]^{1-\frac{1}{q}}\left[\int_0^1(1-t)\left(a^{3(1-t)}b^{3t}\right)\left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2 t}\right)\right|^q dt\right]^{\frac{1}{q}} \\
&+\frac{\ln b-\ln a}{2}\left[\int_0^1 t\left(a^{3(1-t)}b^{3t}\right)dt\right]^{1-\frac{1}{q}}\left[\int_0^1 t\left(a^{3(1-t)}b^{3t}\right)\left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2 t}\right)\right|^q dt\right]^{\frac{1}{q}} \\
\leq\;& \frac{\ln b-\ln a}{2}\left[\int_0^1(1-t)\left(a^{3(1-t)}b^{3t}\right)dt\right]^{1-\frac{1}{q}}\left[\int_0^1(1-t)\left(a^{3(1-t)}b^{3t}\right)\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1(1-t^\alpha)}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2 t^\alpha}dt\right]^{\frac{1}{q}} \\
&+\frac{\ln b-\ln a}{2}\left[\int_0^1 t\left(a^{3(1-t)}b^{3t}\right)dt\right]^{1-\frac{1}{q}}\left[\int_0^1 t\left(a^{3(1-t)}b^{3t}\right)\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1(1-t^\alpha)}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2 t^\alpha}dt\right]^{\frac{1}{q}} \\
\leq\;& b^{\frac{3}{q}}\frac{\ln b-\ln a}{2}\left[\int_0^1(1-t)\left(a^{3(1-t)}b^{3t}\right)dt\right]^{1-\frac{1}{q}}\left[\int_0^1(1-t)\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1(1-t^\alpha)}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2 t^\alpha}dt\right]^{\frac{1}{q}} \\
&+b^{\frac{3}{q}}\frac{\ln b-\ln a}{2}\left[\int_0^1 t\left(a^{3(1-t)}b^{3t}\right)dt\right]^{1-\frac{1}{q}}\left[\int_0^1 t\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1(1-t^\alpha)}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2 t^\alpha}dt\right]^{\frac{1}{q}} \\
=\;& b^{\frac{3}{q}}\frac{\ln b-\ln a}{2}\left[\frac{L\left(a^3,b^3\right)-a^3}{3\left(\ln b-\ln a\right)}\right]^{1-\frac{1}{q}}\left[\int_0^1(1-t)\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1(1-t^\alpha)}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2 t^\alpha}dt\right]^{\frac{1}{q}} \\
&+b^{\frac{3}{q}}\frac{\ln b-\ln a}{2}\left[\frac{b^3-L\left(a^3,b^3\right)}{3\left(\ln b-\ln a\right)}\right]^{1-\frac{1}{q}}\left[\int_0^1 t\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1(1-t^\alpha)}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2 t^\alpha}dt\right]^{\frac{1}{q}} \\
=\;& b^{\frac{3}{q}}\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{m_1}\frac{\ln b-\ln a}{2}\left[\frac{L\left(a^3,b^3\right)-a^3}{3\left(\ln b-\ln a\right)}\right]^{1-\frac{1}{q}} \\
&\times\left\{\frac{\Gamma\left(\frac{1}{\alpha}\right)-\Gamma\left(\frac{1}{\alpha},\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right)}{\alpha\left[\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right]^{\frac{1}{\alpha}}}+\frac{\Gamma\left(\frac{2}{\alpha},\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right)}{\alpha\left[\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right]^{\frac{2}{\alpha}}}\right\}^{\frac{1}{q}}
\end{aligned}$$

$$+b^{\frac{3}{q}}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{m_2}\frac{\ln b-\ln a}{2}\left[\frac{b^3-L\left(a^3,b^3\right)}{3\left(\ln b-\ln a\right)}\right]^{1-\frac{1}{q}}\left\{\frac{\Gamma\left(\frac{2}{\alpha}\right)-\Gamma\left(\frac{2}{\alpha},\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right)}{\alpha\left[\ln\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^{qm_2}-\ln\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^{qm_1}\right]^{\frac{2}{\alpha}}}\right\}^{\frac{1}{q}}.$$

This completes the proof of theorem. □

Corollary 4.13. *Under the assumption of Theorem 4.12 with $m_1 = m_2 = \alpha = 1$ and $q = 1$, we get the following inequality:*

$$\left|\frac{b^2 f(a)-a^2 f(b)}{2}-\int_a^b xf(x)dx\right|\leq b^3\left|f'(a)\right|\frac{\ln b-\ln a}{2}$$
$$\times\left\{\frac{\left[1-\Gamma\left(1,\ln\left|f'(b)\right|-\ln\left|f'(a)\right|\right)\right]}{\left(\ln\left|f'(b)\right|-\ln\left|f'(a)\right|\right)}+\frac{\Gamma\left(2,\ln\left|f'(b)\right|-\ln\left|f'(a)\right|\right)}{\left[\ln\left|f'(b)\right|-\ln\left|f'(a)\right|\right]^2}\right\}$$
$$+b^3\left|f'(b)\right|\frac{\ln b-\ln a}{2}\left\{\frac{1-\Gamma\left(2,\ln\left|f'(b)\right|-\ln\left|f'(a)\right|\right)}{\left[\ln\left|f'(b)\right|-\ln\left|f'(a)\right|\right]^2}\right\}.$$

References

[1] M.K. Bakula, M.E. Özdemir, and J. Pečarić, *Hadamard type inequalities for m-convex and (α, m)-convex functions*, J. Inequal. Pure Appl. Math. **9(4)** (2008), Art. 96, 12 pages.

[2] S.S. Dragomir and C.E.M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Its Applications*, RGMIA Monograph, 2002.

[3] S.S. Dragomir, J. Pečarić and LE.Persson, *Some inequalities of Hadamard Type*, Soochow Journal of Mathematics **21(3)** (2001), 335-341.

[4] J. Hadamard, *Étude sur les propriétés des fonctions entières en particulier d'une fonction considérée par Riemann*, J. Math. Pures Appl. **58** (1893), 171-215.

[5] İ. İşcan, *Some new Hermite-Hadamard type inequalities for geometrically convex functions*, Mathematics and Statistics **1(2)** (2013), 86-91.

[6] İ. İşcan, *New refinements for integral and sum forms of Hölder inequality*, Journal of Inequalities and Applications **2019**:304 (2019), 11 pages.

[7] İ. İşcan and M. Kunt, *Hermite-Hadamard-Fejer type inequalities for quasi-geometrically convex functions via fractional integrals*, Journal of Mathematics **2016**, Article ID 6523041, 7 pages.

[8] İ. İşcan H. Kadakal and M. Kadakal. *Some new integral inequalities for n-times differentiable quasi-convex functions*, Sigma **35(3)** (2017), 363-368.

[9] A.P. Ji, T.Y. Zhang, F. Qi, *Integral inequalities of Hermite-Hadamard type for (α, m)-GA-convex functions*, arXiv preprint arXiv:1306.0852, 4 June 2013.

[10] H. Kadakal, *Multiplicatively P-functions and some new inequalities.* New Trends in Mathematical Sciences, NTMSCI **6** (2018), 111-118.

[11] M. Kadakal, *Some new inequalities for (α, m_1, m_2)-GA convex functions*, Sakarya University Journal of Science **24(4)** (2020), 641-653.

[12] H. Kadakal, *(m1, m2)-Convexity and Some New Hermite-Hadamard Type Inequalities.* International Journal of Mathematical Modelling & Computations, **9(4)** (Fall) (2019): 297-309.

[13] H. Kadakal, *New Inequalities for Strongly r-Convex Functions*, Journal of Function Spaces, Volume **2019**, Article ID 1219237 (2019), 10 pages.

[14] H. Kadakal, *(α, m_1, m_2)-convexity and some inequalities of Hermite-Hadamard type*, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics **68(2)** (2019), 2128-2142.

[15] H. Kadakal and K. Bekar, *(m_1, m_2)-GG convex functions and related inequalities*, Konuralp Journal of Mathematics **8(2)** (2020), 313-321.

[16] M. Kadakal, İ. İşcan, H. Kadakal, and K. Bekar, *On improvements of some integral inequalities*, Miskolc Mathematical Notes **20(2)** (2019), 997-1011.

[17] M. Kadakal, H. Kadakal and İ. İşcan, *Some new integral inequalities for n-times differentiable s-convex functions in the first sense*, Turkish Journal of Analysis and Number Theory **5(2)** (2017), 63-68.

[18] H. Kadakal, M. Kadakal and İ. İşcan, *New type integral inequalities for three times differentiable preinvex and prequasiinvex functions.* Open J. Math. Anal. **2(1)** (2018), 33-46.

[19] S. Maden, H. Kadakal, M. Kadakal and İ İşcan, *Some new integral inequalities for n-times differentiable convex and concave functions*, Journal of Nonlinear Sciences and Applications **10(12)** (2017), 6141-6148.

[20] C.P. Niculescu, *Convexity according to the geometric mean*, Math. Inequal. Appl. **3(2)** (2000), 155-167.

[21] S. Özcan, *Some Integral Inequalities for Harmonically (α, s)-Convex Functions*, Journal of Function Spaces **2019** (2019), Article ID 2394021, 8 pages.

[22] S. Özcan and İ. İşcan, *Some new Hermite-Hadamard type inequalities for s-convex functions and their applications*, Journal of Inequalities and Applications **2019**:201 (2019).

[23] F. Usta, H. Budak and M. Z. Sarıkaya, *Montgomery identities and Ostrowski type inequalities for fractional integral operators*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales, Serie A. Matemáticas **113(2)** (2019), 1059-1080

[24] F. Usta, H. Budak and M. Z. Sarıkaya, *Some new Chebyshev type inequalities utilizing generalized fractional integral operators*, AIMS Mathematics **5(2)** (2020) 1147-1161.

[25] F. Usta, H. Budak, M. Z. Sarıkaya and E. Set, *On generalization of trapezoid type inequalities for s-convex functions with generalized fractional integral operators*, Filomat **32(6)** (2018), 2153-2171.

Bayburt University, Faculty of Applied Sciences, Department of Customs Management, Baberti Campus, 69000 Bayburt-Turkey

Email address: mahirkadakal@gmail.com

NEW INEQUALITIES ON ISOTROPIC SPACELIKE SUBMANIFOLDS IN PSEUDO-RIEMANNIAN SPACE FORMS

ALEXANDRU CIOBANU AND MARIUS MIREA

Abstract. Both spacelike and isotropic submanifolds of pseudo-Riemannian spaces have interesting properties, studied in Mathematics and Physics as well. The paper presents new inequalities for isotropic spacelike submanifolds of pseudo-Riemannian space forms, respectively a corresponding inequality of a generalized Euler inequality and a Ricci inequality.

Mathematics Subject Classification (2010): 53C40, 53C25
Key words: Spacelike submanifolds, isotropic submanifolds, pseudo-Riemannian space forms.

Article history:
Received: November 3, 2021
Received in revised form: December 20, 2021
Accepted: December 28, 2021

1. Preliminaries

Pseudo-Riemannian geometry represents one of the most important subjects of study in Differential Geometry. In particular, the theory of submanifolds of a pseudo-Riemannian manifold plays a crucial role in this topic. We focus in this article on isotropic spacelike submanifolds.

In this section we recall basic definitions and formulae. In section 2 we prove an inequality for an isotropic spacelike submanifold of a pseudo-Riemannian space form, an analogue of the generalized Euler inequality. A consequence gives a characterization for totally umbilical isotropic submanifolds. In section 3 we obtain a Ricci inequality for an isotropic spacelike submanifold of a pseudo-Riemannian space form.

The notion of an *isotropic immersion* was introduced by B. O'Neill [3] as an isometric immersion such that all its normal curvature vectors have the same length. This is a generalization of the totally umbilical submanifolds.

For isotropic submanifolds in the pseudo-Riemannian settings we refer to [1].

Let $(\tilde{M}^m_\nu, g)$ be a pseudo-Riemannian manifold of dimension m and signature ν and $\phi : M^n_s \to \tilde{M}^m_\nu$ an isometric immersion of a pseudo-Riemannian manifold M^n_s of dimension n and signature s.

We use the common notations for H - the mean curvature vector, R - the Riemannian curvature tensor, K - the sectional curvature, Ric - the Ricci curvature, τ - the scalar curvature, etc.

The immersion ϕ is called *pseudo-umbilical* if its second fundamental form h satisfies $g(h(X,Y),H) = \rho g(X,Y)$, for some function ρ. It follows that $\rho = g(H,H)$.

We recall that ϕ is a *totally umbilical* immersion if any point $p \in M^n_s$ is *umbilic*, i.e., there exists a vector $\xi_p \in T^\perp_p M^n_s$ such that for all $u, v \in T_p M^n_s$ one has $h(u,v) = g(u,v)\xi_p$.

Clearly, any totally umbilical immersion is pseudo-umbilical.

The isometric immersion ϕ is called *isotropic at* $p \in M^n_s$ (see [1]) if

$$g(h(u,u), h(u,u)) = \lambda(p) \in \mathbb{R}$$

does not depend on the choice of the unit vector $u \in T_pM_s^n$, and ϕ is said to be *isotropic* if it is isotropic at any point of M_s^n.

If $X \in T_pM_0^n$ is a unit vector, let $\{X = e_1, e_2, ..., e_n\}$ be an orthonormal basis of $T_pM_0^n$. The Ricci curvature of X is defined by

$$Ric(X) = \sum_{i=2}^{n} K(X \wedge e_i),$$

where $K(X \wedge e_i)$ is the sectional curvature of the plane spanned by X and e_i.

2. A Generalized Euler Inequality

In [2] the following inequality (known as a generalized Euler inequality (G.E.)) was proved:

Corollary 2.1. [2] *Let N be a spacelike submanifold of an indefinite real space form $R_r^{n+r}(c)$ of constant curvature c. Then*

$$||H||^2 \leq \frac{2\tau}{n(n-1)} - c, \tag{G.E.}$$

$n = \dim N$, *with equality holding at a point $p \in N$ if and only if p is a totally umbilical point.*

We denote by $(\tilde{R}(Z, X)Y)^T$ the tangential component of $\tilde{R}(Z, X)Y$.

We recall the formula

$$\begin{aligned} 3A_{h(X,Y)}Z = {} & \lambda[g(X, Y)Z + g(Y, Z)X + g(X, Z)Y]+ \\ & +R(Z, X)Y - (\tilde{R}(Z, X)Y)^T + R(Z, Y)X - (\tilde{R}(Z, Y)X)^T, \end{aligned} \tag{2.1}$$

for $X, Y, Z \in \mathfrak{X}(M_s^n)$, which is an equivalent condition for the submanifold M_s^n to be isotropic (see [1], Theorem 3.3).

We prove the following

Proposition 2.2. *Let M_0^n be an isotropic spacelike submanifold of a pseudo-Riemannian space form $\tilde{M}_s^m(c)$. Then*

$$3ng(H, H) = (n + 2)\lambda + 2(n - 1)(\rho - c), \tag{2.2}$$

where $\rho = \frac{2\tau}{n(n-1)}$ is the normalized scalar curvature.

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_pM_0^n$, $p \in M_0^n$.

From (2.1) we have

$$\begin{aligned} 3g(A_{h(X,Y)}Z, W) = {} & \lambda[g(X, Y)g(Z, W) + g(Y, Z)g(X, W) + g(X, Z)g(Y, W)]+ \\ & +g(R(Z, X)Y, W) - g(\tilde{R}(Z, X)Y, W) + g(R(Z, Y)X, W) - g(\tilde{R}(Z, Y)X, W), \end{aligned} \tag{2.3}$$

for $X, Y, Z, W \in \mathfrak{X}(M_s^n)$. For $X = Y = e_i$, $Z = W = e_j$ we obtain from (2.3):

$$\begin{aligned} 3g(A_{h(e_i,e_i)}e_j, e_j) = {} & \lambda[g(e_i, e_i)g(e_j, e_j) + g(e_i, e_j)g(e_i, e_j) + g(e_i, e_j)g(e_i, e_j)]+ \\ & +g(R(e_j, e_i)e_i, e_j) - g(\tilde{R}(e_j, e_i)e_i, e_j) + g(R(e_j, e_i)e_i, e_j) - g(\tilde{R}(e_j, e_i)e_i, e_j). \end{aligned} \tag{2.4}$$

By summation after $i, j \in \{1, ..., n\}$ we get

$$3n^2g(H, H) = \lambda(n^2 + 2n) + 4\tau - 2n(n - 1)c, \tag{2.5}$$

where the mean curvature vector H is given by $H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$, the scalar curvature τ is calculated by $\tau = \sum_{1 \leq i < j \leq n} g(R(e_j, e_i)e_i, e_j)$ and the curvature of $\tilde{M}_s^m(c)$ is $c = g(\tilde{R}(e_j, e_i)e_i, e_j), i \neq j$.

From (2.5), after dividing by n, one obtains

$$3ng(H, H) = (n + 2)\lambda + \frac{4\tau}{n} - 2(n - 1)c,$$

which is equivalent with (2.2), the equality to prove.

□

The following Corollary of Proposition 2.2 coincides with Corollary 4.3 from [1]:

Corollary 2.3. *Let $\phi : M_s^n(k) \to \tilde{M}_s^m(c)$ be an isotropic immersion of a pseudo-Riemannian space form $M_s^n(k)$ into a pseudo-Riemannian space form $\tilde{M}_s^m(c)$. Then*

$$3ng(H,H) = (n+2)\lambda + 2(n-1)(k-c).$$

In particular, if $k = c$ then $\lambda = 0$ if and only if $g(H,H) = 0$.

Remark 2.4. *In the Corollary* 4.3 *from* [1] *it is also proved that ϕ is pseudo-umbilical, i.e. $g(h(X,Y),H) = \rho g(X,Y)$, for a function ρ. In our case, $\rho = g(H,H)$.*

Proposition 2.5. *Let M_0^n be an isotropic spacelike submanifold of a pseudo-Riemannian space form $\tilde{M}_s^{n+s}(c)$. Then $g(H,H) \geq \lambda$.*

Proof. The relation (G.E.) (Chen's Corollary, see [2]) says that

$$g(H,H) \leq \rho - c,$$

with equality holding if and only if M_0^n is a totally umbilical submanifold.

From Proposition 2.2 we have

$$\rho - c = \frac{3ng(H,H) - (n+2)\lambda}{2(n-1)}.$$

Then

$$g(H,H) \leq \frac{3ng(H,H) - (n+2)\lambda}{2(n-1)},$$

which implies

$$[2(n-1) - 3n]g(H,H) \leq -(n+2)\lambda,$$

equivalent with $g(H,H) \geq \lambda$. □

From Chen's Corollary and Proposition 2.5 we obtain the following characterization of a totally umbilical isotropic spacelike submanifold of a pseudo-Riemannian space form.

Theorem 2.6. *Let M_0^n be an isotropic spacelike submanifold of a pseudo-Riemannian space form $\tilde{M}_s^{n+s}(c)$. Then M_0^n is totally umbilical if and only if $g(H,H) = \lambda$.*

3. A Ricci Inequality

Let M_0^n be an isotropic spacelike submanifold of a pseudo-Riemannian space form $\tilde{M}_s^{n+s}(c)$. Denote by $\{X = e_1, e_2, \ldots, e_n\}$ an orthonormal basis of $T_pM_0^n$, $p \in M_0^n$.

By Gauss equation, we have

$$Ric(X) = \tau - \frac{1}{2}(n-1)(n-2)c + \sum_{r=n+1}^{n+s} \sum_{2\leq i<j\leq n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2]. \tag{3.1}$$

On the other hand, the formula (6) from [1] gives the relation

$$n^2 g(H,H) = n(n+2)\lambda - 2g(h,h), \tag{3.2}$$

for every isotropic immersion, where $g(h,h)$ is expressed by

$$g(h,h) = \sum_{i,j=1}^{n} \epsilon_i \epsilon_j g(h(e_i,e_j), h(e_i,e_j)), \epsilon_i = g(e_i,e_i) = 1, \forall i \in \{1, ..., n\}.$$

By using (3.1), we can write

$$n^2g(H,H) = n(n+2)\lambda + 2\sum_{r=n+1}^{n+s}[(h^r_{11})^2 + (h^r_{22}+\ldots+h^r_{nn})^2 + 2\sum_{1\le i<j\le n}(h^r_{ij})^2]- \tag{3.3}$$

$$-4\sum_{r=n+1}^{n+s}\sum_{2\le i<j\le n}h^r_{ii}h^r_{jj} =$$

$$= n(n+2)\lambda + 4\tau - 4Ric(X) - 2(n-1)(n-2)c+$$

$$+\sum_{r=n+1}^{n+s}[(h^r_{11}+\ldots+h^r_{nn})^2 + (h^r_{11}-h^r_{22}-\ldots-h^r_{nn})^2 + 4\sum_{j=2}^{n}(h^r_{1j})^2] \ge$$

$$\ge n(n+2)\lambda + 4\tau - 4Ric(X) - 2(n-1)(n-2)c - n^2g(H,H).$$

Then (3.3) is equivalent with

$$Ric(X) \ge \tau + \frac{1}{4}n(n+2)\lambda - \frac{n^2}{2}g(H,H) - \frac{1}{2}(n-1)(n-2)c.$$

The equality case holds for X if and only if the inequality in (3.3) becomes an equality, i.e., $h^r_{1j}=0$ and $h^r_{11} = h^r_{22}+\ldots+h^r_{nn}$, for any $r\in\{n+1,...,n+s\}$.

The equality case holds for every vector in $T_pM^n_0$ if and only if

$$h^r_{ij}=0,\ \forall 1\le i\ne j\le n, \forall r\in\{n+1,...,n+s\},$$

$$2h^r_{ii} = h^r_{11}+h^r_{22}+\ldots+h^r_{nn};\ \forall r\in\{n+1,...,n+s\}.$$

Summing the above second equations, we get

$$(n-2)(h^r_{11}+h^r_{22}+\ldots+h^r_{nn}) = 0,\ \forall r\in\{n+1,...,n+s\}.$$

We distinguish 2 cases:
i) $n\ne 2$; then p is a totally geodesic point;
ii) $n=2$; then p is a totally umbilical point.

It follows that we proved the following Ricci inequality and we gave the characterization of the equality case.

Theorem 3.1. *Let M^n_0 be an isotropic spacelike submanifold of a pseudo-Riemannian space form $\tilde{M}^{n+s}_s(c)$ of constant curvature c. Then one has the following Ricci inequality:*

$$Ric(X) \ge \tau + \frac{1}{4}n(n+2)\lambda - \frac{n^2}{2}g(H,H) - \frac{1}{2}(n-1)(n-2)c. \tag{3.4}$$

The equality case holds for any vector $X\in T_pM^n_0$ if and only if p is a totally geodesic point or $n=2$ and p is a totally umbilical point.

Corollary 3.2. *Let M^n_0 be an isotropic spacelike submanifold of $\mathbb{E}^{n+s}_s$. Then*

$$\min Ric(X) \ge \tau + \frac{1}{4}n(n+2)\lambda - \frac{n^2}{2}g(H,H). \tag{3.5}$$

References

[1] J. L. Cabrerizo, M. Fernandez and J. S. Gomez, *Isotropic submanifolds of pseudo-Riemannian spaces*, J. Geom. Phys. **62** (2012), 1915-1924.

[2] B.-Y. Chen, *Pseudo-Riemannian Geometry, δ-Invariants and Applications*, World Sci. Singapore, 2011.

[3] B. O'Neill, *Isotropic and Kähler immersions*, Canadian J. Math. **17** (1965), 907-915.

[4] B. O'Neill, *Semi-Riemannian Geometry. With Applications to Relativity*, Acad. Press, New York, 1983.

Interdisciplinary Doctoral School, Transilvania University of Brasov, Brasov, Romania

E-mail address: `alexandru.ciobanu@unitbv.ro`

Doctoral School of Mathematics, University of Bucharest, Bucharest, Romania

E-mail address: `marius-ion.mirea@my.fmi.unibuc.ro`