ABOUT ROUGH COMMUTATORS WITH VARIABLE KERNEL OF FRACTIONAL TYPE ON VANISHING GENERALIZED WEIGHTED MORREY SPACES

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Abstract. In this paper, we obtain the boundedness of rough commutators with variable kernel of fractional type on vanishing generalized weighted Morrey spaces.

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1. Introduction and useful informations

1.1. **Background.** Suppose that $S^{n-1}=\{x\in\mathbb{R}^n:|x|=1\}$ is the unit sphere on $\mathbb{R}^n$ $(n\geq 2)$ equipped with the normalized Lebesgue measure $d\sigma(x')$. We say that a function $\Omega(x,z)$ defined on $\mathbb{R}^n\times\mathbb{R}^n$ belongs to the space $L_\infty(\mathbb{R}^n)\times L_s(S^{n-1})$ for $s>1$, if $\Omega(x,z)$ satisfies the following conditions:

For any $x,\, z\in\mathbb{R}^n$ and $\lambda>0$,

$$\Omega(x,\lambda z)=\Omega(x,z); \tag{1.1}$$

and for any $z\in\mathbb{R}^n\setminus\{0\}$ and $z'=z/|z|$

$$\|\Omega\|_{L_\infty(\mathbb{R}^n)\times L_s(S^{n-1})}:=\sup_{x\in\mathbb{R}^n}\left(\int\limits_{S^{n-1}}|\Omega(x,z')|^s\,d\sigma(z')\right)^{1/s}<\infty. \tag{1.2}$$

Let us consider the following commutators with variable kernel of rough fractional type integral operators with variable kernel defined by

$$\begin{aligned}[b,I_{\Omega,\alpha}]f(x)&\equiv b(x)I_{\Omega,\alpha}f(x)-I_{\Omega,\alpha}(bf)(x)\\&=\int\limits_{\mathbb{R}^n}[b(x)-b(y)]\frac{\Omega(x,x-y)}{|x-y|^{n-\alpha}}f(y)dy,\end{aligned}$$

and

$$\begin{aligned}[b,M_{\Omega,\alpha}]f(x)&\equiv b(x)\,M_{\Omega,\alpha}f(x)-M_{\Omega,\alpha}(bf)(x)\\&=\sup_{t>0}|B(x,t)|^{-1+\frac{\alpha}{n}}\int\limits_{B(x,t)}|b(x)-b(y)|\,|\Omega(x,x-y)|\,|f(y)|dy,\end{aligned}$$

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where f is a suitable function and $\Omega \in L_\infty(\mathbb{R}^n) \times L_s(S^{n-1})$, $s > 1$, is homogeneous of degree zero with respect to the second variable y on $\mathbb{R}^n$.

Recently, rough commutators with variable kernel of fractional type have been receiving more widely attention. Many results about the rough commutators with variable kernel $[b, I_{\Omega,\alpha}]$ and $[b, M_{\Omega,\alpha}]$ on various function spaces have been studied, respectively, see [1, 3] for details. However, the boundedness of these operators on vanishing generalized weighted Morrey spaces has almost never been studied. In this work, it is planned to fill the gap in the existing literature by our original results. That is, the purpose of this paper is to consider the mapping properties for the rough fractional type commutator operators with variable kernel $[b, I_{\Omega,\alpha}]$ and $[b, M_{\Omega,\alpha}]$ on vanishing generalized weighted Morrey spaces.

Now, we need the weight class $A(p,q)$ introduced by Muckenhoupt and Wheeden in [5] to study weighted boundedness of fractional integrals.

We say that $w(x) \in A(p,q)$ for $1 < p < q < \infty$ if and only if

$$\sup_{B(x,r)} \left(|B(x,r)|^{-1} \int\limits_{B(x,r)} w(x)^q dx \right)^{\frac{1}{q}} \left(|B(x,r)|^{-1} \int\limits_{B(x,r)} w(x)^{-p'} dx \right)^{\frac{1}{p'}} < \infty, \tag{1.3}$$

where the supremum is taken over all the balls $B(x,r)$. Note that, by Hölder's inequality, for all balls $B(x,r)$ we have

$$|B(x,r)|^{\frac{1}{p}-\frac{1}{q}-1} \|w\|_{L_q(B(x,r))} \|w^{-1}\|_{L_{p'}(B(x,r))} \geq 1. \tag{1.4}$$

By (1.3), we have

$$\left(\int\limits_{B(x,r)} w(x)^q dx \right)^{\frac{1}{q}} \left(\int\limits_{B(x,r)} w(x)^{-p'} dx \right)^{\frac{1}{p'}} \lesssim |B(x,r)|^{\frac{1}{q}+\frac{1}{p'}}. \tag{1.5}$$

Moreover, if $w(x)^{s'} \in A\left(\frac{p}{s'}, \frac{q}{s'}\right)$, then by (1.4) and (1.5), we obtain

$$\|w^{s'}\|_{L_{\frac{q}{s'}}(B(x,r))} \|w^{-s'}\|_{L_{\left(\frac{p}{s'}\right)'}(B(x,r))} \approx |B(x,r)|^{1+\frac{1}{\frac{q}{s'}}-\frac{1}{\frac{p}{s'}}}. \tag{1.6}$$

Recall that reverse Hölder's inequality is defined by

$$\sup_{B(x,r)} \left(\int\limits_{B(x,r)} w(x)^q dx \right)^{\frac{1}{q}} \left(\int\limits_{B(x,r)} w(x) dx \right)^{-1} < \infty$$

such that $1 < q < \infty$.

It is noteworthy to mention that the vanishing generalized weighted Morrey spaces have been defined by Gürbüz in [2].

Definition 1.1. ***(Vanishing generalized weighted Morrey spaces)*** *Let $1 \leq p < \infty$, $\varphi(x,r) : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ and w is nonnegative measurable function on $\mathbb{R}^n$. Vanishing generalized weighted Morrey space $VM_{p,\varphi}(w) \equiv VM_{p,\varphi}(\mathbb{R}^n, w)$ is defined as the space of functions $f \in VM_{p,\varphi}(w) \equiv VM_{p,\varphi}(\mathbb{R}^n, w)$ such that*

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \frac{1}{\varphi(x,r)} \|f\|_{L_p(B(x,r),w)} = 0. \tag{1.7}$$

Naturally, $\varphi(x,t)$ satisfies the following conditions:

$$\lim_{t \to 0} \sup_{x \in \mathbb{R}^n} \frac{(w(B(x,t)))^{\frac{1}{p}}}{\varphi(x,t)} = 0, \tag{1.8}$$

and

$$\inf_{t>1}\sup_{x\in\mathbb{R}^n}\frac{(w(B(x,t)))^{\frac{1}{p}}}{\varphi(x,t)}>0. \tag{1.9}$$

From now on, we denote by $\varphi\in\mathcal{B}(w)$ if $\varphi(x,r):\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ and satisfies (1.8) and (1.9).

For functions supported on x-centred Euclidean ball $B(x,r)\subset\mathbb{R}^n$, the space of functions of bounded mean oscillation $BMO(\mathbb{R}^n)$ is the set of all $b\in L_1^{loc}(\mathbb{R}^n)$ such that

$$\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|b(x)-b_{B(x,r)}|dx<\infty,$$

where

$$b_{B(x,r)}=\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}b(y)dy$$

is the mean of b over the ball $B(x,r)$ and the supremum is taken over all balls $B(x,r)$. Now, we define

$$BMO(\mathbb{R}^n)=\left\{b\in L_1^{loc}(\mathbb{R}^n):\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|b(x)-b_{B(x,r)}|dx<\infty\right\}$$

and

$$\|b\|_{BMO}=\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|b(x)-b_{B(x,r)}|dx.$$

Let $b\in BMO(\mathbb{R}^n)$. Then, for any $1<p<\infty$, by the John-Nirenberg inequality, we can obtain

$$\|b\|_{BMO}\approx\sup_{x\in\mathbb{R}^n,r>0}\left(\frac{1}{|B(x,r)|}\int\limits_{B(x,r)}|b(x)-b_{B(x,r)}|^p dx\right)^{\frac{1}{p}} \tag{1.10}$$

and for $0<2r<t$ there is a constant $C>0$ such that

$$\left|b_{B(x,r)}-b_{B(x,t)}\right|\leq C\|b\|_{BMO}\ln\frac{t}{r}. \tag{1.11}$$

Finally, $A\lesssim B$ means that $A\leq CB$ with some positive constant C independent of appropriate quantities and if $A\lesssim B$ and $B\lesssim A$, we write $A\approx B$, and also p' and s' always denote the conjugate index of any $p>1$ and $s>1$, that is, $\frac{1}{p'}:=1-\frac{1}{p}$ and $\frac{1}{s'}:=1-\frac{1}{s}$.

2. Main Results

Our result can be stated as follows.

Theorem 2.1. *Suppose that $0<\alpha<n$, $1\leq s'<p<\frac{n}{\alpha}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, $1<q<\infty$, $b\in BMO(\mathbb{R}^n)$, $\Omega(x,z)$ satisfies (1.1) and (1.2) for any $x\in\mathbb{R}^n\setminus\{0\}$. For $p>1$, $w(x)^{s'}\in A\left(\frac{p}{s'},\frac{q}{s'}\right)$ and $s'<p$, the inequality*

$$\|[b,I_{\Omega,\alpha}]f\|_{L_q(B(x_0,r),w^q)}\lesssim\|b\|_{BMO}\left(w^q\left(B\left(x_0,r\right)\right)\right)^{\frac{1}{q}}$$
$$\times\int\limits_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t),w^p)}}{\left(w^q\left(B\left(x_0,r\right)\right)\right)^{\frac{1}{q}}}\frac{dt}{t} \tag{2.1}$$

holds for any ball $B(x_0,r)$ and for all $f \in L^{loc}_{p,w}(\mathbb{R}^n)$. If $\varphi_1 \in \mathcal{B}(w^p)$, $\varphi_2 \in \mathcal{B}(w^q)$ and the pair (φ_1,φ_2) satisfies the following conditions

$$C_{\delta_0} := \int_{\delta}^{\infty} \left(1+\ln\frac{t}{r}\right) \sup_{x\in\mathbb{R}^n} \frac{\varphi_1(x,t)}{\left(w^q\left(B(x,t)\right)\right)^{\frac{1}{q}}} \frac{1}{t} dt < \infty \tag{2.2}$$

for every $\delta > 0$, and

$$\int_{r}^{\infty} \left(1+\ln\frac{t}{r}\right) \frac{\varphi_1(x,t)}{\left(w^q\left(B(x,t)\right)\right)^{\frac{1}{q}}} \frac{1}{t} dt \le C \frac{\varphi_2(x,r)}{\left(w^q\left(B(x,t)\right)\right)^{\frac{1}{q}}}, \tag{2.3}$$

then for $p > 1$, $w(x)^{s'} \in A\left(\frac{p}{s'}, \frac{q}{s'}\right)$ and $s' < p$,

$$\|[b, I_{\Omega,\alpha}]f\|_{VM_{q,\varphi_2}(w^q)} \lesssim \|b\|_{BMO} \|f\|_{VM_{p,\varphi_1}(w^p)}, \tag{2.4}$$

$$\|[b, M_{\Omega,\alpha}]f\|_{VM_{q,\varphi_2}(w^q)} \lesssim \|b\|_{BMO} \|f\|_{VM_{p,\varphi_1}(w^p)}. \tag{2.5}$$

Proof. Let $b \in BMO(\mathbb{R}^n)$. For any $x_0 \in \mathbb{R}^n$, we write as $f = f_1 + f_2$, where $f_1(y) = f(y)\chi_{B(x_0,2r)}(y)$, $f_2(y) = f(y)\chi_{(B(x_0,2r))^C}(y)$, $r > 0$. Then

$$\|[b, I_{\Omega,\alpha}]f\|_{L_q(w^q,B(x_0,r))} \le \|[b, I_{\Omega,\alpha}]f_1\|_{L_q(w^q,B(x_0,r))} + \|[b, I_{\Omega,\alpha}]f_2\|_{L_q(w^q,B(x_0,r))}.$$

Let us estimate $\|[b, I_{\Omega,\alpha}]f_1\|_{L_q(w^q,B(x_0,r))}$ and $\|[b, I_{\Omega,\alpha}]f_2\|_{L_q(w^q,B(x_0,r))}$, respectively.

Since $f_1 \in L_p(w^p,\mathbb{R}^n)$, by the boundedness of $[b, I_{\Omega,\alpha}]$ from $L_p(w^p,\mathbb{R}^n)$ to $L_q(w^q,\mathbb{R}^n)$ (see Theorem 3.6.1 in [4]), (1.6) and since $1 \le s' < p < q$, we get

$$\begin{aligned}
\|[b, I_{\Omega,\alpha}]f_1\|_{L_q(w^q,B(x_0,r))} &\le \|[b, I_{\Omega,\alpha}]f_1\|_{L_q(w^q,\mathbb{R}^n)} \\
&\lesssim \|b\|_{BMO} \|f_1\|_{L_p(w^p,\mathbb{R}^n)} \\
&= \|b\|_{BMO} \|f\|_{L_p(w^p,B(x_0,2r))} \\
&\lesssim \|b\|_{BMO} r^{n-\alpha s'} \|f\|_{L_p(w^p,B(x_0,2r))} \int_{2r}^{\infty} \frac{dt}{t^{n-\alpha s'+1}} \\
&\approx \|b\|_{BMO} \|w^{s'}\|_{L_{\frac{q}{s'}}(B(x_0,r))} \|w^{-s'}\|_{L_{\left(\frac{p}{s'}\right)'}(B(x_0,r))} \\
&\quad \times \int_{2r}^{\infty} \|f\|_{L_p(w^p,B(x_0,t))} \frac{dt}{t^{n-\alpha s'+1}} \\
&\lesssim \|b\|_{BMO} \left(w^q\left(B(x_0,r)\right)\right)^{\frac{1}{q}} \\
&\quad \times \int_{2r}^{\infty} \|f\|_{L_p(w^p,B(x_0,t))} \|w^{-s'}\|_{L_{\left(\frac{p}{s'}\right)'}(B(x_0,t))} \frac{dt}{t^{n-\alpha s'+1}} \\
&\lesssim \|b\|_{BMO} \left(w^q\left(B(x_0,r)\right)\right)^{\frac{1}{q}} \\
&\quad \times \int_{2r}^{\infty} \|f\|_{L_p(w^p,B(x_0,t))} \left[\|w^{s'}\|_{L_{\left(\frac{q}{s'}\right)}(B(x_0,t))}\right]^{-1} \frac{1}{t} dt \\
&\lesssim \|b\|_{BMO} \left(w^q\left(B(x_0,r)\right)\right)^{\frac{1}{q}} \\
&\quad \times \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0,t),w^p)}}{\left(w^q\left(B(x_0,r)\right)\right)^{\frac{1}{q}}} \frac{1}{t} dt. \qquad (2.6)
\end{aligned}$$

For $\|[b,I_{\Omega,\alpha}]f_2\|_{L_q(w^q,B(x_0,r))}$, noting that $|x_0-x|\le r$, $2r\le|x_0-y|$, we have

$$|x_0-y|\le 2\,|x-y|\le 3\,|x_0-y|\,,$$

thus

$$\begin{aligned}
|[b,I_{\Omega,\alpha}]f_2\,(x)| &\lesssim \int_{2r}^{\infty}|b(y)-b\,(y)|\,\frac{|\Omega(x,x-y)|}{|x_0-y|^{n-\alpha}}\,|f(y)|\,dy\\
&\lesssim \int_{2r}^{\infty}\left|b(y)-b_{B(x_0,r)}\right|\frac{|\Omega(x,x-y)|}{|x_0-y|^{n-\alpha}}\,|f(y)|\,dy\\
&+\int_{2r}^{\infty}\left|b(x)-b_{B(x_0,r)}\right|\frac{|\Omega(x,x-y)|}{|x_0-y|^{n-\alpha}}\,|f(y)|\,dy\\
&=: F_1+F_2.
\end{aligned}$$

To estimate F_1, let $1<s,q<\infty$, such that $\frac{1}{s}=\frac{1}{s_1}+\frac{1}{s_2}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. Then, by using Hölder's inequality, (1.10), (1.11) and (2.7) in [2], we obtain

$$\begin{aligned}
F_1 &\lesssim \int_{2r}^{\infty}\left|b(y)-b_{B(x_0,r)}\right||\Omega(x,x-y)|\,|f(y)|\int_{|x_0-y|}^{\infty}\frac{dt}{t^{n-\alpha+1}}dy\\
&\approx \int_{2r}^{\infty}\int_{2r<|x_0-y|<t}\left|b(y)-b_{B(x_0,r)}\right||\Omega(x,x-y)|\,|f(y)|\,dy\frac{dt}{t^{n-\alpha+1}}\\
&\lesssim \int_{2r}^{\infty}\int_{B(x_0,t)}\left|b\,(y)-b_{B(x_0,t)}\right||\Omega(x,x-y)|\,|f\,(y)|\,dy\frac{dt}{t^{n-\alpha+1}}\\
&+\int_{2r}^{\infty}\left|b_{B(x_0,r)}-b_{B(x_0,t)}\right|\int_{B(x_0,t)}|\Omega(x,x-y)|\,|f\,(y)|\,dy\frac{dt}{t^{n-\alpha+1}}
\end{aligned}$$

$$\lesssim \int_{2r}^{\infty} \left\| \left(b(\cdot) - b_{B(x_0,t)}\right) \Omega(x, x-\cdot) \right\|_{L_s(B(x_0,t))} \|f\|_{L_{s'}(B(x_0,t))} \frac{dt}{t^{n-\alpha+1}}$$

$$+ \int_{2r}^{\infty} \left| b_{B(x_0,r)} - b_{B(x_0,t)} \right| \|\Omega(x, x-\cdot)\|_{L_s(B(x_0,t))} \|f\|_{L_{s'}(B(x_0,t))} \frac{dt}{t^{n-\alpha+1}}$$

$$\lesssim \int_{2r}^{\infty} \left\| b(\cdot) - b_{B(x_0,t)} \right\|_{L_{s_1}(B(x_0,t))} \|\Omega(x, x-\cdot)\|_{L_{s_2}(B(x_0,t))} \|f\|_{L_{s'}(B(x_0,t))} \frac{dt}{t^{n-\alpha+1}}$$

$$+ \|b\|_{BMO} \int_{2r}^{\infty} \ln\frac{t}{r} \|\Omega(x, x-\cdot)\|_{L_s(B(x_0,t))} \|f\|_{L_{s'}(B(x_0,t))} \frac{dt}{t^{n-\alpha+1}}$$

$$\lesssim \|b\|_{BMO} \int_{2r}^{\infty} \|f\|_{L_{s'}(B(x_0,t))} \left|B(x_0,t)\right|^{\frac{1}{s_1}} \left|B(x_0,2t)\right|^{\frac{1}{s_2}} \frac{dt}{t^{n-\alpha+1}}$$

$$+ \|b\|_{BMO} \int_{2r}^{\infty} \ln\frac{t}{r} \|f\|_{L_{s'}(B(x_0,t))} \left|B(x_0,2t)\right|^{\frac{1}{s}} \frac{dt}{t^{n-\alpha+1}}$$

$$\lesssim \|b\|_{BMO} \int_{2r}^{\infty} \left(1 + \ln\frac{t}{r}\right) \|f\|_{L_{s'}(B(x_0,t))} \left|B(x_0,2t)\right|^{\frac{1}{s}} \frac{dt}{t^{n-\alpha+1}}$$

$$\lesssim \|b\|_{BMO} \int_{2r}^{\infty} \left(1 + \ln\frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0,t),w^p)}}{\left(w^q\left(B(x_0,r)\right)\right)^{\frac{1}{q}}} \frac{dt}{t},$$

then taking the norm, we have

$$\|F_1\|_{L_q(B(x_0,r),w^q)} \lesssim \|b\|_{BMO} \left(w^q\left(B(x_0,r)\right)\right)^{\frac{1}{q}} \times \int_{2r}^{\infty} \left(1 + \ln\frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0,t),w^p)}}{\left(w^q\left(B(x_0,r)\right)\right)^{\frac{1}{q}}} \frac{dt}{t}.$$

Now turn to estimate F_2. By using Hölder's inequality and from (2.7) in [2], it is easy to see that

$$F_2 \lesssim \int_{2r}^{\infty} \left| b(x) - b_{B(x_0,r)} \right| |\Omega(x, x-y)| |f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n-\alpha+1}} dy$$

$$\lesssim \int_{2r}^{\infty} \left| b(x) - b_{B(x_0,r)} \right| \int_{B(x_0,t)} |\Omega(x, x-y)| |f(y)| \, dy \frac{dt}{t^{n-\alpha+1}}$$

$$\lesssim \left| b(x) - b_{B(x_0,r)} \right| \int_{2r}^{\infty} \|\Omega(x, x-\cdot)\|_{L_s(B(x_0,t))} \|f\|_{L_{s'}(B(x_0,t))} \frac{dt}{t^{n-\alpha+1}}$$

$$\lesssim \left| b(x) - b_{B(x_0,r)} \right| \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t),w^p)}}{\left(w^q(B(x_0,r))\right)^{\frac{1}{q}}} \frac{dt}{t}.$$

Then, applying reverse Hölder's inequality and by (1.10), we get the following

$$\|F_2\|_{L_q(B(x_0,r),w^q)} \lesssim \left(\int_{B(x_0,r)} \left| b(x) - b_{B(x_0,r)} \right|^q w^q(x) \, dx \right)^{\frac{1}{q}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t),w^p)}}{\left(w^q(B(x_0,r))\right)^{\frac{1}{q}}} \frac{dt}{t}$$

$$\lesssim \left(\int\limits_{B(x_0,r)} w^{qs}(x)\,dx \right)^{\frac{1}{qs}} \left(\int\limits_{B(x_0,r)} \left| b(x) - b_{B(x_0,r)} \right|^{\frac{qs}{s-1}} dx \right)^{\frac{s-1}{qs}}$$
$$\times \int\limits_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t),w^p)}}{(w^q(B(x_0,r)))^{\frac{1}{q}}} \frac{dt}{t}$$
$$\approx \left[\left(\int\limits_{B(x_0,r)} w^{qs}(x)\,dx \right)^{\frac{1}{s}} \right]^{\frac{1}{q}} |B(x_0,r)|^{\frac{s-1}{qs}} \left(\frac{\int\limits_{B(x_0,r)} \left| b(x) - b_{B(x_0,r)} \right|^{\frac{qs}{s-1}} dx}{|B(x_0,r)|} \right)^{\frac{s-1}{qs}}$$
$$\times \int\limits_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t),w^p)}}{(w^q(B(x_0,r)))^{\frac{1}{q}}} \frac{dt}{t}$$
$$\approx \|b\|_{BMO} \left(w^q\left(B\left(x_0,r\right)\right)\right)^{\frac{1}{q}} \int\limits_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t),w^p)}}{(w^q(B(x_0,r)))^{\frac{1}{q}}} \frac{dt}{t}.$$

Thus, combining all the estimates for $\|F_1\|_{L_q(B(x_0,r),w^q)}$ and $\|F_2\|_{L_q(B(x_0,r),w^q)}$, we get

$$\|[b,I_{\Omega,\alpha}]f_2\|_{L_q(w^q,B(x_0,r))} \lesssim \|b\|_{BMO} \left(w^q\left(B(x_0,r)\right)\right)^{\frac{1}{q}}$$
$$\times \int\limits_{2r}^{\infty} \left(1 + \ln\frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0,t),w^p)}}{\left(w^q\left(B\left(x_0,r\right)\right)\right)^{\frac{1}{q}}} \frac{1}{t} dt. \tag{2.7}$$

At last, from (2.6) and (2.7), the proof of (2.1) is completed.

Moreover, by the definition of vanishing generalized weighted Morrey spaces, (2.1) and (2.3), we have

$$\|[b,I_{\Omega,\alpha}]f\|_{VM_{q,\varphi_2}(w^q)} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(x,r)} \|[b,I_{\Omega,\alpha}]f\|_{L_q(w^q,B(x_0,r))}$$
$$\lesssim \|b\|_{BMO} \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(x,r)} \left(w^q\left(B\left(x_0,r\right)\right)\right)^{\frac{1}{q}}$$
$$\times \int\limits_{r}^{\infty} \left(1 + \ln\frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0,t),w^p)}}{\left(w^q\left(B\left(x_0,r\right)\right)\right)^{\frac{1}{q}}} \frac{dt}{t}$$
$$\lesssim \|b\|_{BMO} \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(x,r)} \left(w^q\left(B\left(x_0,r\right)\right)\right)^{\frac{1}{q}}$$
$$\times \int\limits_{r}^{\infty} \left[\varphi_1(x,t)^{-1} \|f\|_{L_p(B(x_0,t),w^p)}\right] \left(1 + \ln\frac{t}{r}\right) \frac{\varphi_1(x,t)}{\left(w^q\left(B\left(x,t\right)\right)\right)^{\frac{1}{q}}} \frac{1}{t} dt$$
$$\lesssim \|b\|_{BMO} \|f\|_{VM_{p,\varphi_1}(w^p)} \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(x,r)} \left(w^q\left(B\left(x_0,r\right)\right)\right)^{\frac{1}{q}}$$
$$\times \int\limits_{r}^{\infty} \left(1 + \ln\frac{t}{r}\right) \frac{\varphi_1(x,t)}{\left(w^q\left(B\left(x,t\right)\right)\right)^{\frac{1}{q}}} \frac{1}{t} dt$$
$$\lesssim \|b\|_{BMO} \|f\|_{VM_{p,\varphi_1}(w^p)}.$$

At last, it is sufficient to prove that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \frac{1}{\varphi_1(x,r)} \|f\|_{L_p(w^p,B(x_0,r))} = 0$$

implies

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\frac{1}{\varphi_2(x,r)}\|[b,I_{\Omega,\alpha}]f\|_{L_q(w^q,B(x_0,r))}=0.$$

To show that

$$\sup_{x\in\mathbb{R}^n}\frac{1}{\varphi_2(x,r)}\|[b,I_{\Omega,\alpha}]f\|_{L_q(w^q,B(x_0,r))}<\epsilon$$

for any small $r>0$, we split the right hand side of (2.1) as follow

$$\frac{1}{\varphi_2(x,r)}\|[b,I_{\Omega,\alpha}]f\|_{L_q(w^q,B(x_0,r))}\le C_0\left[\mathcal{F}_\psi(x,r)+\mathcal{G}_\psi(x,r)\right], \tag{2.8}$$

where $0<r<\psi$, and

$$\mathcal{F}_\psi(x,r):=\|b\|_{BMO}\frac{\left(w^q\left(B\left(x_0,r\right)\right)\right)^{\frac{1}{q}}}{\varphi_2(x,r)}$$
$$\times\int_r^\psi\left(1+\ln\frac{t}{r}\right)\frac{\varphi_1(x,t)}{\left(w^q\left(B\left(x_0,r\right)\right)\right)^{\frac{1}{q}}}\sup_{0<r<t}\left[\frac{\|f\|_{L_p(B(x_0,t),w^p)}}{\varphi_1(x,t)}\right]\frac{1}{t}dt$$

and

$$\mathcal{G}_\psi(x,r):=\|b\|_{BMO}\frac{\left(w^q\left(B\left(x_0,r\right)\right)\right)^{\frac{1}{q}}}{\varphi_2(x,r)}$$
$$\times\int_\psi^\infty\left(1+\ln\frac{t}{r}\right)\frac{\varphi_1(x,t)}{\left(w^q\left(B\left(x_0,r\right)\right)\right)^{\frac{1}{q}}}\sup_{0<r<t}\left[\frac{\|f\|_{L_p(B(x_0,t),w^p)}}{\varphi_1(x,t)}\right]\frac{1}{t}dt.$$

Since $f\in VM_{p,\varphi_1}(w^p,\mathbb{R}^n)$, for all $0<r<\psi$, we can choose any fixed $\psi>0$ such that

$$\sup_{x\in\mathbb{R}^n}\sup_{0<r<\psi}\frac{\|f\|_{L_p(w^p,B(x,r))}}{\varphi_1(x,r)}<\frac{\epsilon}{2CC_0\|b\|_{BMO}},$$

where the constants C and C_0 come from (2.3) and (2.8), respectively. Then, for $0<r<\psi$, by (2.3), we have

$$\sup_{x\in\mathbb{R}^n}C_0\mathcal{F}_\psi(x,r)<\frac{\epsilon}{2C}\frac{\left(w^q\left(B\left(x_0,r\right)\right)\right)^{\frac{1}{q}}}{\varphi_2(x,r)}$$
$$\times\int_r^\psi\left(1+\ln\frac{t}{r}\right)\frac{\varphi_1(x,t)}{\left(w^q\left(B\left(x_0,r\right)\right)\right)^{\frac{1}{q}}}\frac{1}{t}dt<\frac{\epsilon}{2}.$$

The estimation of $\mathcal{G}_\psi(x,r)$ may be obtained by choosing r sufficiently small. Indeed, it follows from (2.2) that

$$\sup_{x\in\mathbb{R}^n}C_0\mathcal{G}_\psi(x,r)\le C_{\delta_0}\|b\|_{BMO}\frac{\left(w^q\left(B\left(x_0,r\right)\right)\right)^{\frac{1}{q}}}{\varphi_2(x,r)}\|f\|_{VM_{p,\varphi_1}(w^p,\mathbb{R}^n)},$$

where C_{δ_0} is the constant from (2.2).

Then, since $\varphi_2\in\mathcal{B}(w^q)$, it suffices to choose r small enough such that

$$\sup_{x\in\mathbb{R}^n}\frac{\left(w^q\left(B\left(x_0,r\right)\right)\right)^{\frac{1}{q}}}{\varphi_2(x,r)}<\frac{\epsilon}{2C_0C_\psi\|b\|_{BMO}\|f\|_{VM_{p,\varphi_1}(w^p,\mathbb{R}^n)}}.$$

Hence,

$$\sup_{x\in\mathbb{R}^n}C\mathcal{G}_\psi(x,r)<\frac{\epsilon}{2}.$$

Thus,

$$\frac{\|[b, I_{\Omega,\alpha}]f\|_{L_q(w^q,B(x_0,r))}}{\varphi_2(x,r)} < \epsilon,$$

which means that

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\frac{1}{\varphi_2(x,r)}\|[b, I_{\Omega,\alpha}]f\|_{L_q(w^q,B(x_0,r))} = 0,$$

which completes the proof of (2.4). On the other hand, since $[b, M_{\Omega,\alpha}]f(x) \le [b, I_{|\Omega|,\alpha}]\left(|f|\right)(x)$, $x \in \mathbb{R}^n$ (see Remark 3.6.2 in [4]) we can also use the same method for $[b, M_{\Omega,\alpha}]$, so we omit the details. As a result, we complete the proof of Theorem 2.1. □

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COMPLEX INTERPOLATION AND COMMUTATORS ACTING ON MORREY SPACES

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Abstract. The goal of this note is to improve the boundedness result of commutators generated by the fractional integral operator I_α of order α, $0 < \alpha < n$ and BMO functions by the use of the complex interpolation. In particular, we prove the boundedness of commutators generated by BMO functions and fractional integral operators from the Calderón–Lozanovskiĭ product between Morrey spaces to Morrey spaces. Moreover, we also discuss the compactness of these commutators. The results concern the boundedness property of commutators acting on the complex interpolation spaces of Morrey spaces. However, the actual proof uses the Calderón–Lozanovskiĭ product and the complex interpolation is hidden behind the Calderón–Lozanovskiĭ product.

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1. Introduction

The goal of this note is to refine a result on the Morrey-boundedness of commutators generated by BMO functions and the fractional integral operator I_α of order $\alpha \in (0, n)$ in terms of the complex interpolation. To this end, we will start with the definition of the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ for $1 \le q \le p < \infty$. We write $Q(x, r) \equiv \left\{ y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n : \max\limits_{j=1,2,\ldots,n} |x_j - y_j| \le r \right\}$ when $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $r > 0$. Denote by $\mathcal{Q}$ the set of all cubes of the form $Q(x, r)$ for some $x \in \mathbb{R}^n$ and $r > 0$. Let $1 \le q \le p < \infty$. Then the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all measurable functions f on $\mathbb{R}^n$ for which

$$\|f\|_{\mathcal{M}^p_q} \equiv \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} |Q(x, r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q(x,r)} |f(y)|^q \mathrm{d}y \right)^{\frac{1}{q}} < \infty.$$

As we mentioned we handle commutators generated by BMO functions and the fractional integral operator I_α, $\alpha \in (0, n)$. To this end, we next recall the definition of the related function spaces and operators. We start with I_α, $\alpha \in (0, n)$. Let I_α be the fractional integral operator of order α given by

$$I_\alpha f(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} \mathrm{d}y \quad (x \in \mathbb{R}^n), \tag{1.1}$$

which is defined for a suitable measurable function f. Next we recall the definition of $\mathrm{BMO}(\mathbb{R}^n)$. If E has positive measure and f is integrable over E, Then denote by $m_E(f)$ the average of f over E. $|E|$ denotes the volume of E. Define $\|f\|_* \equiv \sup\limits_{Q\in\mathcal{Q}} m_Q(|f - m_Q(f)|)$ for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. One says that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$

has bounded mean oscillation (abbreviated to $f \in \mathrm{BMO}(\mathbb{R}^n)$), if $\|f\|_* < \infty$. In this paper, we handle the commutator $[a, I_\alpha]$ defined by $[a, I_\alpha]f(x) \equiv \int_{\mathbb{R}^n} \frac{a(x)-a(y)}{|x-y|^{n-\alpha}} f(y)dy$ for $\alpha \in (0,n)$ and $a \in \mathrm{BMO}(\mathbb{R}^n)$. Here, f is a suitable function that will be chosen so that the right-hand side makes sense.

We are interested in the improvement of the following theorem due to Di Fazio and Ragusa [9] on the boundedness of the commutator $[a, I_\alpha]$, initially considered by Chanillo [6].

Proposition 1.1. *Let $a \in \mathrm{BMO}(\mathbb{R}^n)$ and $0 < \alpha < n$. Assume that the parameters $p, q, s, t \in (1, \infty)$ satisfy $q \le p$, $t \le s$, $\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}$ and $\frac{q}{p} = \frac{t}{s}$. Then $[a, I_\alpha]$ is a bounded linear operator from $\mathcal{M}^p_q(\mathbb{R}^n)$ to $\mathcal{M}^s_t(\mathbb{R}^n)$, that is, for any $f \in \mathcal{M}^p_q(\mathbb{R}^n)$, the integral defining $[a, I_\alpha]f(x)$ converges absolutely for almost all $x \in \mathbb{R}^n$ and the mapping $f \in \mathcal{M}^p_q(\mathbb{R}^n) \mapsto [a, I_\alpha]f \in \mathcal{M}^s_t(\mathbb{R}^n)$ is a bounded linear operator.*

The proof of Proposition 1.1 heavily hinges on the boundedness of I_α, initially proved by Adams [1]. See the inclusive textbook [42] for more about the action of fractional integral operators on Morrey spaces.

Proposition 1.2. *The conclusion of Proposition 1.1 remains valid if we replace $[a, I_\alpha]$ by I_α.*

One of the techniques to prove Proposition 1.2 is to use Hedberg's inequality [21]. A standard argument shows that Hedberg's inequality can be refined by the use of the Morrey norm $\|\cdot\|_{\mathcal{M}^p_1}$, see [38]. As is established in [32, 36], we can measure how strongly we can use the Morrey norm $\|\cdot\|_{\mathcal{M}^p_1}$ by the use of the complex interpolation or the Calderón–Lozanovskiĭ product, which we recall now.

Let $0 < \theta < 1$, and let $\mathcal{X}(\mathbb{R}^n)$ and $\mathcal{Y}(\mathbb{R}^n)$ be Banach lattices. Then the Calderón-Lozanovskiĭ product $(\mathcal{X}(\mathbb{R}^n))^{1-\theta}(\mathcal{Y}(\mathbb{R}^n))^\theta$, which is due to Calderón [5, §13.5] and Lozanovskiĭ [28, 29], is the set of all measurable functions f for which $|f| \le |f_0|^{1-\theta}|f_1|^\theta$ for some $f_0 \in \mathcal{X}(\mathbb{R}^n)$ and $f_1 \in \mathcal{Y}(\mathbb{R}^n)$. The norm of $f \in (\mathcal{X}(\mathbb{R}^n))^{1-\theta}(\mathcal{Y}(\mathbb{R}^n))^\theta$ is given by

$$\|f\|_{(\mathcal{X})^{1-\theta}(\mathcal{Y})^\theta} = \inf\{(\|f_0\|_\mathcal{X})^{1-\theta}(\|f_1\|_\mathcal{Y})^\theta\},$$

where f_0 and f_1 move over all functions in $\mathcal{X}(\mathbb{R}^n)$ and $\mathcal{Y}(\mathbb{R}^n)$ satisfying $|f| \le |f_0|^{1-\theta}|f_1|^\theta$. See [34] for more about the Calderón–Lozanovskiĭ product.

We recall a result of [36].

Proposition 1.3. *Assume that the parameters $p, q, s, t \in (1, \infty)$ satisfy $q \le p$, $t \le s$, $\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}$ and $\frac{q}{p} = \frac{t}{s}$. Write $\theta \equiv \frac{\alpha}{n} \in (0,1)$. Then the fractional integral operator I_α maps $(\mathcal{M}^p_q(\mathbb{R}^n))^{1-\theta}(\mathcal{M}^p_1(\mathbb{R}^n))^\theta$ boundedly to $\mathcal{M}^s_t(\mathbb{R}^n)$.*

The goal of this note is to obtain an analog of Proposition 1.3 for the commutator $[a, I_\alpha]$. Due to the singularity of BMO functions, we need to replace $\mathcal{M}^p_1(\mathbb{R}^n)$ by a slightly smaller space $\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)$. Motivated by [37], we write

$$\|f\|_{\mathrm{L}\log\mathrm{L};Q(x,r)} = \inf\left\{\lambda > 0 \,:\, \frac{1}{|Q(x,r)|}\int_{Q(x,r)} \frac{|f(y)|}{\lambda}\log\left(3 + \frac{|f(y)|}{\lambda}\right) \mathrm{d}y \le 1\right\}$$

for a measurable function f. The quantity $\|f\|_{\mathrm{L}\log\mathrm{L};Q(x,r)}$ is called the Orlicz average of f. The Orlicz–Morrey space $\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)$, $p > 1$, is the set of all measurable functions f for which $\|f\|_{\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}} \equiv \sup_{x\in\mathbb{R}^n, r>0} |Q(x,r)|^{\frac{1}{p}}\|f\|_{\mathrm{L}\log\mathrm{L};Q(x,r)}$ is finite.

We seek to prove the following theorem:

Theorem 1.4. *Let $a \in \mathrm{BMO}(\mathbb{R}^n)$ and $0 < \alpha < n$. Assume that the parameters $p, q, s, t \in (1, \infty)$ satisfy $q \le p$, $t \le s$, $\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}$ and $\frac{q}{p} = \frac{t}{s}$. Write $\theta \equiv \frac{\alpha p}{n} \in (0,1)$. Then $[a, I_\alpha]$ maps $(\mathcal{M}^p_q(\mathbb{R}^n))^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))^\theta$ boundedly to $\mathcal{M}^s_t(\mathbb{R}^n)$. Furthermore, the estimate*

$$\|[a, I_\alpha]\|_{(\mathcal{M}^p_q)^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}})^\theta \to \mathcal{M}^s_t} \lesssim \|a\|_* \tag{1.2}$$

holds.

It seems that our results can be extended to the generalized settings [10, 12, 13, 14, 15, 27]. However, to simplify, we content ourselves with the Euclidean space.

Remark 1.5. *Since $\mathcal{X}(\mathbb{R}^n)\cap\mathcal{Y}(\mathbb{R}^n)\subseteq\mathcal{X}(\mathbb{R}^n)^{1-\theta}\mathcal{Y}(\mathbb{R}^n)^\theta$ for any Banach lattices $\mathcal{X}(\mathbb{R}^n)$ and $\mathcal{Y}(\mathbb{R}^n)$ and $\mathcal{M}^p_q(\mathbb{R}^n)\subseteq\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)$ for any $p>1$, we see that $\mathcal{M}^p_q(\mathbb{R}^n)\subseteq(\mathcal{M}^p_q(\mathbb{R}^n))^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))^\theta$. Thus, Theorem 1.4 can be viewed as an improvement of Proposition 1.1.*

We do not discuss seriously whether $[a,I_\alpha]f(x)$ can be defined almost everywhere. In fact, we can also consider operators having singularity slightly stronger than commutators defined above. We define the linear operator $\tilde{C}[a,I_\alpha]$ by

$$\tilde{C}[a,I_\alpha]f(x)\equiv\int_{\mathbb{R}^n}\frac{|a(x)-a(y)|}{|x-y|^{n-\alpha}}f(y)dy$$

for a measurable function f as long as the integral makes sense for almost all $x\in\mathbb{R}^n$. This definition goes back to the paper [4]. Usually, due to the positivity of the integral kernel, we may assume that f is non-negative almost everywhere. However, under some extra integrability condition, we will mainly consider the case where f is not always non-negative.

Then we can prove the boundedness of $\tilde{C}[a,I_\alpha]$.

Theorem 1.6. *The same conclusion remains valid in Theorem 1.4 if we replace $[a,I_\alpha]$ by $\tilde{C}[a,I_\alpha]$.*

Theorem 1.4 will follow immediately once we prove Theorem 1.6: We concentrate on Theorem 1.6.

We can also discuss the compactness of commutators in Theorem 1.4.

Theorem 1.7. *In addition to the assumption of Theorem 1.4, if $a\in\mathrm{VMO}(\mathbb{R}^n)$, then $[a,I_\alpha]$ is a compact operator from $(\mathcal{M}^p_q(\mathbb{R}^n))^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))^\theta$ to $\mathcal{M}^s_t(\mathbb{R}^n)$.*

Remark 1.8. *The space $\mathrm{VMO}(\mathbb{R}^n)$ is defined to be the set of all functions $a\in\mathrm{BMO}(\mathbb{R}^n)$ for which $\lim\limits_{r\to0^+}\sup\limits_{x\in\mathbb{R}^n}m_{Q(x,r)}(|a-m_{Q(x,r)}(a)|)=0$.*

The converse of Theorem 1.7 is also available.

Corollary 1.9. *In addition to the assumption of Theorem 1.4 if $[a,I_\alpha]$ is a compact operator from $(\mathcal{M}^p_q(\mathbb{R}^n))^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))^\theta$ to $\mathcal{M}^s_t(\mathbb{R}^n)$ then $a\in\mathrm{VMO}(\mathbb{R}^n)$.*

In fact, this is a direct corollary of a result obtained in [8] asserting that $a\in\mathrm{VMO}(\mathbb{R}^n)$ if $[a,I_\alpha]$ is a compact operator from $\mathcal{M}^p_q(\mathbb{R}^n)$ to $\mathcal{M}^s_t(\mathbb{R}^n)$.

We can paraphrase our theorems in terms of the complex interpolation functors. We focus on the complex interpolation of $\mathcal{M}^p_q(\mathbb{R}^n)$ and $\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)$. Remark that there are two different complex interpolation functors, both of which we recall. We write $S\equiv\{z\in\mathbb{C}:0<\mathrm{Re}(z)<1\}$ and let $\bar{S}$ be its closure. For $j=0,1$, we set $j+i\mathbb{R}\equiv\{z\in\mathbb{C}:\mathrm{Re}(z)=j\}$. Also, for a Banach space X, the space $\mathrm{Lip}(\mathbb{R};X)$ stands for the Banach space (modulo constants) of all continuous functions $f:\mathbb{R}\to X$ for which $\|f\|_{\mathrm{Lip}(\mathbb{R};X)}=\sup\limits_{s,t\in\mathbb{R},s\neq t}\dfrac{\|f(s)-f(t)\|_X}{|s-t|}$ is finite.

Definition 1.10. *Let $1<q\le p<\infty$.*

(1) *The space $\mathcal{F}(\mathcal{M}^p_q(\mathbb{R}^n),\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))$ is defined as the set of all functions $F:\bar{S}\to\mathcal{M}^p_q(\mathbb{R}^n)+\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)=\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)$ such that*
   (a) *F is continuous on $\bar{S}$ and $\sup\limits_{z\in\bar{S}}\|F(z)\|_{\mathcal{M}^p_q+\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}}<\infty$,*
   (b) *F is holomorphic on S,*
   (c) *the functions $t\in\mathbb{R}\mapsto F(it)\in\mathcal{M}^p_q(\mathbb{R}^n)$ and $t\in\mathbb{R}\mapsto F(1+it)\in\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)$ are bounded and continuous on $\mathbb{R}$.*

The space $\mathcal{F}(\mathcal{M}^p_q(\mathbb{R}^n),\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))$ is equipped with the norm

$$\|F\|_{\mathcal{F}(\mathcal{M}^p_q,\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}})}\equiv\max\left(\sup_{t\in\mathbb{R}}\|F(it)\|_{\mathcal{M}^p_q},\sup_{t\in\mathbb{R}}\|F(1+it)\|_{\mathcal{M}^p_{L\log L}}\right).$$

(2) Let $\theta \in (0,1)$. The first/lower complex interpolation space $[\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]_\theta$ with respect to the couple $(\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))$ is defined to be the Banach lattice of all functions $f \in \mathcal{M}^p_q(\mathbb{R}^n) + \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n) = \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)$ such that f is realized as $f = F(\theta)$ for some element $F \in \mathcal{F}(\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))$. The norm on $[\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]_\theta$ is defined by

$$\|f\|_{[\mathcal{M}^p_q, \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}]_\theta} \equiv \inf\{\|F\|_{\mathcal{F}(\mathcal{M}^p_q, \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}})} : f = F(\theta) \text{ for some } F \in \mathcal{F}(\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))\}.$$

Definition 1.11. *Let $1 < q \le p < \infty$. Also let $\theta \in (0,1)$.*

(1) The space $\mathcal{G}(\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))$ stands for the set of all functions $G : \bar{S} \to \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)$ such that

(a) G is continuous on $\bar{S}$ and $\sup\limits_{z\in\bar{S}} \left\| \frac{G(z)}{1+|z|} \right\|_{\mathcal{M}^p_q + \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}} < \infty$,

(b) G is holomorphic on S,

(c) the functions $t \in \mathbb{R} \mapsto G(it) - G(0) \in \mathcal{M}^p_q(\mathbb{R}^n)$ and $t \in \mathbb{R} \mapsto G(1+it) - G(1) \in \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)$ are Lipschitz continuous on $\mathbb{R}$ for each $j = 0, 1$.

The space $\mathcal{G}(\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))$ is equipped with the norm

$$\|G\|_{\mathcal{G}(\mathcal{M}^p_q, \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}})} \equiv \max\left\{ \|G(i\cdot)\|_{\mathrm{Lip}(\mathbb{R}, \mathcal{M}^p_q)},\ \|G(1+i\cdot)\|_{\mathrm{Lip}(\mathbb{R}, \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}})} \right\}. \tag{1.3}$$

(2) The second/upper complex interpolation space $[\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]^\theta$ with respect to the couple $(\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))$ is defined to be the linear space of all functions $f \in \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)$ such that $f = G'(\theta)$ for some $G \in \mathcal{G}(\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))$. For $f \in [\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]^\theta$, its norm on $[\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]^\theta$ is defined by

$$\|f\|_{[\mathcal{M}^p_q, \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}]^\theta} \equiv \inf\{\|G\|_{\mathcal{G}(\mathcal{M}^p_q, \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}})} : f = G'(\theta) \text{ for some } G \in \mathcal{G}(\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))\}.$$

See [3] for these definitions. According to the general theory in [3],

$$[\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]^\theta \supset [\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]_\theta$$

for $1 < q \le p < \infty$. More precisely, it is important that $[\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]_\theta$ is the closure of $\mathcal{M}^p_q(\mathbb{R}^n) \cap \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)$ in $[\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]^\theta$ with coincidence of norms [2]. In general this inclusion is strict; see [22, 23] as well as [16, 19, 20, 30, 43]. It is remarkable that based on [22, 23], much more is investigated on smoothness Morrey spaces in [17, 18, 41, 43]

In [31] we obtained the following expression:

$$[\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]^\theta = (\mathcal{M}^p_q(\mathbb{R}^n))^{1-\theta} (\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))^\theta. \tag{1.4}$$

Thus, we can rephrase our theorems in terms of the complex interpolation functors.

Theorem 1.12. *Let $a \in \mathrm{BMO}(\mathbb{R}^n)$ and $0 < \alpha < n$. Assume that the parameters $p, q, s, t \in (1, \infty)$ satisfy $q \le p$, $t \le s$, $\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}$ and $\frac{q}{p} = \frac{t}{s}$. Write $\theta \equiv \frac{\alpha p}{n} \in (0,1)$. Under the assumption of Theorem 1.4 $[a, I_\alpha]$ maps $[\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]^\theta$ boundedly to $\mathcal{M}^s_t(\mathbb{R}^n)$. Furthermore, the estimate*

$$\|[a, I_\alpha]\|_{[\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]^\theta \to \mathcal{M}^s_t} \lesssim \|a\|_* \tag{1.5}$$

holds.

Theorem 1.13. *In addition to the assumption of Theorem 1.4, if $a \in \mathrm{VMO}(\mathbb{R}^n)$, then $[a, I_\alpha]$ is a compact operator from $[\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]^\theta$ to $\mathcal{M}^s_t(\mathbb{R}^n)$.*

The converse is also available.

Corollary 1.14. *In addition to the assumption of Theorem 1.4 if* $[a, I_\alpha]$ *is a compact operator from* $[\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]_\theta$ *to* $\mathcal{M}^s_t(\mathbb{R}^n)$ *then* $a \in \mathrm{VMO}(\mathbb{R}^n)$. *In particular, under the assumption of Theorem 1.4 if* $[a, I_\alpha]$ *is a compact operator from* $[\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]^\theta$ *to* $\mathcal{M}^s_t(\mathbb{R}^n)$ *then* $a \in \mathrm{VMO}(\mathbb{R}^n)$.

We remark that the compactness of $[a, I_\alpha]$ from $[\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]^\theta$ to $\mathcal{M}^s_t(\mathbb{R}^n)$ guarantees the one from $[\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]_\theta$ to $\mathcal{M}^s_t(\mathbb{R}^n)$ thanks to the embedding from $[\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]_\theta$ into $[\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]^\theta$.

In the light of the result in the famous paper [23], there is a gap between Corollaries 1.9 and 1.14. In fact, $[\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]^\theta$ and $[\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)]_\theta$ are different. Since Chen, Ding and Wang used compactly supported functions for the proof of [8, Theorem 1.2], we can close this gap.

The remaining part of this paper is organized as follows: Section 2 collects preliminary facts, while Section 3 and Section 4 prove Theorems 1.6 and 1.7, respectively.

2. Preliminaries

2.1. A vector-valued maximal inequality.

We invoke the following extension of the Fefferman–Stein vector-valued inequality for the Hardy–Littlewood maximal operator M.

Lemma 2.1. [38, 39] *Let* $1 < t \le s < \infty$ *and* $1 < r < \infty$. *Then for any sequence* $\{f_j\}_{j=1}^\infty \subset \mathcal{M}^s_t(\mathbb{R}^n)$,

$$\left\| \left(\sum_{j=1}^\infty (M f_j)^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}^s_t} \lesssim \left\| \left(\sum_{j=1}^\infty |f_j|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}^s_t}$$

2.2. Fractional Orlicz maximal operators.

As an auxiliary step, we will investigate the boundedness property of the fractional Orlicz maximal operator given by

$$M_{\alpha,\mathrm{L}\log\mathrm{L}} f(x) \equiv \sup_{Q \in \mathcal{Q}} \chi_Q(x) \ell(Q)^\alpha \|f\|_{\mathrm{L}\log\mathrm{L};Q}$$

for $f \in L^0(\mathbb{R}^n)$, where $L^0(\mathbb{R}^n)$ stands for the linear space of all measurable functions on $\mathbb{R}^n$. If $\alpha = 0$, then abbreviate $M_{\alpha,\mathrm{L}\log\mathrm{L}}$ to $M_{\mathrm{L}\log\mathrm{L}}$. We also remark that this operator is slightly bigger than the fractional maximal operator given by

$$M_\alpha f(x) \equiv \sup_{Q \in \mathcal{Q}} \chi_Q(x) \ell(Q)^{\alpha - n} \|f\|_{L^1(Q)}.$$

In fact, we have $M_\alpha f \le M_{\alpha,\mathrm{L}\log\mathrm{L}} f$ for any $f \in L^0(\mathbb{R}^n)$.

Lemma 2.2. *Let* $0 < \alpha < n$. *Assume that the parameters* $p, q, s, t \in (1, \infty)$ *satisfy* $q \le p$, $t \le s$, $\dfrac{1}{s} = \dfrac{1}{p} - \dfrac{\alpha}{n}$ *and* $\dfrac{q}{p} = \dfrac{t}{s}$. *Write* $\theta \equiv \dfrac{\alpha p}{n} \in (0,1)$. *Then* $M_{\alpha,\mathrm{L}\log\mathrm{L}}$ *maps* $(\mathcal{M}^p_q(\mathbb{R}^n))^{1-\theta} (\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))^\theta$ *boundedly to* $\mathcal{M}^s_t(\mathbb{R}^n)$.

Proof. The proof resembles that of the main theorem in [36]. Here we supply the proof for the completeness. Let $f \in (\mathcal{M}^p_q(\mathbb{R}^n))^{1-\theta} (\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))^\theta$. We may assume $f \ne 0$; otherwise the conclusion $M_{\alpha,\mathrm{L}\log\mathrm{L}} f = 0 \in (\mathcal{M}^p_q(\mathbb{R}^n))^{1-\theta} (\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))^\theta$. Recall that

$$\|f\|_{(\mathcal{M}^p_q)^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}})^\theta} = \inf\{\|f_0\|_{\mathcal{M}^p_q}^{1-\theta} \|f_1\|_{\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}}^\theta \,:\, f_0 \in \mathcal{M}^p_q(\mathbb{R}^n), f_1 \in \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)\}.$$

It follows directly from the definition at least that

$$\{\|f_0\|_{\mathcal{M}^p_q}^{1-\theta} \|f_1\|_{\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}}^\theta \,:\, f_0 \in \mathcal{M}^p_q(\mathbb{R}^n), f_1 \in \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)\} = (\|f\|_{(\mathcal{M}^p_q)^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}})^\theta}, \infty)$$

or that

$$\{\|f_0\|_{\mathcal{M}^p_q}^{1-\theta} \|f_1\|_{\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}}^\theta \,:\, f_0 \in \mathcal{M}^p_q(\mathbb{R}^n), f_1 \in \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)\} = [\|f\|_{(\mathcal{M}^p_q)^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}})^\theta}, \infty),$$

which yields

$$2\|f\|_{(\mathcal{M}^p_q)^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}})^\theta} \in \mathrm{Int}(\{\|f_0\|^{1-\theta}_{\mathcal{M}^p_q}\|f_1\|^\theta_{\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}} \,:\, f_0 \in \mathcal{M}^p_q(\mathbb{R}^n), f_1 \in \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)\})$$
$$= (\|f\|_{(\mathcal{M}^p_q)^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}})^\theta}, \infty).$$

Consequently there exist $f_0 \in \mathcal{M}^p_q(\mathbb{R}^n)$ and $f_1 \in \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)$ such that $|f| \le |f_0|^{1-\theta}|f_1|^\theta$ and that

$$\|f_0\|^{1-\theta}_{\mathcal{M}^p_q}\|f_1\|^\theta_{\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}} \le 2\|f\|_{(\mathcal{M}^p_q)^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}})^\theta}. \tag{2.1}$$

Let $x \in \mathbb{R}^n$ and suppose that $Q \in \mathcal{Q}$ contains x. Thanks to the generalized H́older inequality for Orlicz spaces, or thanks to the inequality $a^{1-\theta}b^\theta \le a+b$ for $a, b \ge 0$,

$$\|f\|_{\mathrm{L}\log\mathrm{L};Q} \lesssim (\|f_0\|_{\mathrm{L}\log\mathrm{L};Q})^{1-\theta}(\|f_1\|_{\mathrm{L}\log\mathrm{L};Q})^\theta.$$

Therefore,

$$\ell(Q)^\alpha\|f\|_{\mathrm{L}\log\mathrm{L};Q} \lesssim M_{\mathrm{L}\log\mathrm{L}}f_0(x)^{1-\theta}|Q|^{\frac{\alpha}{n}-\frac{\theta}{p}}\|f_1\|^\theta_{\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}} = (M_{\mathrm{L}\log\mathrm{L}}f_0(x))^{\frac{p}{s}}\|f_1\|^\theta_{\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}}.$$

Consequently,

$$M_{\alpha,\mathrm{L}\log\mathrm{L}}f(x) \lesssim (M_{\mathrm{L}\log\mathrm{L}}f_0(x))^{\frac{p}{s}}\|f_1\|^\theta_{\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}}.$$

Taking the Morrey norm $\|\cdot\|_{\mathcal{M}^s_t}$, we get

$$\|M_{\alpha,\mathrm{L}\log\mathrm{L}}f\|_{\mathcal{M}^s_t} \lesssim \|M_{\mathrm{L}\log\mathrm{L}}f_0\|^{\frac{p}{s}}_{\mathcal{M}^p_q}\|f_1\|^\theta_{\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}}. \tag{2.2}$$

Since $M_{\mathrm{L}\log\mathrm{L}}f(x) \lesssim_\varepsilon M[|f|^{1+\varepsilon}](x)^{\frac{1}{1+\varepsilon}}$ for any $\varepsilon > 0$, we have

$$\|M_{\mathrm{L}\log\mathrm{L}}f_0\|_{\mathcal{M}^p_q} \lesssim \|f_0\|_{\mathcal{M}^p_q}.$$

Combining this inequality, (2.1), (2.2) and the identity $\dfrac{p}{s} = 1-\theta$, we have the desired result. □

2.3. **Mean oscillation.** For the proof of the theorem, we will use the mean oscillation [26].

Recall that the decreasing rearrangement of $f \in L^0(\mathbb{R}^n)$ is given as follows:

$$f^*(t) \equiv \inf\{\lambda > 0 \,:\, |\{|f| > \lambda\}| < t\} \quad (t > 0).$$

Let $Q \in \mathcal{Q}$, and $f \in L^0(Q)$ be a real-valued function.

We use the following notation: for a right-open cube Q^0, which is not always a dyadic cube, $\mathcal{D}(Q^0)$ is the set of all dyadic cubes with respect to a cube Q^0. The mean oscillation of $f \in L^0(Q)$ of level $\lambda \in (0,1)$ is given by $\omega_\lambda(f;Q) \equiv \inf\limits_{c\in\mathbb{C}}((f-c)\chi_Q)^*(\lambda|Q|)$, where * denotes the decreasing rearrangement for functions. We will use

$$\omega_\lambda(af+b;Q) = |a|\omega_\lambda(f;Q) \tag{2.3}$$

for $a, b \in \mathbb{C}$ and $f \in L^0(\mathbb{R}^n)$.

Before we go further, a useful remark is in order. Let g be a measurable function defined on a cube Q. Then since g^* is decreasing,

$$g^*(\lambda|Q|) \le \frac{1}{\lambda|Q|}\int_0^{\lambda|Q|} g^*(t)dt \le \frac{1}{\lambda|Q|}\int_0^{|Q|} g^*(t)dt = \frac{1}{\lambda|Q|}\int_Q |g(x)|dx. \tag{2.4}$$

Lemma 2.3. *Let $Q = Q(z,r)$ be a fixed cube. Let $0 < \lambda < 1$, $a \in \mathrm{BMO}(\mathbb{R}^n)$, $0 < \alpha < n$ and $f \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$. Then*

$$\omega_\lambda(\tilde{C}[a, I_\alpha]f;Q) \lesssim \|a\|_* \inf_{w\in Q} I_\alpha[|f|](w) + \|a\|_* \inf_{w\in Q} M_{\alpha,\mathrm{L}\log\mathrm{L}}f(w). \tag{2.5}$$

Here $L^\infty_{\mathrm{c}}(\mathbb{R}^n)$ stands for the space of all compactly supported essentially bounded functions.

We employ the following notation for the proof: For $\alpha > 0$, $r > 0$ and $x \in \mathbb{R}^n$, we write $\alpha Q(x,r) \equiv Q(x,\alpha r)$, so that $\alpha Q(x,r)$ is the α-times expansion of $Q(x,r)$. Denote by $\mathrm{W}L^p(\mathbb{R}^n)$ the weak L^p space.

Proof of Lemma 2.3. We decompose

$$\tilde{C}[a, I_\alpha]f = \tilde{C}[a, I_\alpha][\chi_{3Q}f] + \tilde{C}[a, I_\alpha][\chi_{\mathbb{R}^n\setminus 3Q}f]. \tag{2.6}$$

Fix $x \in Q$. We handle the first term in the right-hand side of (2.6). We calculate

$$|\tilde{C}[a, I_\alpha][\chi_{3Q}f](x)| \le |a(x) - m_Q(a)| I_\alpha[\chi_{3Q}|f|](x) + I_\alpha[|a - m_Q(a)| \cdot \chi_{3Q}|f|](x).$$

Since $I_\alpha[\chi_{3Q}|f|]$ is an A_1-weight according to [7, Lemma 5.2(2)], more precisely, $M[I_\alpha[\chi_{3Q}|f|]] \lesssim_\alpha I_\alpha[\chi_{3Q}|f|]$, it satisfies the reverse Hölder inequality:

$$m_Q(I_\alpha[\chi_{3Q}|f|]^{1+\varepsilon})^{\frac{1}{1+\varepsilon}} \lesssim m_Q(I_\alpha[\chi_{3Q}|f|])$$

for $\varepsilon > 0$. Thus, by [7, Lemma 5.2(1)], we have

$$m_Q(|a - m_Q(a)| I_\alpha[\chi_{3Q}|f|]) \lesssim \|a\|_* \ell(3Q)^\alpha m_{3Q}(|f|).$$

Meanwhile, we estimate $I_\alpha[|a - m_Q(a)| \cdot \chi_{3Q}|f|](x)$ by using the John–Nirenberg inequality, Hölder's inequality for weak/strong Lebesgue spaces over a probability space (see [11, Exercise 1.1.11]) and the duality $\mathrm{L}\log\mathrm{L}$–$\mathrm{Exp(L)}$ over a probability space. We recall

$$\|f\|_{\mathrm{Exp(L)};Q(x,r)} = \inf\left\{\lambda > 0 \,:\, \frac{1}{|Q(x,r)|}\int_{Q(x,r)} \left(\exp\left(3 + \frac{|f(y)|}{\lambda}\right) - 1\right) \mathrm{d}y \le 1\right\}$$

for a measurable function f. As well as the above inequalities, by the use of the Hardy–Littlewood–Sobolev inequality, which asserts that I_α maps $L^1(\mathbb{R}^n)$ to $\mathrm{W}L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$, we have

$$\begin{aligned} m_Q(I_\alpha[|a - m_Q(a)| \cdot \chi_{3Q}|f|]) &\lesssim \frac{\|I_\alpha[|a - m_Q(a)| \cdot \chi_{3Q}|f|]\|_{\mathrm{W}L^{\frac{n}{n-\alpha}}}}{\|\chi_Q\|_{L^{\frac{n}{n-\alpha}}}} \\ &\lesssim \frac{\|(a - m_Q(a)) \cdot \chi_{3Q}|f|\|_{L^1}}{\|\chi_Q\|_{L^{\frac{n}{n-\alpha}}}} \\ &\lesssim \|a\|_* \ell(Q)^\alpha m_{\mathrm{L}\log\mathrm{L},3Q}(|f|). \end{aligned}$$

Combining these estimates with (2.4), we obtain

$$\omega_{\lambda/2}(|\tilde{C}[a, I_\alpha][\chi_{3Q}f]|; Q) \lesssim m_Q(|\tilde{C}[a, I_\alpha][\chi_{3Q}f]|) \lesssim \|a\|_* \ell(Q)^\alpha m_{\mathrm{L}\log\mathrm{L},3Q}(|f|). \tag{2.7}$$

For the second term of (2.6), we estimate

$$\begin{aligned} &\left|\tilde{C}[a, I_\alpha][\chi_{\mathbb{R}^n\setminus 3Q}f](x) - \int_{\mathbb{R}^n\setminus 3Q} \frac{|m_Q(a) - a(y)|}{|z-y|^{n-\alpha}}|f(y)|dy\right| \\ &\le \left|\tilde{C}[a, I_\alpha][\chi_{\mathbb{R}^n\setminus 3Q}f](x) - \int_{\mathbb{R}^n\setminus 3Q} \frac{|m_Q(a) - a(y)|}{|x-y|^{n-\alpha}}|f(y)|dy\right| \\ &\quad + \left|\int_{\mathbb{R}^n\setminus 3Q} \frac{|m_Q(a) - a(y)|}{|x-y|^{n-\alpha}}|f(y)|dy - \int_{\mathbb{R}^n\setminus 3Q} \frac{|m_Q(a) - a(y)|}{|z-y|^{n-\alpha}}|f(y)|dy\right| \\ &\lesssim |a(x) - m_Q(a)| \int_{\mathbb{R}^n\setminus 3Q} \frac{|f(y)|}{|z-y|^{n-\alpha}}dy + r\int_{\mathbb{R}^n\setminus 3Q} \frac{|m_Q(a) - a(y)|}{|z-y|^{n+1-\alpha}}|f(y)|dy. \end{aligned}$$

Note that the quantities

$$\int_{\mathbb{R}^n\setminus 3Q} \frac{|f(y)|}{|z-y|^{n-\alpha}}dy, \quad r\int_{\mathbb{R}^n\setminus 3Q} \frac{|m_Q(a) - a(y)|}{|z-y|^{n+1-\alpha}}|f(y)|dy$$

are constants. Thus, from (2.3), we deduce

$$\omega_{\lambda/2}\left(|a-m_Q(a)|\int_{\mathbb{R}^n\setminus 3Q}\frac{|f(y)|}{|z-y|^{n-\alpha}}dy+r\int_{\mathbb{R}^n\setminus 3Q}\frac{|m_Q(a)-a(y)|}{|z-y|^{n+1-\alpha}}|f(y)|dy;Q\right)$$
$$=\omega_{\lambda/2}\left(|a-m_Q(a)|;Q\right)\int_{\mathbb{R}^n\setminus 3Q}\frac{|f(y)|}{|z-y|^{n-\alpha}}dy+r\int_{\mathbb{R}^n\setminus 3Q}\frac{|m_Q(a)-a(y)|}{|z-y|^{n+1-\alpha}}|f(y)|dy.$$

Note also that

$$\int_{\mathbb{R}^n\setminus 3Q}\frac{|f(y)|}{|z-y|^{n-\alpha}}dy\lesssim\int_{\mathbb{R}^n\setminus 3Q}\frac{|f(y)|}{|w-y|^{n-\alpha}}dy\le I_\alpha[|f|](w)$$

for each $w\in Q=Q(z,r)$. Therefore, we can estimate the second term of (2.6) as follows:

$$\omega_{\lambda/2}\left(\left|\tilde{C}[a,I_\alpha][\chi_{\mathbb{R}^n\setminus 3Q}f](x)-\int_{\mathbb{R}^n\setminus 3Q}\frac{|m_Q(a)-a(y)|}{|z-y|^{n-\alpha}}|f(y)|dy\right|;Q\right)$$
$$\lesssim\omega_{\lambda/2}(|a-m_Q(a)|;Q)\int_{\mathbb{R}^n\setminus 3Q}\frac{|f(y)|}{|z-y|^{n-\alpha}}dy+r\int_{\mathbb{R}^n\setminus 3Q}\frac{|m_Q(a)-a(y)|}{|z-y|^{n+1-\alpha}}|f(y)|dy$$
$$\lesssim\|a\|_*\inf_{w\in Q}I_\alpha[|f|](w)+r\int_{\mathbb{R}^n\setminus 3Q}\frac{|m_Q(a)-a(y)|}{|z-y|^{n+1-\alpha}}|f(y)|dy.$$

By the inclusion

$$\mathbb{R}^n\setminus 3Q\subset\bigcup_{j=1}^{\infty}(2^jQ\setminus 2^{j-1}Q)$$

and Hölder's inequality for Orlicz spaces (the duality $\mathrm{L}\log\mathrm{L}$–$\mathrm{Exp(L)}$ over a probability space), we obtain

$$\int_{\mathbb{R}^n\setminus 3Q}\frac{|m_Q(a)-a(y)|}{|z-y|^{n+1-\alpha}}|f(y)|dy\lesssim\sum_{j=1}^{\infty}\frac{1}{(2^jr)^{n+1-\alpha}}\int_{2^jQ}|f(y)|\cdot|a(y)-m_Q(a)|dy$$
$$\lesssim\sum_{j=1}^{\infty}\frac{1}{(2^jr)^{1-\alpha}}\|f\|_{\mathrm{L}\log\mathrm{L};2^jQ}\|a-m_Q(a)\|_{\mathrm{Exp(L)};2^jQ}$$
$$\lesssim\inf_{w\in Q}M_{\alpha,\mathrm{L}\log\mathrm{L}}f(w)\sum_{j=1}^{\infty}\frac{j}{2^jr}\|a\|_*$$
$$\lesssim r^{-1}\|a\|_*\inf_{w\in Q}M_{\alpha,\mathrm{L}\log\mathrm{L}}f(w).$$

In total,

$$\omega_{\lambda/2}\left(\left|\tilde{C}[a,I_\alpha][\chi_{\mathbb{R}^n\setminus 3Q}f](x)-\int_{\mathbb{R}^n\setminus 3Q}\frac{|m_Q(a)-a(y)|}{|z-y|^{n-\alpha}}|f(y)|dy\right|;Q\right)$$
$$\lesssim\|a\|_*\inf_{w\in Q}I_\alpha[|f|](w)+\|a\|_*\inf_{w\in Q}M_{\alpha,\mathrm{L}\log\mathrm{L}}f(w).\tag{2.8}$$

By combining (2.7) and (2.8), we obtain (2.5). □

For the proof of Theorem 1.6 we employ the Lerner–Hytönen decomposition from [24, Theorem 1.1] and [25, Theorem 4.5]; see also the textbook [33].

Lemma 2.4. *Let $f:Q^0\to\mathbb{R}$ be a measurable function defined on a right-open cube Q^0. Then there exists a family $\{Q_k^j\}_{j\in\mathbb{N}_0,\,k\in K_j}\subset\mathcal{D}(Q^0)$ such that $\{Q_k^0\}_{k\in K_0}=\{Q^0\}$, that*

$$\chi_{\bigcup_{k\in K_{j+1}}Q_k^{j+1}}\le\chi_{\bigcup_{k\in K_j}Q_k^j}\le\chi_{Q^0},$$

that

$$\left|\bigcup_{k' \in K_{j+1}} Q_{k'}^{j+1}\right| \le \frac{1}{2}|Q_k^j|,$$

and that

$$\chi_{Q^0}|f - \mathrm{Med}(f;Q^0)| \le \sum_{j=0}^{\infty}\sum_{k \in K_j} \omega_{2^{-n-2}}(f;Q_k^j)\chi_{Q_k^j}.$$

Here, the inequality is understood as the one for almost every point and $\mathrm{Med}(f;Q^0)$ *stands for a real number satisfying*

$$|\{x \in Q^0 \,:\, f(x) > \mathrm{Med}(f;Q^0)\}|, |\{x \in Q^0 \,:\, f(x) < \mathrm{Med}(f;Q^0)\}| \le \frac{1}{2}|Q^0|.$$

2.4. **Compactness criterion.** Our proof of the compactness of operators hinges on the following simple observation, which is a direct consequence of Kolmogorov's theorem. See [35].

Lemma 2.5. *Let* $k \in L^\infty_{\mathrm{c}}(\mathbb{R}^n \times \mathbb{R}^n)$. *Then the integral operator* T, *given by*

$$Tf(x) \equiv \int_{\mathbb{R}^n} k(x,y)f(y)dy,$$

is a compact operator from $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$ *for all* $1 \le p,q < \infty$.

In particular, T *is a compact operator from* $\mathcal{M}^p_q(\mathbb{R}^n)$ *to* $\mathcal{M}^s_t(\mathbb{R}^n)$ *whenever* $1 \le q \le p < \infty$ *and* $1 \le t \le s < \infty$.

2.5. **An estimate of Welland type.** We write

$$(I_\alpha)_R f(x) \equiv \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}}\chi_{(R,\infty)}(|x-y|)f(y)dy$$

and

$$(M_\alpha)_R f(x) \equiv \sup_{r>R} r^{\alpha-n}\int_{Q(x,r)} |f(y)|dy$$

for $f \in L^0(\mathbb{R}^n)$.

Lemma 2.6. *Let* $0 < \alpha < n$ *and* $\delta \in (0, \min(\alpha, n-\alpha))$. *Then*

$$(I_\alpha)_R f(x) \lesssim \sqrt{(M_{\alpha+\delta})_{R/n}f(x)(M_{\alpha-\delta})_{R/n}f(x)}$$

for all non-negative measurable functions f *and for all* $R > 0$.

Proof. Welland [40] proved that

$$I_\alpha f(x) \lesssim \sqrt{M_{\alpha+\delta}f(x)M_{\alpha-\delta}f(x)}$$

for all non-negative measurable functions f. Remark also that $Q(x,R/n) \subset B(x,R)$. Since

$$\begin{aligned} M_{\alpha+\delta}[\chi_{\mathbb{R}^n\setminus B(x,R)}f](x) &\le c_n \sup_{r>0}(2r)^{\alpha+\delta-n}\int_{Q(x,r)\setminus B(x,R)} |f(y)|dy \\ &\le c_n \sup_{r>R/n}(2r)^{\alpha+\delta-n}\int_{Q(x,r)\setminus B(x,R)} |f(y)|dy \\ &= c_n (M_{\alpha+\delta})_{R/n}f(x) \end{aligned}$$

for some constant c_n depending only on n, if we replace f by $\chi_{\mathbb{R}^n\setminus B(x,R)}f$, then we obtain

$$\begin{aligned}(I_\alpha)_R f(x) &= I_\alpha[\chi_{\mathbb{R}^n\setminus B(x,R)}f](x)\\ &\lesssim \sqrt{M_{\alpha+\delta}[\chi_{\mathbb{R}^n\setminus B(x,R)}f](x)M_{\alpha-\delta}[\chi_{\mathbb{R}^n\setminus B(x,R)}f](x)}\\ &\lesssim \sqrt{(M_{\alpha+\delta})_{R/n}f(x)(M_{\alpha-\delta})_{R/n}f(x)},\end{aligned}$$

as required. □

3. Proof of Theorem 1.6

By the monotone convergence theorem and the fact that the integral kernel $\tilde{C}[a,I_\alpha]$ is non-negative, we may assume that $f \in L^\infty_{\rm c}(\mathbb{R}^n)$. By decomposing f into the positive part and the negative part, we may assume that f is non-negative. In this case, $\tilde{C}[a,I_\alpha]f \in L^{p_0}(\mathbb{R}^n)$ for some $p_0 \in (1,\infty)$ and hence $\mathrm{Med}(\tilde{C}[a,I_\alpha]f;[-J,J)^n) \to 0$ as $J\to\infty$. Therefore,

$$\|\tilde{C}[a,I_\alpha]f\|_{\mathcal{M}^s_t} \le \liminf_{J\to\infty}\|\chi_{[-J,J)^n}(\tilde{C}[a,I_\alpha]f-\mathrm{Med}(\tilde{C}[a,I_\alpha]f;[-J,J)^n))\|_{\mathcal{M}^s_t}.$$

Fix $J\in\mathbb{N}$ here and below. Thanks to the Lerner–Hytönen decomposition, (Lemma 2.4) we obtain

$$|\chi_{[-J,J)^n}(x)(\tilde{C}[a,I_\alpha]f(x)-\mathrm{Med}(\tilde{C}[a,I_\alpha]f;[-J,J)^n))| \le \sum_{j=0}^\infty\sum_{k\in K_j}\omega_{2^{-n-2}}(\tilde{C}[a,I_\alpha]f;Q^j_k)\chi_{Q^j_k}(x)$$

for some collection $\{Q^j_k\}_{j\in\mathbb{N}_0,k\in K_j}$ of cubes as in Lemma 2.4. If we use Lemma 2.3, then we obtain

$$\begin{aligned}&|\chi_{[-J,J)^n}(x)(\tilde{C}[a,I_\alpha]f(x)-\mathrm{Med}(\tilde{C}[a,I_\alpha]f;[-J,J)^n))|\\ &\lesssim \|a\|_*\sum_{j=0}^\infty\sum_{k\in K_j}\inf_{z\in Q^j_k}I_\alpha f(z)\chi_{Q^j_k}(x)+\|a\|_*\sum_{j=0}^\infty\sum_{k\in K_j}\inf_{z\in Q^j_k}M_{\alpha,\mathrm{L}\log\mathrm{L}}f(z)\chi_{Q^j_k}(x)\end{aligned}$$

Write

$$E^j_k \equiv Q^j_k\setminus\bigcup_{k'\in K_{j+1}}Q^{j+1}_{k'} \quad (j\in\mathbb{N}_0, k\in K_j).$$

Since $2|E^j_k|\ge|Q^j_k|$, we have

$$M\chi_{E^j_k}(x)\ge\frac{|E^j_k|}{|Q^j_k|}\chi_{Q^j_k}(x)\ge\frac{1}{2}\chi_{Q^j_k}(x) \quad (x\in\mathbb{R}^n)$$

and hence

$$\begin{aligned}&|\chi_{[-J,J)^n}(x)(\tilde{C}[a,I_\alpha]f(x)-\mathrm{Med}(\tilde{C}[a,I_\alpha]f;[-J,J)^n))|\\ &\lesssim \|a\|_*\sum_{j=0}^\infty\sum_{k\in K_j}\inf_{z\in Q^j_k}I_\alpha f(z)(M\chi_{E^j_k}(x))^2+\|a\|_*\sum_{j=0}^\infty\sum_{k\in K_j}\inf_{z\in Q^j_k}M_{\alpha,\mathrm{L}\log\mathrm{L}}f(z)(M\chi_{E^j_k}(x))^2\end{aligned}$$

Meanwhile, by Lemma 2.1, we obtain

$$\begin{aligned}\left\|\sum_{j=0}^\infty\sum_{k\in K_j}\inf_{z\in Q^j_k}I_\alpha f(z)(M\chi_{E^j_k})^2\right\|_{\mathcal{M}^s_t} &= \left\{\left\|\left(\sum_{j=0}^\infty\sum_{k\in K_j}\inf_{z\in Q^j_k}I_\alpha f(z)(M\chi_{E^j_k})^2\right)^{\frac12}\right\|_{\mathcal{M}^{2s}_{2t}}\right\}^2\\ &\lesssim \left\{\left\|\left(\sum_{j=0}^\infty\sum_{k\in K_j}\inf_{z\in Q^j_k}I_\alpha f(z)\chi_{E^j_k}\right)^{\frac12}\right\|_{\mathcal{M}^{2s}_{2t}}\right\}^2.\end{aligned}$$

Likewise once again by Lemma 2.1, we have

$$\begin{aligned}\left\|\sum_{j=0}^{\infty}\sum_{k\in K_j}\inf_{z\in Q^j_k} M_{\alpha,\mathrm{L}\log\mathrm{L}}f(z)(M\chi_{E^j_k})^2\right\|_{\mathcal{M}^s_t} &= \left\{\left\|\left(\sum_{j=0}^{\infty}\sum_{k\in K_j}\inf_{z\in Q^j_k} M_{\alpha,\mathrm{L}\log\mathrm{L}}f(z)(M\chi_{E^j_k})^2\right)^{\frac12}\right\|_{\mathcal{M}^{2s}_{2t}}\right\}^2\\ &\lesssim \left\{\left\|\left(\sum_{j=0}^{\infty}\sum_{k\in K_j}\inf_{z\in Q^j_k} M_{\alpha,\mathrm{L}\log\mathrm{L}}f(z)\chi_{E^j_k}\right)^{\frac12}\right\|_{\mathcal{M}^{2s}_{2t}}\right\}^2.\end{aligned}$$

As a result,

$$\begin{aligned}&\|\chi_{[-J,J)^n}(\tilde{C}[a,I_\alpha]f-\mathrm{Med}(\tilde{C}[a,I_\alpha]f;[-J,J)^n))\|_{\mathcal{M}^s_t}\\ &\lesssim \|a\|_*\left\|\sum_{j=0}^{\infty}\sum_{k\in K_j}\inf_{z\in Q^j_k} I_\alpha f(z)(M\chi_{E^j_k})^2+\sum_{j=0}^{\infty}\sum_{k\in K_j}\inf_{z\in Q^j_k} M_{\alpha,\mathrm{L}\log\mathrm{L}}f(z)(M\chi_{E^j_k})^2\right\|_{\mathcal{M}^s_t}\\ &\lesssim \|a\|_*\left\|\sum_{j=0}^{\infty}\sum_{k\in K_j}\inf_{z\in Q^j_k} I_\alpha f(z)\chi_{E^j_k}+\sum_{j=0}^{\infty}\sum_{k\in K_j}\inf_{z\in Q^j_k} M_{\alpha,\mathrm{L}\log\mathrm{L}}f(z)\chi_{E^j_k}\right\|_{\mathcal{M}^s_t}.\end{aligned}$$

Since $\{E^j_k\}_{j\in\mathbb{N}_0,k\in K_j}$ is disjoint, it follows that

$$\begin{aligned}&\|\chi_{[-J,J)^n}(\tilde{C}[a,I_\alpha]f-\mathrm{Med}(\tilde{C}[a,I_\alpha]f;[-J,J)^n))\|_{\mathcal{M}^s_t}\\ &\lesssim \|a\|_*\,\|I_\alpha f+M_{\alpha,\mathrm{L}\log\mathrm{L}}f\|_{\mathcal{M}^s_t}.\end{aligned}\tag{3.1}$$

It remains to use Proposition 1.3 and Lemma 2.2. In fact, according to Proposition 1.3 and the inclusion $\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)\subseteq\mathcal{M}^p_1(\mathbb{R}^n)$, we have

$$\|I_\alpha f\|_{\mathcal{M}^s_t}\lesssim\|f\|_{(\mathcal{M}^p_q)^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}})^\theta}\tag{3.2}$$

Meanwhile, by virtue of Lemma 2.2, we have

$$\|M_{\alpha,\mathrm{L}\log\mathrm{L}}f\|_{\mathcal{M}^s_t}\lesssim\|f\|_{(\mathcal{M}^p_q)^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}})^\theta}.\tag{3.3}$$

Thus, the desired result follows from (3.1)–(3.3).

4. Proof of Theorem 1.7

We will reduce matters to a couple of steps. Let

$$[a,I_\alpha]_\varepsilon f(x)\equiv\int_{\mathbb{R}^n}\frac{a(x)-a(y)}{|x-y|^{n-\alpha}}\chi_{(\varepsilon,\infty)}(|x-y|)f(y)dy$$

for $\varepsilon>0$. Remark that any linear operator T from $(\mathcal{M}^p_q(\mathbb{R}^n))^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))^\theta$ to $\mathcal{M}^s_t(\mathbb{R}^n)$ is compact if T is realized as the norm limit of the sequence of compact operators from $(\mathcal{M}^p_q(\mathbb{R}^n))^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))^\theta$ to $\mathcal{M}^s_t(\mathbb{R}^n)$.

Before we come to the proof of Theorem 1.7, based on the above principle, we explain the plan of the proof as follows:

(1) We may assume $a\in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$; see Section 4.1.
(2) We have only to deal with $[a,I_\alpha]_\varepsilon$ for $\varepsilon>0$; see Section 4.2.
(3) As a key step, we will show that $[a,I_\alpha]_R\to0$, $R\to\infty$ in the operator norm; see Section 4.3.

(4) Using Lemma 2.5, we prove the compactness of $[a, I_\alpha]_\varepsilon - [a, I_\alpha]_R$ for $R > \varepsilon > 0$; see Section 4.4. Once this is done, we first conclude that $[a, I_\alpha]_\varepsilon$ is compact from $(\mathcal{M}^p_q(\mathbb{R}^n))^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))^\theta$ to $\mathcal{M}^s_t(\mathbb{R}^n)$ and then also conclude that $[a, I_\alpha]$ is compact from $(\mathcal{M}^p_q(\mathbb{R}^n))^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))^\theta$ to $\mathcal{M}^s_t(\mathbb{R}^n)$.

4.1. **Reduction to the case of $a \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$.** In view of (1.2), we may assume that $a \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$. In fact, since $a \in \mathrm{VMO}(\mathbb{R}^n)$, there exists a sequence $\{a_j\}_{j=1}^\infty \subset C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ such that $\|a - a_j\|_* \le 2^{-j}$. Thus, $\|[a, I_\alpha] - [a_j, I_\alpha]\|_{(\mathcal{M}^p_q)^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}})^\theta \to \mathcal{M}^s_t} \lesssim 2^{-j}$ thanks to (1.2). Thus, once we show that $[a_j, I_\alpha]$ is compact from $(\mathcal{M}^p_q(\mathbb{R}^n))^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))^\theta$ to $\mathcal{M}^s_t(\mathbb{R}^n)$, it follows that $[a, I_\alpha]$ is compact from $(\mathcal{M}^p_q(\mathbb{R}^n))^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))^\theta$ to $\mathcal{M}^s_t(\mathbb{R}^n)$.

4.2. **Reduction to the compactness of $[a, I_\alpha]_\varepsilon$.** Observe that

$$|[a, I_\alpha]_\varepsilon f(x) - [a, I_\alpha] f(x)| \lesssim \varepsilon I_\alpha[|f|](x)$$

by the mean value theorem. Since I_α maps $(\mathcal{M}^p_q(\mathbb{R}^n))^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))^\theta$ boundedly to $\mathcal{M}^s_t(\mathbb{R}^n)$ thanks to Proposition 1.3, we have only to show the compactness of the operator $[a, I_\alpha]_\varepsilon$.

4.3. **Reduction to the compactness of $[a, I_\alpha]_\varepsilon - [a, I_\alpha]_R$.** The key observation for the proof is the following estimate:

Proposition 4.1. *Let $a \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$. Then the operator norm of $[a, I_\alpha]_R$ from $(\mathcal{M}^p_q(\mathbb{R}^n))^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))^\theta$ to $\mathcal{M}^s_t(\mathbb{R}^n)$ converges to 0 as $R \to \infty$.*

Proof. Let $f \in (\mathcal{M}^p_q(\mathbb{R}^n))^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))^\theta$. Note that

$$[a, I_\alpha]_R f(x) = a(x)(I_\alpha)_R f(x) - (I_\alpha)_R(af)(x) \quad (x \in \mathbb{R}^n).$$

We will take care of each term. Choose $\delta > 0$ sufficiently small so that $\alpha + \delta < \frac{n}{p}$. Define $u^\pm$ and $v^\pm$ by $\frac{1}{u^\pm} = \frac{1}{p} - \frac{\alpha\pm\delta}{n} = \frac{1}{s} \mp \frac{\delta}{n}$ and $\frac{v^\pm}{u^\pm} = \frac{q}{p}$. Then we have

$$|a(x)(I_\alpha)_R f(x)| \le R^{-\delta}|a(x)| I_{\alpha+\delta}[|f|](x).$$

Since $a \in L^\infty_{\mathrm{c}}(\mathbb{R}^n)$, it follows from Hölder's inequality for Morrey spaces and Proposition 1.3 that

$$\|a \cdot I_{\alpha+\delta}[|f|]\|_{\mathcal{M}^s_t} \lesssim \|I_{\alpha+\delta}[|f|]\|_{\mathcal{M}^{u^+}_{v^+}} \lesssim \|f\|_{(\mathcal{M}^p_q)^{1-\frac{(\alpha+\delta)p}{n}}(\mathcal{M}^p_1)^{\frac{(\alpha+\delta)p}{n}}} \lesssim \|f\|_{(\mathcal{M}^p_q)^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}})^\theta}.$$

Meanwhile, by Lemma 2.6 and Hölder's inequality for Morrey spaces,

$$\begin{aligned} \|(I_\alpha)_R[a\cdot f]\|_{\mathcal{M}^s_t} &\lesssim \sqrt{\|(M_{\alpha+\delta})_{R/n}[a\cdot f]\|_{\mathcal{M}^{u^+}_{v^+}} \|(M_{\alpha-\delta})_{R/n}[a\cdot f]\|_{\mathcal{M}^{u^-}_{v^-}}} \\ &\lesssim \sqrt{\|M_{\alpha+\delta}[a\cdot f]\|_{\mathcal{M}^{u^+}_{v^+}} \|(M_{\alpha-\delta})_{R/n}[a\cdot f]\|_{\mathcal{M}^{u^-}_{v^-}}}. \end{aligned} \tag{4.1}$$

We can handle with ease $\|M_{\alpha+\delta}[a\cdot f]\|_{\mathcal{M}^{u^+}_{v^+}}$ by the use of the Adams theorem. In fact, we have

$$\|M_{\alpha+\delta}[a\cdot f]\|_{\mathcal{M}^{u^+}_{v^+}} \lesssim \|a\cdot f\|_{\mathcal{M}^p_q} \lesssim \|f\|_{\mathcal{M}^p_q}. \tag{4.2}$$

We will move on to the estimate of $\|(M_{\alpha-\delta})_{R/n}[a\cdot f]\|_{\mathcal{M}^{u^-}_{v^-}}$. Let $f_0 \in \mathcal{M}^p_q(\mathbb{R}^n)$ and $f_1 \in \mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n)$ satisfy $|f| \le |f_0|^{1-\theta}|f_1|^\theta$. Then we have

$$\begin{aligned} \ell(Q)^{\alpha-\delta} m_Q(|a\cdot f|) &\le \ell(Q)^{\alpha-\delta} m_Q(|a\cdot f_0|)^{1-\theta} m_Q(|f_1|)^\theta \\ &= \ell(Q)^{-\delta}(m_Q(|a\cdot f_0|))^{1-\theta}(\ell(Q)^{\frac{n}{p}} m_Q(|f_1|))^\theta \\ &\lesssim R^{-\delta} M[a\cdot f_0](x)^{1-\theta}(\|f_1\|_{\mathcal{M}^p_1})^\theta \end{aligned}$$

for any cube Q with $x \in Q$ and $n\ell(Q) \ge R$. Therefore,

$$(M_{\alpha-\delta})_{R/n}[a\cdot f](x) \lesssim R^{-\delta} M[a\cdot f_0](x)^{1-\theta}(\|f_1\|_{\mathcal{M}^p_1})^\theta.$$

By taking the norm $\|\cdot\|_{\mathcal{M}^{u^-}_{v^-}}$ on both sides, we obtain

$$\|(M_{\alpha-\delta})_{R/n}[a\cdot f]\|_{\mathcal{M}^{u^-}_{v^-}} \le R^{-\delta}(\|M[a\cdot f_0]\|_{\mathcal{M}^{(1-\theta)u^-}_{(1-\theta)v^-}})^{1-\theta}(\|f_1\|_{\mathcal{M}^p_1})^\theta$$
$$\lesssim R^{-\delta}(\|a\cdot f_0\|_{\mathcal{M}^{(1-\theta)u^-}_{(1-\theta)v^-}})^{1-\theta}(\|f_1\|_{\mathcal{M}^p_1})^\theta.$$

Arithmetic shows that $p>(1-\theta)u^-$ and that $q>(1-\theta)v^-$. Thus, we are in the position of choosing U and V so that

$$\frac{1}{p}+\frac{1}{U}=\frac{1}{(1-\theta)u^-}, \quad \frac{1}{q}+\frac{1}{V}=\frac{1}{(1-\theta)v^-}.$$

By applying Hölder's inequality for Morrey spaces and the fact that $a\in\mathcal{M}^U_V(\mathbb{R}^n)$, we obtain

$$\|(M_{\alpha-\delta})_{R/n}[a\cdot f]\|_{\mathcal{M}^{u^-}_{v^-}} \lesssim R^{-\delta}(\|a\|_{\mathcal{M}^U_V})^{1-\theta}(\|f_0\|_{\mathcal{M}^p_q})^{1-\theta}(\|f_1\|_{\mathcal{M}^p_1})^\theta \lesssim R^{-\delta}(\|f_0\|_{\mathcal{M}^p_q})^{1-\theta}(\|f_1\|_{\mathcal{M}^p_1})^\theta.$$

As a result, since f_0 and f_1 are arbitrary, we obtain

$$\|(M_{\alpha-\delta})_{R/n}[a\cdot f]\|_{\mathcal{M}^{u^-}_{v^-}} \lesssim R^{-\delta}\|f\|_{(\mathcal{M}^p_q)^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}})^\theta}. \tag{4.3}$$

Combining (4.1), (4.2) and (4.3), we conclude

$$\|(I_\alpha)_R[a\cdot f]\|_{\mathcal{M}^s_t} \lesssim R^{-\delta}\|f\|_{(\mathcal{M}^p_q)^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}})^\theta}$$

for all $f\in(\mathcal{M}^p_q(\mathbb{R}^n))^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))^\theta$, proving the lemma. □

4.4. **Conclusion of the proof of Theorem 1.7.** Note that the integral kernel k of $[a,I_\alpha]_\varepsilon-[a,I_\alpha]_R$ is given by

$$k(x,y)=\frac{a(x)-a(y)}{|x-y|^{n-\alpha}}\chi_{(\varepsilon,R)}(|x-y|)$$

and belongs to $L^\infty_{\mathrm{c}}(\mathbb{R}^n\times\mathbb{R}^n)$ thanks to the fact that $a\in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$. Thus, in view of Proposition 4.1 and Lemma 2.5, the operator $[a,I_\alpha]_\varepsilon-[a,I_\alpha]_R$ is compact from $(\mathcal{M}^p_q(\mathbb{R}^n))^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))^\theta$ to $\mathcal{M}^s_t(\mathbb{R}^n)$. Hence, $[a,I_\alpha]$ is compact from $(\mathcal{M}^p_q(\mathbb{R}^n))^{1-\theta}(\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}(\mathbb{R}^n))^\theta$ to $\mathcal{M}^s_t(\mathbb{R}^n)$.

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ATOMIC AND MOLECULAR DECOMPOSITIONS OF WEIGHTED TRIEBEL-LIZORKIN-TYPE SPACES

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Abstract. Weighted General Triebel-Lizorkin spaces are introduced and studied with the use of discrete wavelet transforms. This study extends the methods of dyadic φ-transforms of Frazier and Jawerth [12] and [39]. We consider the classes of almost diagonal operators on some appropriate Sequence Spaces and we obtain atomic and molecular decompositions of Weighted Triebel-Lizorkin-type Spaces.
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0. Introduction

Function Spaces play a crucial role in the genesis of functional analysis and widely used in the development of the modern analysis of partial differential equations. For instance, Morrey [26] study the local regularity of solutions of some partial differential equations in an appropriate space, called Morrey space. This local regularity of solutions is more precise than on the familiar Lebesgue spaces.
During the last decades various classical operators of harmonic analysis, such as maximal, singular, and potential operators were widely investigated both in classical and generalized Morrey spaces, we refer the reader to [17, 18, 20, 19, 21, 22, 24, 25, 29] and the references therein.

The classical Besov and Triebel-Lizorkin spaces are class of function spaces containing many well-known classical function spaces and are more suitable in the treatment of a large type of partial differential equations, see for instance [7, 11]. A comprehensive treatment of these function spaces and their history can be found in Triebel's monographs [37, 38] and in the fundamental paper of M. Frazier and B. Jawerth [12].

In recent years, there has been increasing interest in a new family of function spaces, called New class of Besov and Triebel-Lizorkin spaces. These spaces unify and generalize many classical spaces including Besov spaces, Morrey spaces, Triebel-lizorkin spaces, see for instance [39, 30].

0.1. **Some background tools.** In this section, w denotes a weight function $\mathbb{R}^n$ i.e, w is an almost every (a.e) positive locally integrable function in $\mathbb{R}^n$. A function $f \in L^p(w)$, $0 < p < \infty$ if and only if

$$||f||_{p,w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} < \infty.$$

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1

A weight function w is said to be in the Muckenhoupt classes A_p, $1 \leq p < \infty$ if there exists a constant $C_p > 0$ such that for every cube Q,

$$\frac{1}{|Q|}\int_Q w dy \left(\frac{1}{|Q|}\int_Q w^{1-p'} dy\right)^{p-1} \leq C_p$$

when $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, well for $p = 1$,

$$\frac{1}{|Q|}\int_Q w(y)dy \leq C_1 w(x),$$

for a.e. $x \in Q$, or equivalently $Mw(x) \leq C_1 w(x)$ for a.e. $x \in \mathbb{R}^n$, where M is the Hardy-Littlewood maximal operator defined, for a locally integrable function f, by

$$Mf(x) = \sup_{x\in Q}\frac{1}{|Q|}\int_Q |f(y)|dy.$$

The supremum is taken over all cubes containing x.

The classe A_p was introduced by Muckenhoupt, B. [27] in order to characterize the boundedness of the Hardy-Littlewodd maximal operator M on the weighted Lebesgue spaces, see also [8, 15, 33, 34]. The pioneering work of Muckenhoupt, B. [27] showed that

$$M : \quad L_p(w) \to L_p(w)$$

if and only if $w \in A_p$ when $1 < p < \infty$, and

$$M : \quad L_1(w) \to L_{1,\infty}(w),$$

if and only if $w \in A_1$.

$L_{q,\infty}(w)$ denotes the space of all measurable functions f such that

$$\sup_{\lambda>0} \left(w\left\{x \in \mathbb{R}^n : f(x) > \lambda\right\}\right)^{\frac{1}{q}} < \infty.$$

Moreover, if $1 < p < \infty$, $1 < q \leq \infty$ and $w \in A_p$, then there exists a positive constant C such that for all sequences $\{f_k\}_{k\in\mathbb{Z}}$ of locally integrable functions on $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n}\left(\sum_{k\in\mathbb{Z}}[Mf_k(x)]^q\right)^{p/q} w(x)dx \leq C\int_{\mathbb{R}^n}\left(\sum_{k\in\mathbb{Z}}|f_k(x)|^q\right)^{p/q} w(x)dx. \tag{0.1}$$

The inequality 0.1 is the well known Fefferman-Stein vector-valued inequality, see for instance [10, 15, 16, 33].

The following important properties of Muckenhoupt weights will be widely used in this work.

Lemma 0.1. *Let $w \in A_p$. Then, there exist $\delta > 0$, $C > 0$ s.t, every time we have a measurable subset A of a cube Q, the following "δ-reverse doubling" inequality holds*

$$\frac{w(A)}{w(Q)} \leq C\left(\frac{|A|}{|Q|}\right)^{\delta} \tag{0.2}$$

and also the following "p- doubling" inequality holds

$$\frac{w(Q)}{w(A)} \leq C\left(\frac{|Q|}{|A|}\right)^{p}. \tag{0.3}$$

Remark 0.1. *If $w \in A_p$, then $w^{\frac{-1}{p-1}} \in A_{\frac{p}{p-1}}$ and satisfies the same "δ-reverse doubling" inequality.*

2

The reverse condition is known as A_∞ - condition and the class of the weights w satisfying A_∞-condition is denoted by A_∞. It is well known that $A_\infty = \cup_{p\geq 1} A_p$, which motivates the notation A_∞, see for instance [15, Corollary 2.13, pp.403-404].

Throughout this work, w will be a fixed weight in A_∞ and we denote by r_0 the number $r_0 = \inf\{s > 0 : w \in A_s\}$. If we choose $0 < r < p/r_0,\ 0 < p < \infty$, then, in particular $w \in A_{\frac{p}{r}}$ and $w^{-\frac{r}{p-r}} \in A_{\frac{p}{p-r}}$. We also denote by δ the same reverse doubling constant of w and $w^{-\frac{r}{p-r}}$. (See Lemma 0.1 and Remark 0.1). Let $\mathcal{S}(\mathbb{R}^n)$ to be the space of all Schwartz functions on $\mathbb{R}^n$ with the classical topology generated by the family of semi-norms

$$||\nu||_{k,N} = sup_{x\in\mathbb{R}^n} sup_{|\beta|\leq N} \left(1+|x|\right)^k |\partial^\beta \nu(x)| \quad k, N \in \mathbb{N}_0, \quad \nu \in \mathcal{S}(\mathbb{R}^n).$$

The topological dual space, $\mathcal{S}'(\mathbb{R}^n)$ of $\mathcal{S}(\mathbb{R}^n)$ is the set of all continuous linear functional $\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}$ endowed with the weak $\star$-topology. We denote by $\mathcal{S}_\infty(\mathbb{R}^n)$, the topological subspace of functions in $\mathcal{S}(\mathbb{R}^n)$ having all vanishing moments :

$$\mathcal{S}_\infty(\mathbb{R}^n) = \Big\{\nu \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} x^\beta \nu(x)dx = 0, \qquad \forall \beta \in \mathbb{N}^n\Big\}.$$

$\mathcal{S}'_\infty(\mathbb{R}^n)$ denotes the topological dual space of $\mathcal{S}_\infty(\mathbb{R}^n)$, namely, the set of all continuous linear functional on $\mathcal{S}'_\infty(\mathbb{R}^n)$. The space $\mathcal{S}'_\infty(\mathbb{R}^n)$ is also endowed with the weak $\star$-topology. It is well known that $\mathcal{S}'_\infty(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ as topological spaces, where $\mathcal{P}(\mathbb{R}^n)$ denotes the set of all polynomials on $\mathbb{R}^n$, see for example, [43, Proposition 8.1].

Similarly, for any $R \in \mathbb{N}$, the space $\mathcal{S}_R(\mathbb{R}^n)$ is defined to be the set of all Schwartz functions having vanishing moments of order R and $\mathcal{S}'_R(\mathbb{R}^n)$ is its topological dual space. We write $\mathcal{S}_{-1}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n)$.

The Fourier transform, $\mathcal{F}\nu = \hat{\nu}$, of Schwartz function ν is defined by

$$\hat{\nu}(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\xi.x} \nu(x)dx.$$

The convolution of two function $\nu, \mu \in \mathcal{S}(\mathbb{R}^n)$ is defined by

$$\nu \star \mu(x) = \int_{\mathbb{R}^n} \nu(x-y)\mu(y)dy$$

and still belongs to $\mathcal{S}(\mathbb{R}^n)$.
The convolution operator can be extended to $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n)$ via $\quad \nu \star f(x) = \langle f, \mu(x - .)\rangle$. It makes sense pointwise and is a C^∞ function in $\mathbb{R}^n$ of at most polynomial growth. For simplifying notation, we write often $\nu f = \nu \star f$.

Throughout this paper, C denotes unspecified positive constant, possibly different at each occurrence; the symbol $A \preceq B$ means that $A \leq CB$. If $A \preceq B$ and $B \preceq A$, then we write $A \simeq B$. The symbol $\lfloor s \rfloor$ denotes the maximal integer no more than s and $s^\star = s - \lfloor s \rfloor$.

For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, we denote by Q_{jk} the dyadic cube $2^{-j}\left([0,1]^n + k\right)$, $l(Q_{jk}) = 2^{-j}$ is its side length, $x_{Q_{jk}} = 2^{-j}k$ is its lower "left-corner" and $c_{Q_{jk}}$ is its center. We set $\mathcal{Q} = \{Q_{jk} : j \in \mathbb{Z},\ k \in \mathbb{Z}^n\}$, and $j_Q = -log_2 l(Q)$ for all $Q \in \mathcal{Q}$.

When the dyadic cube Q appears as an index, such as $\sum_{Q\in\mathcal{Q}}$, it is understood that Q runs over all dyadic cubes in $\mathbb{R}^n$.

For a function ν and dyadic cube $Q = Q_{jk}$, set

$$\nu_Q(x) = |Q|^{-1/2}\nu(2^j x - k) = |Q|^{1/2}\nu_j(x - x_Q),$$

for all $x \in \mathbb{R}^n$, where $\nu_j(x) = 2^{nj}\nu(2^j x)$.

Definition 0.1. *A Schwartz function $\nu : \mathbb{R}^n \longrightarrow \mathbb{C}$ is a Littlewood-Paley function if $\hat{\nu}$ is a real-valued function and satisfies:*

$$supp \quad \hat{\nu} \subset \{\xi \in \mathbb{R}^n \quad : \quad 1/2 \leq |\xi| \leq 2\} \tag{0.4}$$

$$|\hat{\nu}(\xi)| \geq C > 0 \quad if \quad 3/5 \leq |\xi| \leq 5/3. \tag{0.5}$$

3

The function $\hat{\mu}(\xi) = \hat{\nu}/\eta$ *with* $\eta(\xi) = \sum_{j\in\mathbb{Z}} \hat{\nu}(2^{-j}\xi)\hat{\nu}(2^{-j}\xi)$, *is a Littlewood-Paley dual function related to* ν *and it is itself a Littlewood-Paley function, satisfying moreover*

$$\sum_{j\in\mathbb{Z}} \hat{\mu}(2^{-j}\xi)\hat{\nu}(2^{-j}\xi) = 1 \quad for \quad all \quad \xi \neq 0. \tag{0.6}$$

Lemma 0.2 (Reproducing Calderón Formula)**.**

(1) Let $\nu \in \mathcal{S}(\mathbb{R}^n)$ *be such that supp* $\hat{\nu}$ *is compact, bounded away from the origin and satisfying* $\sum_{j\in\mathbb{Z}} \nu(2^j\xi) = 1$ *for all* $\xi \neq 0$*. Then, for any* $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$,

$$f = \sum_{j\in\mathbb{Z}} \tilde{\nu}_j \star f. \tag{0.7}$$

(2) Let $\mu,\ \nu \in \mathcal{S}(\mathbb{R}^n)$ *such that supp* $\hat{\mu}$, *supp* $\hat{\nu}$ *are compact and bounded away from the origin and 0.6 holds. Then for any* $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$.

$$f = \sum_{j\in\mathbb{Z}} 2^{-jn} \sum_{k\in\mathbb{Z}} \tilde{\nu}_j \star f(2^{-j}k)\mu_j(. - 2^{-j}k) = \sum_{Q\in\mathcal{Q}} \langle f, \nu_Q\rangle \mu_Q \tag{0.8}$$

where and in what follows $\tilde{\nu}_j(x) = \overline{\nu(-x)}$.

For any $\varphi \in S(\mathbb{R}^n)$, define

$$||\varphi||_{\mathcal{S}_{M+1}} = sup_{|\beta|\leq M} sup_{x\in\mathbb{R}^n} |\partial^\beta \varphi(x)| \left(1+|x|\right)^{n+M+\beta}.$$

Then the following estimate holds (see [39]).

Lemma 0.3. *For any* $M \in \mathbb{N}$, *there exists a positive constant* $C = C(M,n)$ *such that for all* $\varphi,\psi \in S_\infty(\mathbb{R}^n), i,j \in \mathbb{Z}$ *and* $x \in \mathbb{R}^n$,

$$|\varphi_i \star \psi_j(x)| \leq C||\varphi||_{\mathcal{S}_{M+1}}||\psi||_{\mathcal{S}_{M+1}} 2^{-|i-j|M} \frac{2^{-min(i,j)M}}{\left(2^{-min(i,j)} + |x|\right)^{n+M}}. \tag{0.9}$$

0.2. **Classical Triebel-Lizorkin spaces.**

Definition 0.2. *Let* $w \in A_\infty$, $0 < p,q \leq \infty$, $\gamma \in \mathbb{R}$ *and* $\nu \in \mathcal{S}(\mathbb{R}^n)$ *satisfies 0.4 and 0.5. The homogeneous Triebel-Lizorkin space* $\dot{F}^{\gamma,q}_{p,w}$ *is the set of all distribution* $f \in \mathcal{S}'_\infty$ *such that*

$$||f||_{\dot{F}^{\gamma,q}_{p,w}} = \Big|\Big| \left(\sum_{j\in\mathbb{Z}} 2^{j\gamma q}|\nu_j f|^q\right)^{\frac{1}{q}} \Big|\Big|_{p,w} < \infty, \qquad 0 < p,q < \infty$$

and

$$||f||_{\dot{F}^{\gamma,q}_{\infty,w}} = \sup_Q \Big\{ \frac{1}{w(Q)} \int_Q \sum_{j=j_Q}^{\infty} 2^{j\gamma q}|\nu_j f|^q w(x)dx \Big\}^{\frac{1}{q}} < \infty, 0 < q \leq \infty$$

with the interpretation that when $q = \infty$,

$$||f||_{\dot{F}^{\gamma,\infty}_{\infty,w}} = \sup_Q \sup_{j\geq j_Q} \frac{1}{w(Q)} \int_Q 2^{j\gamma}|\nu_j f| w(x)dx < \infty.$$

Moreover, it is well known that the space $\dot{F}^{\gamma,q}_{p,w}$ *is independent of the choice of* ν *(see, for example,* [4, 5, 6, 12]*).*

It has long been known that many classical smoothness spaces are covered by the Triebel-Lizorkin spaces. We recall some examples,

4

(1) $\dot{F}^{0,2}_{p,w} = H_{p,w}, \quad 0 < p < \infty,$
where $H_{p,w}$ denotes the weighted Hardy spaces of $f \in \mathcal{S}'$ for which

$$||f||_{H_{p,w}} = ||\sup_{t>0} |\mu_t \star f|||_{p,w} < \infty,$$

where μ is a fixed function in $\mathcal{S}$ with $\int_{\mathbb{R}^n} \mu(x)dx \neq 0$. By the fundamental work of Fefferman, C. and Stein, E. [9] adapted to the weighted case, $H_{p,w}$ does not depend on the choices of μ in its definition. In particular

$$\dot{F}^{0,2}_{p,w} = L_{p,w}, \quad 1 < p < \infty,$$

see also [5] for a counter-part result related to the local version of weighted Hardy space $h_{p,w}$, the space of $f \in \mathcal{S}'$ for which

$$||f||_{h_{p,w}} = ||\sup_{0<t<1} |\mu_t \star f|||_{p,w} < \infty,$$

where μ is as in definition of $H_{p,w}$.

(2) $\dot{F}^{\alpha,2}_{p,w} = L^{\alpha}_{p,w}$, $1 < p < \infty$, where $L^{\alpha}_{p,w}$ denotes the weighted Bessel potential space defined by

$$||f||_{L^{\alpha}_{p,w}} = ||\mathcal{F}^{-1}(1 + |\xi|)^{\alpha/2}\mathcal{F}f||_{L_{p,w}}.$$

In particular, when the exponent is a natural number, say $\alpha = N \in \mathbb{N}$, then the weighted Bessel potential space can be identified with the classical Sobolev space

$$W^{N}_{p,w} = \{f \in L_{p,w} : ||\sum_{|\gamma|\leq N} \partial^{\gamma} f||_{L_{p,w}} < \infty\},$$

where all identities have to be understood in the sense of equivalent quasi-norms.

0.3. **Weighted Triebel-Lizorkin-type spaces.** Triebel-Lizorkin spaces $\dot{F}^{\gamma,\tau}_{p,q}$ were introduced and investigated in [30, 39, 41, 43, 42, 44]. These spaces unify and generalize many classical function spaces such as classical Triebel-Lizorkin spaces, Triebel-Lizorkin Morrey spaces, Q spaces, Hardy spaces ....

We now define Weighted Triebel-Lizorkin-type spaces $\dot{F}^{\gamma,\tau}_{p,q,w}$, as follows.

Definition 0.3. *Let $w \in A_\infty$, $0 < p, q \leq \infty$, $\gamma \in \mathbb{R}$ and $\nu \in \mathcal{S}(\mathbb{R}^n)$ and satisfies 0.4 and 0.5. The homogeneous Triebel-Lizorkin space $\dot{F}^{\gamma,\tau}_{p,q,w}$ is the set of all distribution $f \in \mathcal{S}'_\infty$ such that*

$$||f||_{\dot{F}^{\gamma,\tau}_{p,q,w}} = ||f||_{\dot{F}^{\gamma,\tau,\nu}_{p,q,w}} = \sup_{Q\in\mathcal{Q}} \frac{1}{[w(Q)]^\tau} \left[\int_Q \left(\sum_{j=j_Q}^{\infty} 2^{j\gamma q}|\nu_j f|^q\right)^{\frac{p}{q}} w(x)dx\right]^{\frac{1}{p}} < \infty.$$

We note that in his paper [36] Tang has defined $\dot{F}^{\gamma,\tau}_{p,q,w}$ as the space of $f \in \mathcal{S}'_\infty$ such that

$$||f||_{\dot{F}^{\gamma,\tau}_{p,q,w}} = \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau} \left[\int_Q \left(\sum_{j=j_Q}^{\infty} 2^{j\gamma q}|\nu_j f|^q\right)^{\frac{p}{q}} w(x)dx\right]^{\frac{1}{p}} < \infty.$$

These spaces cannot be compared to ours except in the case where the weight w is identically equal to 1.

In this work, we extend some fundamental results obtained in the unweighted spaces such as the φ-transforms characterizations, the boundedness of the ϵ-almost diagonal operators, molecular and atomic decomposition in the weighted Triebel-Lizorkin-type spaces. See for instance [12, 30, 31, 32, 35, 39, 41].

This paper is organized as follows. In Section 1 we establish the φ-transforms of the space $\dot{F}^{\gamma,\tau}_{p,q,w}$. In Section 2 we prove that ϵ-almost diagonal operators are bounded on the Triebel-Lizorkin sequence spaces. And in Section 3 we study the molecular and atomic decomposition of the space $\dot{F}^{\gamma,\tau}_{p,q,w}$.

5

1. The φ-transform characterizations of Triebel-Lizorkin-sequence spaces

In this section, we establish the φ-transform characterizations of the spaces $\dot{F}^{\gamma,\tau}_{p,q,w}$. To this end, we introduce their corresponding sequence spaces as follows.

Definition 1.1. *The discrete Triebel-Lizorkin sequence space $\dot{f}^{\gamma,\tau}_{p,q,w}$ is defined to be the collection of all complex-valued sequences $t = \{t_Q\}_{Q\in\mathcal{Q}}$ such that*

$$||t||_{\dot{f}^{\gamma,\tau}_{p,q,w}} = \sup_{P\in\mathcal{Q}} \frac{1}{[w(P)]^\tau}\left[\int_P \left(\sum_{Q\subset P} |Q|^{-\frac{\gamma}{n}q}|t_Q\tilde{\chi}_Q|^q\right)^{\frac{p}{q}} w(x)dx\right]^{\frac{1}{p}} < \infty.$$

where $\tilde{\chi}_Q = |Q|^{-\frac{1}{2}}\chi_Q$ is the L^2-normalized characteristic function of the dyadic cube Q.

Remark 1.1. *Note that if $P, Q \in \mathcal{Q}$ with $l(Q) \leq l(P)$ then either $Q \subset P$ or do not overlap(by which we mean that their interiors are disjoint).*

It follows that

$$||t||_{\dot{f}^{\gamma,\tau}_{p,q,w}} = \sup_{P\in\mathcal{Q}} \frac{1}{[w(P)]^\tau}\left[\int_P \Big(\sum_{j_Q\geq j_P} |Q|^{-\frac{\gamma}{n}q}|t_Q\tilde{\chi}_Q|^q\Big)^{\frac{p}{q}} w(x)dx\right]^{\frac{1}{p}}.$$

Definition 1.2. *Suppose that $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ are s.t supp$\hat{\varphi}$, supp$\hat{\psi}$ are compact and bounded away from the origin. The φ-transform S_φ is defined to be the map taking each $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ to the sequence $S_\varphi f = \{(S_\varphi f)_Q\}_{Q\in\mathcal{Q}}$ where $(S_\varphi f)_Q = \langle f, \varphi_Q\rangle$ for all dyadic cubes Q; the inverse ψ-transform T_ψ is defined to be the map taking a sequence $t = \{t_Q\}_{Q\in\mathcal{Q}}$ to $T_\psi t = \sum_{Q\in\mathcal{Q}} t_Q\psi_Q$; see, for example,* [12, 13, 14].

The next result is a generalization of the fundamental result of Frazier and Jawerth saying that the following diagram is commutative :

$$\begin{array}{ccc} \dot{F}^{\gamma,\tau}_{p,q,w} & \xrightarrow{id} & \dot{F}^{\gamma,\tau}_{p,q,w} \\ {\scriptstyle S_\varphi}\downarrow & \nearrow {\scriptstyle T_\psi} & \\ \dot{f}^{\gamma,\tau}_{p,q,w} & & \end{array} \tag{1.1}$$

Theorem 1.1. *Let $0 < p,\ q \leq \infty, w \in A_\infty$ and $\varphi,\ \psi$ satisfying 0.4 and 0.6. Then*

$$S_\psi : \dot{F}^{\gamma,\tau,\psi}_{p,q,w} \longrightarrow \dot{f}^{\gamma,\tau}_{p,q,w}$$

and

$$T_\varphi : \dot{f}^{\gamma,\tau}_{p,q,w} \longrightarrow \dot{F}^{\gamma,\tau,\varphi}_{p,q,w}$$

are bounded.Furthermore, $T_\psi \circ S_\varphi$ is the identity on $\dot{F}^{\gamma,\tau,\psi}_{p,q,w} = \dot{F}^{\gamma,\tau,\varphi}_{p,q,w}$.

Corollary 1.1. *The space $\dot{F}^{\gamma,\tau}_{p,q,w}$ is independent of the particular choice of the function ν. The quasi-norms arising from different ν are equivalent.*

The proof of Theorem 1.1 is based on some technical lemmas.

Lemma 1.1. [2, Lemma 2.11]. *Let $\delta \in \mathbb{R}$ and $w \in A_{r_0}$. Then, there exist positive constant C such that for all $j \in \mathbb{Z}$ and all $L > r_0|\delta| + 1$*

$$\sum_{Q\in\mathcal{Q}, l(Q)=2^{-j}} [w(Q)]^\delta (1 + \frac{|x_Q|^n}{max(|Q|,1)}\Big)^{-L} \leq C2^{n(2r_0|\delta|+1)|j|}. \tag{1.2}$$

To show that T_ψ is well defined for all $t \in \dot{f}^{\gamma,\tau}_{p,q,w}$, we have the following conclusions.

Lemma 1.2. *Let $\gamma \in \mathbb{R}, 0 \leq p < \infty, 0 < q \leq \infty$ and $\psi \in S_\infty$. Then, for all $t \in \dot{f}^{\gamma,\tau}_{p,q,w}$, $\quad T_\psi = \sum\limits_{Q\in\mathcal{Q}} t_Q \psi_Q$, converges in $\mathcal{S}'_\infty(\mathbb{R}^n)$. Moreover, the operator*

$$T_\psi : \dot{f}^{\gamma,\tau}_{p,q,w} \longrightarrow S'_\infty(\mathbb{R}^n)$$

is continuous.

Proof. Note first, that for any $t \in \dot{f}^{\gamma,\tau}_{p,q,w}$ we have for $0 < p < \infty$,

$$|t_Q| \leq ||t||_{\dot{f}^{\gamma,\tau}_{p,q,w}} |Q|^{\frac{\gamma}{n}+\frac{1}{2}} [w(Q)]^{\tau-\frac{1}{p}}. \tag{1.3}$$

On the other, hand for any $L > 0$ there exist constants $N,\ C > 0$ s.t for all $Q,\ P \in \mathcal{Q}$ we have the following well known estimate, see for instance [2].

$$|\langle \psi_Q, \phi_P\rangle| \leq C||\psi||_N||\phi||_N \left(1 + \frac{|x_Q - x_P|^n}{max(|Q|, |P|)}\right)^{-L} min\left(\frac{|Q|}{|P|}, \frac{|P|}{|Q|}\right)^L \tag{1.4}$$

where the constant C depends only on L and

$$||\phi||_N = sup_{x\in\mathbb{R}^n} sup_{|\beta|\leq N} \left(1 + |x|\right)^N |\partial^\beta \phi(x)|.$$

In particular, if $P = [0, 1]^n$ then $\phi_P = \phi$ and

$$|\langle \psi_Q, \phi\rangle| \leq C||\psi||_N||\phi||_N \left(1 + \frac{|x_Q|^n}{max(|Q|, 1)}\right)^{-L} min\left(|Q|, |Q|^{-1}\right)^L.$$

Combining the above estimates and the estimate 1.2 to obtain

$$\begin{aligned}
&\sum_{Q\in\mathcal{Q}} |t_Q||\langle \psi_Q, \phi\rangle| \\
&\leq C||\phi||_N||t||_{\dot{f}^{\gamma,\tau}_{p,q,w}} \sum_{Q\in\mathcal{Q}} |Q|^{\frac{\gamma}{n}+\frac{1}{2}} [w(Q)]^{\tau-\frac{1}{p}} \Big(1 + \frac{|x_Q|^n}{max(|Q|, 1)}\Big)^{-L} min\left(|Q|, |Q|^{-1}\right)^L \\
&\leq C||\phi||_N||t||_{\dot{f}^{\gamma,\tau}_{p,q,w}} \\
&\times \sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{Q}, l(Q)=2^{-j}} |Q|^{\frac{\gamma}{n}+\frac{1}{2}} [w(Q)]^{\tau-\frac{1}{p}} \Big(1 + \frac{|x_Q|^n}{max(|Q|, 1)}\Big)^{-L} min\left(|Q|, |Q|^{-1}\right)^L \\
&\leq C||\phi||_N||t||_{\dot{f}^{\gamma,\tau}_{p,q,w}} \sum_{j\in\mathbb{Z}} 2^{-|j|nL} 2^{-j(\gamma+n/2)} 2^{|j|n(2r_0|\tau-1/p|+1)} \\
&\leq C||\phi||_N||t||_{\dot{f}^{\gamma,\tau}_{p,q,w}},
\end{aligned}$$

for sufficiently large L. It follows that $T_\psi = \sum\limits_{Q\in\mathcal{Q}} t_Q \psi_Q$ converges in $\mathcal{S}'_\infty$ and

$$|\langle T_\psi, \phi\rangle| = \left|\sum_{Q\in\mathcal{Q}} t_Q \langle \psi_Q, \phi\rangle\right| \leq C||\phi||_N||t||_{\dot{f}^{\gamma,\tau}_{p,q,w}}. \tag{1.5}$$

This shows the continuity of T_ψ. □

For a sequence $t = \{t_Q\}_{Q\in\mathcal{Q}},\ 0 < r \leq \infty$ and a fixed $\lambda > 0$, set

$$(t^\star_{r,\lambda})_Q = \left(\sum_{l(R)=l(Q)} |t_R|^r \left(1 + \frac{|x_R - x_Q|}{l(R)}\right)^{-\lambda}\right)^{1/r}.$$

Lemma 1.3. *Let $\gamma \in \mathbb{R}$, $0 \leq p,\ q,\ \tau < \infty$ and $w \in A_{r_0}$. Then, for any $r > 0$ and $\lambda/n > max(1, r/q, rr_0/p)$ if $\tau - 1/p < 0$ or $\lambda/n > max(1, r/q, rr_0/p, r(r_0 - \delta)(\tau - 1/p))$ if $\tau - 1/p \geq 0$, then*

$$||t^\star_{r,\lambda}||_{\dot{f}^{\gamma,\tau}_{p,q,w}} \simeq ||t||_{\dot{f}^{\gamma,\tau}_{p,q,w}}.$$

Proof. The inequality $||t^\star_{r,\lambda}||_{\dot{f}^{\gamma,\tau}_{p,q,w}} \geq ||t||_{\dot{f}^{\gamma,\tau}_{p,q,w}}$ is immediate from the definition of $t^\star_{r,\lambda}$.

To see the converse, fix a dyadic cube P. Let $s_Q = t_Q$ if $Q \subset 3P$ and $s_Q = 0$ otherwise, and let $u_Q = t_Q - s_Q$. Then, for any dyadic cube and $r > 0$ we have

$$(t^\star_{r,\lambda})^r_Q = (s^\star_{r,\lambda})^r_Q + (u^\star_{r,\lambda})^r_Q. \tag{1.6}$$

Suppose $r > 0$ and $\lambda/n > max(1, r/q, rr_0/p)$, then by Lemma 3.3 in [3] we have

$$\begin{aligned} I_P &= \frac{1}{[w(P)]^\tau}\left[\int_P \Big(\sum_{Q\subset P} |Q|^{-\frac{\gamma}{n}q}|(s^\star_{r,\lambda})_Q\tilde{\chi}_Q|^q\Big)^{\frac{p}{q}} w(x)dx\right]^{\frac{1}{p}} \\ &\leq C\frac{1}{[w(P)]^\tau}||s^\star_{r,\lambda}||_{\dot{f}^{\gamma}_{p,q,w}} \leq C\frac{1}{[w(P)]^\tau}||s||_{\dot{f}^{\gamma}_{p,q,w}} \leq C||t||_{\dot{f}^{\gamma,\tau}_{p,q,w}}. \end{aligned}$$

On the other hand, let $Q \subset P$ be a dyadic cube with $l(Q) = 2^{-i}l(P)$ for some $i \in \mathbb{N}$. Suppose R is any dyadic cube with $l(R) = l(Q) = 2^{-i}l(P)$, $R \subset P_k = P + kl(P)$ and $R \not\subset 3P$ for some $k \in \mathbb{Z}^n$, where $P + kl(P) = \{x + kl(P) : x \in P\}$. Then $|k| \geq 2$ and $(1 + l(R)^{-1}|x_Q - x_R|) \simeq 2^i|k|$. Set

$$A(i,k,P) = \{R \in \mathcal{Q}{:}l(R) = l(Q) = 2^{-i}l(P),\ R \subset P + kl(P),\ R \cap (3P) = \emptyset\}$$

and

$$(u_{i,k,P})^r = \sum_{R\in A(i,k,P)} (|R|^{-\gamma/n-1/2}|u_R|)^r \left(1 + \frac{|x_R - x_Q|}{l(R)}\right)^{-\lambda}.$$

Then, we have the following results:
Suppose $0 < a \leq r < \infty$. Then, for all $x \in Q$

$$u_{i,k,P} \preceq 2^{-i(\lambda/r-n/a)}|k|^{-(\lambda/r-n/a)}\left[M\left(\sum_{R\in A(i,k,P)} (|R|^{-\gamma/n}|u_R\tilde{\chi}_R|)^a\right)(x)\right]^{1/a}, \tag{1.7}$$

and for all $x \in P$

$$u_{i,k,P} \preceq 2^{-i(\lambda/r-n/a)}|k|^{-\lambda/r}\left[M\left(\sum_{R\in A(i,k,P)} (|R|^{-\gamma/n}|u_R\tilde{\chi}_R|)^a\right)(x + kl(P))\right]^{1/a}. \tag{1.8}$$

The proof of 1.7 is given in [12] while the estimate 1.8 is a consequence of Remark A.3 in[12]. See also [40, p 461].

Proof of Lemma 1.3: continued. If $\tau - 1/p < 0$ then, using 1.7, Lemma 4.2 and arguing as in [2, 12] to get the result. To prove Lemma 1.3 when $\tau - 1/p \geq 0$, we suppose $r > 0$ and $\lambda/n > max(1, r/q, rr_0/p, r(r_0 - \delta)(\tau - 1/p))$. If $r < min(q, p/r_0)$, then set $a = r$. Otherwise, if $r \geq min(q, p/r_0)$, then take a such that $nr/\lambda < a < min(r, q, p/r_0)$. It is possible to choose such an a, since $\lambda/n > max(1, r/q, rr_0/p)$; implies $nr/\lambda < a < min(r, q, p/r_0)$. In both cases we have that

$$0 < a \leq r < \infty,\ \ \lambda > max(nr/a, nr(r_0 - \delta)(\tau - 1/p)),\ \ q/a > 1,\ \ p/a > r_0.$$

Then, by 1.8 we have for all $x \in P$

$$u_{i,k,P} \preceq 2^{-i(\lambda-nr/a)/r}|k|^{-\lambda/r}\left[M\left(\sum_{R\in A(i,k,P)} (|R|^{-\gamma/n}|u_R\tilde{\chi}_R|)^a\right)(x + kl(P))\right]^{1/a}.$$

8

Rising this inequality to the power of p and integrating over the cube P with respect to $w(x+kl(P))dx$, we get

$$u_{i,k,P}\left(\int_P w(x+kl(P))dx\right)^{1/p}$$
$$\preceq 2^{-i(\lambda-nr/a)/r}|k|^{-\lambda/r}$$
$$\times\left(\int_P\left[M\left(\sum_{R\in A(i,k,P)}(|R|^{-\gamma/n}|u_R\tilde{\chi}_R|)^a\right)(x+kl(P))\right]^{p/a}w(x+kl(P))dx\right)^{1/p}$$

$$\preceq 2^{-i(\lambda-nr/a)/r}|k|^{-\lambda/r}\left(\int_{\mathbb{R}^n}\left[M\left(\sum_{R\in A(i,k,P)}(|R|^{-\gamma/n}|u_R\tilde{\chi}_R|)^a\right)(x)\right]^{p/a}w(x)dx\right)^{1/p}.$$

Since $0<a<p/r_0$ implies $w\in A_{p/a}$, then the boundedness of the maximal operator M on $L_{p/a}(w)$ and the Hölder's inequality for $q/a>1$ leads to

$$u_{i,k,P}\left(\int_{P_k} w(x)dx\right)^{1/p}$$
$$\preceq 2^{-i(\lambda-nr/a)/r}|k|^{-\lambda/r}\left(\int_{\mathbb{R}^n}\left[\sum_{R\in A(i,k,P)}(|R|^{-\gamma/n}|u_R\tilde{\chi}_R|)^a(x)\right]^{p/a}w(x)dx\right)^{1/p}$$
$$\preceq 2^{-i(\lambda-nr/a)/r}|k|^{-\lambda/r}$$
$$\times\left(\int_{\mathbb{R}^n}\left[\sum_{R\in A(i,k,P)}(|R|^{-\gamma/n}|u_R\tilde{\chi}_R|)^q(x)\right]^{p/q}\left[\sum_{R\in A(i,k,P)}\chi_R(x)\right]^{p(1/a-1/q)}w(x)dx\right)^{1/p}$$
$$\preceq 2^{-i(\lambda-nr/a)/r}|k|^{-\lambda/r}\left(\int_{P_k}\left[\sum_{R\subset P_k}(|R|^{-\gamma/n}|u_R\tilde{\chi}_R|)^q(x)\right]^{p/q}w(x)dx\right)^{1/p}$$
$$\preceq 2^{-i(\lambda-nr/a)/r}|k|^{-\lambda/r}[w(P_k)]^{\tau}||t||_{\dot{f}^{\gamma,\tau}_{p,q,w}}.$$

On the other hand, since $P_k=P+kl(P)\subset B(x_P,c_n|k|l(P))=B_k$ for some constant $c_n>1$, we have

$$w(P_k)\preceq\left(\frac{|P_k|}{|B_k|}\right)^{\delta}w(B_k)\preceq|k|^{n(r_0-\delta)}w(P).$$

The above estimates lead to

$$u_{i,k,P}\preceq 2^{-i(\lambda-nr/a)/r}|k|^{-\lambda/r+n(r_0-\delta)(\tau-1/p)}[w(P)]^{|\tau-1/p|}||t||_{\dot{f}^{\gamma,\tau}_{p,q,w}}.$$

So that if $-\lambda/r+n(r_0-\delta)(\tau-1/p)<0$ then

$$|Q|^{-\frac{\gamma}{n}-\frac{1}{2}}(u^{\star}_{r,\lambda})_Q\preceq 2^{-i(\lambda-nr/a)/r}[w(P)]^{\tau-1/p}||t||_{\dot{f}^{\gamma,\tau}_{p,q,w}}\sum_{k\in\mathbb{Z}^n,|k|\geq 2}|k|^{-\lambda/r+n(r_0-\delta)(\tau-1/p)}$$
$$\preceq 2^{-i(\lambda-nr/a)/r}[w(P]^{\tau-1/p}||t||_{\dot{f}^{\gamma,\tau}_{p,q,w}}.$$

9

It follows that

$$\frac{1}{[w(P)]^{\tau}}\left[\int_P \Big(\sum_{Q\subset P}|Q|^{-\frac{\gamma}{n}q}|(u^{\star}_{r,\lambda})_Q\tilde{\chi}_Q(x)|^q\Big)^{\frac{p}{q}}w(x)dx\right]^{1/p}$$

$$\preceq \frac{1}{[w(P)]^{\tau}}\left[\int_P \Big(\sum_{i\geq 0}\sum_{l(Q)=2^{-i}l(P)}|Q|^{-\frac{\gamma}{n}q}|(u^{\star}_{r,\lambda})_Q\tilde{\chi}_Q(x)|^q\Big)^{\frac{p}{q}}w(x)dx\right]^{1/p}$$

$$\preceq [w(P)]^{-1/p}||t||_{\dot{f}^{\gamma,\tau}_{p,q,w}}\left[\int_P \Big(\sum_{i\geq 0}\sum_{l(Q)=2^{-i}l(P)}|2^{-i(\lambda-nr/a)/r}\chi_Q(x)|^q\Big)^{\frac{p}{q}}w(x)dx\right]^{1/p}$$

$$\preceq ||t||_{\dot{f}^{\gamma,\tau}_{p,q,w}}.$$

Using the identity 1.6 to finish the proof. □

Let φ to be a Schwartz and satisfy 0.4 and 0.5 . Since $\tilde{\varphi}(x)=\varphi(-x)$ also satisfies 0.4 and 0.5 , we may take $\tilde{\varphi}$ in place of φ in the definition of $\dot{F}^{\gamma,\tau,\varphi}_{p,q,w}$. For $f\in S'_{\infty}(\mathbb{R}^n)$ and $Q\in\mathcal{Q}$ with $l(Q)=2^{-j}$, define the sequence $sup(f)=\{sup_Q(f)\}_Q$ by setting

$$sup_Q(f)=|Q|^{1/2}sup_{y\in Q}|\tilde{\varphi}_j * f(y)|,$$

and for any $m\in\mathbb{N}$, the sequence $\inf_m(f)=\{\inf_{Q,m}(f)\}_Q$ by setting

$$\inf_{Q,m}(f)=|Q|^{1/2}max\{\inf_{y\in R}|\tilde{\varphi}_j * f(y)| : l(R)=2^{-m}l(Q), R\subset Q\}.$$

Then, we have the following estimates.

Lemma 1.4. *Let $\gamma\in\mathbb{R}, 0<p<\infty, 0<q\leq\infty$, $w\in A_{r_0}$ and λ as in Lemma 1.3. Then* $||f||_{\dot{F}^{\gamma,\tau}_{p,q,w}}\simeq||sup(f)||_{\dot{f}^{\gamma,\tau}_{p,q,w}}\simeq||\inf_m(f)||_{\dot{f}^{\gamma,\tau}_{p,q,w}}$.

Proof. We adapt here the the proof given in [12, 39]. The estimate

$$||f||_{\dot{F}^{\gamma,\tau}_{p,q,w}}\leq||sup(f)||_{\dot{f}^{\gamma,\tau}_{p,q,w}}$$

follows from the definition.

To prove the converse, define $\{t_R\}_R$ by

$$t_R=|R|^{1/2}\inf_{y\in R}|\tilde{\varphi}_{i-m}\star f(y)| \quad \text{for all } R\in\mathcal{Q}\in \text{ with } l(R)=2^{-i}.$$

Then

$$\inf_{Q,m}(f)\tilde{\chi}_Q\leq C_n 2^{m\lambda}\sum_{\substack{R\subset Q\\ l(R)=2^{-m}l(Q)}}(t^{\star}_{1,\lambda})_R\tilde{\chi}_R.$$

Applying Lemma 1.3 to get

$$||\inf_m(f)||_{\dot{f}^{\gamma,\tau}_{p,q,w}}\preceq||t^{\star}_{1,\lambda}||_{\dot{f}^{\gamma,\tau}_{p,q,w}}\simeq||t||_{\dot{f}^{\gamma,\tau}_{p,q,w}}\preceq||f||_{\dot{F}^{\gamma,\tau}_{p,q,w}}.$$

Now, let $j\in\mathbb{Z}$ and apply Lemma A.4 in [12] to the function $\tilde{\varphi}_j * f(2^{-j}x)$, to obtain for all dyadic cube Q with $l(Q)=2^{-j}$,

$$(sup(f)^{\star}_{1,\lambda})_Q\simeq(\inf(f)^{\star}_{1,\lambda})_Q.$$

Thus

$$||sup(f)^{\star}_{1,\lambda}||_{\dot{f}^{\gamma,\tau}_{p,q,w}}\simeq||\inf(f)^{\star}_{1,\lambda}||_{\dot{f}^{\gamma,\tau}_{p,q,w}},$$

which together with Lemma 1.3 yield

$$||sup(f)||_{\dot{F}^{\gamma,\tau}_{p,q,w}}\simeq||\inf(f)||_{\dot{F}^{\gamma,\tau}_{p,q,w}}.$$

□

10

Proof of Theorem 1.1. The boundedness of S_φ follows from Lemma 1.4, since, if $Q = Q_{jk}$,

$$|(S_\varphi f)_Q| = |\langle f, \varphi_Q\rangle| = |Q|^{1/2}|\tilde{\varphi_j} \star f(2^{-jk})| \leq sup_Q(f).$$

To prove the boundedness of T_ψ, take any $t = \{t_Q\}_Q$. Then, by Lemma 1.2, $f = T_\psi t = \sum_{Q\in\mathcal{Q}} t_Q\psi_Q$ converges in S'_∞. Therefore, the following estimate established in [12] holds: for any $r > 0$ and $\lambda > n$,

$$|\tilde{\varphi_j} \star f(x)| \leq C|Q|^{-1/2}\left((t^\star_{r,\lambda})_{Q^\star} + (t^\star_{r,\lambda})_Q + (t^\star_{r,\lambda})_{Q^{\star\star}}\right)\chi_{Q^\star}(x)$$

for all $x \in Q^\star \subset Q \subset Q^{\star\star}$, where $Q^\star, Q$ and $Q^{\star\star}$ are dyadic cubes with $l(Q^\star) = 2^{-j-1}, l(Q) = 2^{-j}$ and $l(Q^{\star\star}) = 2^{-j+1}$. Let $r_0 = \inf\{\tilde{r} : w \in A_{\tilde{r}}\}$ and using Lemma 1.3 with the same λ, to obtain

$$||T_\psi t||_{\dot{F}^{\gamma,\tau}_{p,q,w}} = ||f||_{\dot{F}^{\gamma,\tau}_{p,q,w}} \preceq ||t||_{\dot{f}^{\gamma,\tau}_{p,q,w}},$$

which shows the boundedness of T_ψ.

Finally, if we assume additionally that φ and ψ satisfy 0.4 and 0.5, then, by Lemma 0.2 $T_\psi \circ S_\varphi$ is the identity on $\dot{F}^{\gamma,\tau}_{p,q,w}$. More precisely, $\dot{F}^{\gamma,\tau,\tilde{\varphi}}_{p,q,w} \hookrightarrow \dot{F}^{\gamma,\tau,\varphi}_{p,q,w}$ is a bounded inclusion. Hence, by reversing the roles of $\tilde{\varphi}$ and φ we have

$$\dot{F}^{\gamma,\tau,\tilde{\varphi}}_{p,q,w} = \dot{F}^{\gamma,\tau,\varphi}_{p,q,w},$$

which completes the proof of Theorem 1.1. □

Proposition 1.1. *The inclusion map $i : \dot{F}^{\gamma,\tau}_{p,q,w} \longrightarrow S'_\infty$ is continuous. Moreover, $\dot{F}^{\gamma,\tau}_{p,q,w}$ equipped with $||f||_{\dot{F}^{\gamma,\tau}_{p,q,w}}$ is a quasi-Banach space, i.e., $\dot{F}^{\gamma,\tau}_{p,q,w}$ is a complete quasi-normed space.*

Proof. Suppose that φ and ψ satisfy 0.4 and 0.5. By Lemma 1.2 the map $T_\psi : \dot{f}^{\gamma,\tau}_{p,q,w} \longrightarrow S'_\infty$ is continuous and by Theorem 1.1 the map $S_\varphi : \dot{F}^{\gamma,\tau}_{p,q,w} \longrightarrow \dot{f}^{\gamma,\tau}_{p,q,w}$ is also continuous. Hence, by Lemma 0.2 $i = T_\psi \circ S_\varphi : \dot{F}^{\gamma,\tau}_{p,q,w} \longrightarrow S'_\infty$ is a continuous inclusion. □

Theorem 1.2. *Let $w \in A_\infty$, $0 < p < \infty$, $0 < q \leq \infty$, $r_0 = \inf\{s \geq 1 : w \in A_s\}$ and $\delta > 0$ is as in Lemma 0.1. If $f \in \dot{F}^{\gamma,\tau}_{p,q,w}$, then, there exists a canonical way to find a representation of f s.t $f \in S'_L$, where $L \equiv max\,(-1, \lfloor\gamma + r_0 n(\tau - 1/p)\rfloor)$ if $\tau - 1/p \geq 0$ and $L \equiv max\,(-1, \lfloor\gamma + n(r_0 - \delta)(\tau - 1/p)\rfloor)$ if $\tau - 1/p < 0$.*
More precisely, assume for instance that $\tau - 1/p \geq 0$ and let $\varphi = \varphi_1$, $\psi = \psi_1 \in \mathcal{S}(\mathbb{R}^n)$ satisfying 0.4, 0.5 and 0.6. Then, there exists a sequence of polynomials $\{P^1_N\}^\infty_{N=1}$, with degree of each P^1_N no more than $L \equiv \lfloor\gamma + r_0 n(\tau - 1/p)\rfloor$ and $g_1 \in \mathcal{S}'(\mathbb{R}^n)$ s.t

$$g_1 = \lim_{N\to\infty}\left(\sum_{j=-N\leq j\leq N} \tilde{\psi_j} \star \varphi_j \star f + P^1_N\right) \quad in \quad \mathcal{S}'(\mathbb{R}^n). \tag{1.9}$$

Moreover, if g_2 is the corresponding limit in 1.9 for some other φ_2, ψ_2 satisfying the same conditions as φ_1, ψ_1, then,

$$g_1 - g_2 \quad \in \quad \mathcal{P} \quad and \quad deg(g_1 - g_2) \leq L. \tag{1.10}$$

We can take g_1 as a representation of the equivalent class $f + P(R^n)$ and we identify f with its representative g_1. In the sense, $f \in S'_L$, with

$$L \equiv max(-1, \lfloor\gamma + r_0 n(\tau - 1/p)\rfloor).$$

Similar conclusion holds whenever $\tau - 1/p < 0$ by taking

$$L \equiv max(-1, \lfloor\gamma + (r_0 - \delta)n(\tau - 1/p)\rfloor).$$

The proof of Theorem 1.2 is based on some several technical lemma's.

Lemma 1.5. *Let $\nu \in S(\mathbb{R}^n)$ with supp $\hat{\nu} \subset \{\xi : 1/2 < |\xi| \leq 2\}$. Suppose $f \in \dot{F}^{\gamma,\tau}_{p,q,w}$, then, for any $0 < r < p/r_0$ and for any multi-indices β*

$$sup_{z\in Q_{jk}}|\partial^\beta(\nu_j f(z))|^r \leq C2^{jn}2^{jr|\beta|}2^{-j\gamma r}||f||^r_{\dot{F}^{\gamma,\tau}_{p,q,w}} \times \left(\int_{B(2^{-j}k,2^{-j})} w(x)dx\right)^{r\tau}\left(\int_{B(2^{-j}k,2^{-j})} w^{-\frac{r}{p-r}}dx\right)^{1-r/p}.$$

Proof. Notice that *supp* $\hat{\nu_j} f \subset \{\xi : |\xi| \leq 2^{j+1}\}$. It follows from the proof of the Lemma 2.4 in [13, p.782] that for any $r > 0$, for any multi-indices β and $N > 0$ sufficiently large there exists $C = C_{r,M,\beta,N} > 0$ s.t,

$$sup_{z\in Q_{jk}}|\partial^\beta(\nu_j f(z))|^r \leq C2^{jn}2^{jr|\beta|}\sum_{j\in\mathbb{Z}}(1+|l|)^{-N}\int_{Q_{j,k+l}}|\nu_j f(x)|^r dx. \tag{1.11}$$

Let $r > 0$ be s.t $w \in A_{p/r}$. Hölder's inequality implies

$$\begin{aligned} sup_{z\in Q_{jk}}|\partial^\beta(\nu_j f(z))|^r &\leq C2^{jn}2^{jr|\beta|}\sum_{l\in\mathbb{Z}^n}(1+|l|)^{-N} \\ &\times \Big(\int_{Q_{j,k+l}}|\nu_j f(x)|^p w(x)dx\Big)^{r/p}\Big(\int_{Q_{j,k+l}} w^{-\frac{r}{p-r}}dx\Big)^{1-r/p} \\ &\leq C2^{jn}2^{jr|\beta|}2^{-j\gamma r}||f||^r_{\dot{F}^{\gamma,\tau}_{p,q,w}}\sum_{l\in\mathbb{Z}^n}(1+|l|)^{-N} \\ &\times \Big(\int_{Q_{j,k+l}} w(x)dx\Big)^{r\tau}\Big(\int_{Q_{j,k+l}} w^{-\frac{r}{p-r}}dx\Big)^{1-r/p}. \end{aligned} \tag{1.12}$$

Noting that $Q_{j,k+l} \subset B(2^{-j}k, \sqrt{n}2^{-j}(1+|l|))$. It follows that

$$\begin{aligned} &\Big(\int_{Q_{j,k+l}} w(x)dx\Big)^{r\tau}\Big(\int_{Q_{j,k+l}} w^{-\frac{r}{p-r}}dx\Big)^{1-r/p} \\ &\leq C(1+|l|)^{n\delta(r\tau+1-r/p)}\Big(\int_{B(2^{-j}k,2^{-j})} w(x)dx\Big)^{r\tau}\Big(\int_{B(2^{-j}k,2^{-j})} w^{-\frac{r}{p-r}}dx\Big)^{1-r/p}. \end{aligned}$$

Choose $N \geq n\delta(r\tau + 1 - r/p) + n + 1$ and using 1.12 to finish the proof. ☐

Remark 1.2. *Lemma 1.5 corresponds to unweighted version of Lemma 2.4 in* [41].

Corollary 1.2. *Under the same assumptions in the lemma 1.5, there exists $N > 0$ s.t for all j and k*

$$sup_{z\in Q_{jk}}|\nu_j f(z)|^r \leq C2^{jn}2^{-j\gamma r}2^{-min(j,0)N}||f||^r_{\dot{F}^{\gamma,\tau}_{p,q,w}}\inf_{x\in Q_{jk}}(1+|x|)^N. \tag{1.13}$$

Consequently, there exist $C, N > 0$ s.t

$$sup_{x\in\mathbb{R}^n}\frac{|\nu_j f(x)|}{(1+|x|)^N} \leq c^{|j|+1}||f||_{\dot{F}^{\gamma,\tau}_{p,q,w}}. \tag{1.14}$$

Proof. Fix $x \in Q_{jk}$. Then, we have the following elementary inclusions

$$\begin{aligned} B(2^{-j}k, \sqrt{n}2^{-j}) &\subset B(0, \sqrt{n}2^{-j}(1+|k|)) \subset B(0, c_n 2^{-j}(1+2^j|x|)) \\ &\subset B(0, c_n 2^{-min(j,0)}(1+|x|)), \end{aligned} \tag{1.15}$$

for some constant $c_n > 1$. Applying Lemma 0.1 and 1.15, we obtain

$$\begin{aligned} \int_{B(2^{-j}k,2^{-j})} w(y)dy &\preceq \int_{B(0,c_n 2^{-min(j,0)}(1+|x|))} w(y)dy \\ &\preceq 2^{-min(j,0)np/r}(1+|x|)^{np/r}\int_{B(0,1)} w(y)dy \end{aligned}$$

12

and similarly

$$\int_{B(2^{-j}k,2^{-j})} w^{-\frac{r}{p-r}} dy \preceq 2^{-min(j,0)np/(p-r)}(1+|x|)^{np/(p-r)} \int_{B(0,1)} w^{-\frac{r}{p-r}} dy.$$

Thus, by Lemma 1.5

$$sup_{z\in Q_{jk}}|\nu_j f(z)|^r \leq C2^{jn}2^{-j\gamma r}2^{-min(j,0)n(p\tau+1)}(1+|x|)^{n(p\tau+1)}||f||^r_{\dot{F}^{\gamma,\tau}_{p,q,w}}. \tag{1.16}$$

Taking $N = n(p\tau + 1)$ to finish the proof. □

Remark 1.3. *It is interesting to note that Corollary 1.2 implies in a direct way that the inclusion* i : $\dot{F}^{\gamma,\tau}_{p,q,w} \to S_\infty(R^n)$ *is continuous and this can be very useful in other circumstances.*

In fact. Recall first that $S_\infty(\mathbb{R}^n)$ can be defined as a collection of $\nu \in S(\mathbb{R}^n)$ such that semi-norms

$$||\nu||_M = sup_{|\beta|\leq M} sup_{\xi\in R^n}|\partial^\beta \hat{\nu})|(|\xi|^M + |\xi|^{-M}) < \infty \quad \text{for any } M \in \mathbb{N}. \tag{1.17}$$

Moreover, semi-norms $||_{M:M\in N}$ generate a topology of a locally convex space on $S_\infty(\mathbb{R}^n)$ which coincides with the topology of $S_\infty(\mathbb{R}^n)$ as a subspace of a locally convex space $S(\mathbb{R}^n)$. Thus, the proof of the continuity of inclusion i is equivalent to prove

$$|\langle \nu f, \phi\rangle| \leq C||f||_{\dot{F}^{\gamma,\tau}_{p,q,w}}||\phi||_M. \tag{1.18}$$

As a consequence of Corollary 1.2, there exist $c, N > 0$ s.t for any $j \in Z$ and $\phi \in S_\infty(\mathbb{R}^n)$, we have

$$\begin{aligned} |\langle \nu_j f, \phi\rangle| &\leq c^{|j|+1}||f||_{\dot{F}^{\gamma,\tau}_{p,q,w}}||(1+|x|)^N \phi||_\infty \\ &\leq c^{|j|+1}||f||_{\dot{F}^{\gamma,\tau}_{p,q,w}} sup_{|\alpha|\leq n+1,|\beta|\leq N}||\hat{\phi}||_{\alpha,\beta}. \end{aligned} \tag{1.19}$$

Let $\mu \in S(\mathbb{R}^n)$ be such that $\hat{\mu}(\xi) = 1$ for all $\xi \in supp\ \hat{\nu}$ and $supp\ \hat{\mu} \subset \{\xi : 1/2 < |\xi| < 2\}$ and replace the semi-norms $||\phi||_M$ in 1.19 by $||\hat{\mu}(2^{-j}.)\hat{\phi}||_{\alpha,\beta}$ to get

$$|\langle \nu_j f, \phi\rangle| = |\langle \nu_j f, \mu_j \phi\rangle| \leq c^{|j|+1}|||f||_{\dot{F}^{\gamma,\tau}_{p,q,w}} sup_{|\alpha|\leq n+1,|\beta|\leq N}||\hat{\mu}(2^{-j}.)\hat{\phi}||_{\alpha,\beta}.$$

On the other hand, we have, for any $\lambda > 0$ there exists $M > 0$ such that

$$||\hat{\mu}(2^{-j}.)\hat{\phi}||_{\alpha,\beta} \leq C2^{-|j|\lambda}||\phi||_M.$$

For more details of these kinds of estimates, see the proof of [3, Lemma 2.6.]. Combining the last two estimates we deduce the existence of $M > 0$ and $\lambda_1 > 0$ such that

$$|\langle \nu_j f, \phi\rangle| \leq |C|2^{-|j|\lambda_1}||f||_{\dot{F}^{\gamma,\tau}_{p,q,w}}||\phi||_M. \tag{1.20}$$

Now, the estimate 1.20 implies

$$\sum_{j\in\mathbb{Z}}|\langle \nu_j f, \phi\rangle| \leq C\sum_{j\in\mathbb{Z}} 2^{-|j|\lambda_1}||f||_{\dot{F}^{\gamma,\tau}_{p,q,w}}||\phi||_M \leq C||f||_{\dot{F}^{\gamma,\tau,\nu}_{p,q,w}}||\phi||_M \simeq ||f||_{\dot{F}^{\gamma,\tau}_{p,q,w}}||\phi||_M. \tag{1.21}$$

Let $\phi \in S(\mathbb{R}^n)$ and $\psi, \varphi \in S_\infty(\mathbb{R}^n)$ satisfying 0.4, 0.5 and 0.6. Then, using 1.21 with $\nu = \psi \star \varphi$ and Lemma 0.2 to get

$$|\langle f, \phi\rangle| = |\langle \sum_{j\in\mathbb{Z}} \nu_j f, \phi\rangle| \leq C||f||_{\dot{F}^{\gamma,\tau}_{p,q,w}}||\phi||_M. \tag{1.22}$$

□

Proof of Theorem 1.2. Let $\psi = \psi_1,\ \varphi = \varphi_1 \in \mathcal{S}(\mathbb{R}^n)$ satisfying 0.4, 0.5 and 0.6. We claim that for $f \in \dot{F}^{\gamma,\tau}_{p,q,w}$, the series $\sum_{j\geq 0} \tilde{\psi}_j \star \varphi_j \star f$ converges in $S'(\mathbb{R}^n)$. To see this, we need the following estimate, see [39], there exists $M \in \mathbb{N}$, s.t for all $\phi \in S(\mathbb{R}^n)$, $j \in N$ and $x \in \mathbb{R}^n$,

$$|\psi_j \star \phi(x)| \preceq ||\phi||_{\mathcal{S}_{M+1}}||\psi||_{S_{M+1}}2^{-jM}\frac{1}{(1+|x|)^{n+M}}$$

13

which together with 1.14 imply that there exists some constant $M_1 > 0$ s.t,

$$\sum_{j\geq 0} |\langle \tilde{\psi}_j \star \varphi_j \star f, \phi\rangle| \preceq ||\phi||_{\mathcal{S}_{M+1}} ||\psi||_{\mathcal{S}_{M+1}} \sum_{j\geq 0} 2^{-jM} \int_{\mathbb{R}^n} \frac{|\varphi_j \star f(x)|}{(1+|x|)^{n+M}} dx$$
$$\preceq ||\phi||_{\mathcal{S}_{M+1}} ||\psi||_{\mathcal{S}_{M+1}} \sum_{j\geq 0} 2^{-jM_1} ||f||_{\dot{F}^{\gamma,\tau}_{p,q,w}}$$
$$\preceq ||\phi||_{\mathcal{S}_{M+1}} ||\psi||_{\mathcal{S}_{M+1}} ||f||_{\dot{F}^{\gamma,\tau}_{p,q,w}}, \tag{1.23}$$

hence $\sum_{j\geq 0} \tilde{\psi}_j \star \varphi_j \star f$ converges in $\mathcal{S}'(\mathbb{R}^n)$. Now, let $j \in \mathbb{Z}_-$ and $x \in Q_{jk}$, using again the inclusion 1.15 and the A_{r_0}-condition, to obtain

$$\int_{B(2^{-j}k,2^{-j})} w(y)dy \preceq (1+|x|)^{-n\delta} \int_{B(0,2^{-j}(1+|x|)} w(y)dy$$
$$\preceq 2^{-jnr_0}(1+|x|)^{n(r_0-\delta)} \int_{B(0,1)} w(y)dy \simeq 2^{-jnr_0}(1+|x|)^{n(r_0-\delta)}$$

On the other hand, since $j \in \mathbb{Z}_-$ we have $B(0,1) \subset B(0, 2^{-j}(1+|x|)$. Then,

$$\int_{B(0,1)} w(y)dy \preceq 2^{jn\delta}(1+|x|)^{-n\delta} \int_{B(0,2^{-j}(1+|x|))} w(y)dy$$
$$\preceq 2^{jn(\delta-r_0)}(1+|x|)^{n(r_0-\delta)} \int_{B(2^{-j}k,2^{-j})} w(y)dy.$$

It follows from the last inequalities and A_{r_0} properties that

$$\Big(\int_{B(2^{-j}k,2^{-j})} w(x)dx\Big)^{r\tau}\Big(\int_{B(2^{-j}k,2^{-j})} w^{-\frac{r}{p-r}}dx\Big)^{1-r/p} \preceq 2^{-jn}\Big(\int_{B(2^{-j}k,2^{-j})} w(x)dx\Big)^{r(\tau-1/p)}$$
$$\preceq 2^{-jn}2^{-jnr_0r(\tau-1/p)}(1+|x|)^{nr_0r(\tau-1/p)}$$

if $\tau - 1/p \geq 0$ and

$$\Big(\int_{B(2^{-j}k,2^{-j})} w(x)dx\Big)^{r\tau}\Big(\int_{B(2^{-j}k,2^{-j})} w^{-\frac{r}{p-r}}dx\Big)^{1-r/p}$$
$$\preceq 2^{-jn}2^{-jnr(\delta-r_0)(\tau-1/p)}(1+|x|)^{-nr(r_0-\delta)(\tau-1/p)},$$

if $\tau - 1/p < 0$. Using Corollary 1.2 to conclude that

(1) if $\tau - 1/p \geq 0$, then

$$\sup_{x\in\mathbb{R}^n} |\partial^\beta(\tilde{\psi}_j \star \varphi_j \star f(x))|(1+|x|)^{-n(r_0-\delta)(\tau-1/p)} \preceq 2^{j(|\beta|-\gamma-nr_0(\tau-1/p))}||f||_{\dot{F}^{\gamma,\tau}_{p,q,w}} \tag{1.24}$$

(2) and if $\tau - 1/p < 0$, then

$$\sup_{x\in\mathbb{R}^n} |\partial^\beta(\tilde{\psi}_j \star \varphi_j \star f(x))|(1+|x|)^{n(r_0-\delta)(\tau-1/p)} \preceq 2^{j(|\beta|-\gamma-n(r_0-\delta)(\tau-1/p))}||f||_{\dot{F}^{\gamma,\tau}_{p,q,w}}. \tag{1.25}$$

Therefore, by 1.24 and 1.25 $\sum_{j<0} \partial^\beta(\tilde{\psi}_j \star \varphi_j \star f(x))$ converges in $S'(\mathbb{R}^n)$ whenever

(1) $\tau - 1/p \geq 0$ and $\beta > \gamma + nr_0(\tau - 1/p) > 0$
or
(2) $\tau - 1/p < 0$ and $\beta > \gamma + n(r_0 - \delta)(\tau - 1/p) > 0$.

14

Consequently, for $\tau - 1/p \geq 0$, Proposition 2.7 in [3] (see also [12] pp. 153-155) yields a sequence of polynomials $\{P_N^1\}_{N=1}^{\infty}$, with degree no more than $L = \lfloor \gamma + r_0 n(\tau - 1/p) \rfloor$ and $g \in S'(\mathbb{R}^n)$ s.t

$$g_1 = \lim_{N\to\infty} \left(\sum_{j=-N\leq j\leq N} \tilde{\psi}_j \star \varphi_j \star f \quad + P_N^1 \right) \quad in \quad \mathcal{S}'(\mathbb{R}^n). \tag{1.26}$$

Similar conclusion holds whenever $\tau - 1/p < 0$ by taking

$$L = max(-1, \lfloor \gamma + (r_0 - \delta)n(\tau - 1/p) \rfloor),$$

The proof of 1.10 is very similar to the proof of [3, Proposition 3.8]. We deduce that there exists a sequence $\{P_N\}_{N=1}^{\infty}$ of polynomials, with degree no more than L same as above such that

$$g = \lim_{N\to\infty} \left(\sum_{j=-N\leq j\leq N} \tilde{\psi}_j \star \varphi_j \star f + P_N \right) \quad in \quad \mathcal{S}'(\mathbb{R}^n)$$

and g is a representation of the equivalent class $f + P(R^n)$. We identify f with its representative g. In this sense, $f \in S'_L$, which completes the proof of Theorem 1.2. □

Remark 1.4. *If we assume that $w \in A_1$ then, we can take $\delta = 0$ and $r_0 = 1$ so that $L = max(-1, \lfloor \gamma + n(\tau - 1/p) \rfloor)$ in both case. This corresponds to the result in [23] when $w = 1$.*

Corollary 1.3. *Let $w \in A_\infty$ and $f \in \dot{F}^{\gamma,\tau}_{p,q,w}$ with $0 < p < \infty$, $0 < q \leq \infty$. Let φ_1, τ and L as in Theorem 1.2. Then, there exists a sequence $\{P_N\}_{N=1}^{\infty}$ of polynomials with $degP_N \leq L$ and $g_1 \in S'(\mathbb{R}^n)$ s.t such that*

$$g_1 = \lim_{k\to\infty} \left(\sum_{2^{-nk}\leq l(Q)\leq 2^{nk}} \langle f, \varphi_Q \rangle \psi_Q + P_k \right) \quad in \quad \mathcal{S}'(\mathbb{R}^n). \tag{1.27}$$

Moreover, if g_2 is the corresponding limit in 1.27 for some other φ_2, ψ_2 satisfying the same conditions as φ_1, ψ_1, then,

$$g_1 - g_2 \quad \in \quad \mathcal{P} \quad and \quad deg(g_1 - g_2) \leq L. \tag{1.28}$$

2. Almost Diagonal Operators

In this section, we study the class of almost diagonal operators on $\dot{f}^{\gamma,\tau}_{p,q,w}$ which was introduced by Frazier and Jawerth [12]. The interest of these operators on $\dot{f}^{\gamma,\tau}_{p,q,w}$ arises from their close connection to many operators in analysis. For a quasi-Banach space X, let $\mathcal{L}(X)$ be the space of bounded linear operators on X with the operator norm. Define the maps $S^\star_\varphi : \mathcal{L}(\dot{F}^{\gamma,\tau}_{p,q,w}) \longrightarrow \mathcal{L}(\dot{f}^{\gamma,\tau}_{p,q,w})$ and $T^\star_\psi : \mathcal{L}(\dot{f}^{\gamma,\tau}_{p,q,w}) \longrightarrow \mathcal{L}(\dot{F}^{\gamma,\tau}_{p,q,w})$ by

$$\begin{aligned} S^\star_\varphi =& S_\varphi \circ \mathcal{B} \circ T_\psi, \quad for \quad \mathcal{B} \in \mathcal{L}(\dot{F}^{\gamma,\tau}_{p,q,w}) \\ T^\star_\psi =& T_\psi \circ \mathcal{A} \circ S_\varphi, \quad for \quad \mathcal{A} \in \mathcal{L}(\dot{f}^{\gamma,\tau}_{p,q,w}). \end{aligned}$$

Repeating verbatim the arguments in [12, Section 3] and using Theorem 1.1 to obtain the following commutative diagram:

$$\begin{array}{ccc} \mathcal{L}(\dot{F}^{\gamma,\tau}_{p,q,w}) & \xrightarrow{id} & \mathcal{L}(\dot{F}^{\gamma,\tau}_{p,q,w}) \\ {\scriptstyle S^\star_\varphi}\downarrow & \nearrow {\scriptstyle T^\star_\psi} & \\ \mathcal{L}(\dot{f}^{\gamma,\tau}_{p,q,w}) & & \end{array} \tag{2.1}$$

Moreover, if $0 < q < \infty$, then any $A \in \mathcal{L}(\dot{f}^{\gamma,\tau}_{p,q,w})$ is represented by a matrix $\{a_{QP}\}_{Q,P\in\mathcal{Q}}$ where $a_{QP} = (\mathcal{A}e^P)_Q$. Here, e^P denotes the standard unit vector in $\dot{f}^{\gamma,\tau}_{p,q,w}$ defined by $(e^P)_Q = 1$ if $Q = P$ and $(e^P)_Q = 0$ otherwise.

Definition 2.1. *Let $\gamma \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, $w \in A_\infty$ and $J = n \times max(1, r_0/p, 1/q)$. We say that an operator $\mathcal{A}$ with an associated matrix $\{a_{QP}\}_{Q,P\in\mathcal{Q}}$ where $a_{QP} = (\mathcal{A}e^P)_Q$ is an ϵ-almost diagonal operator on $\dot{f}^{\gamma,\tau}_{p,q,w}$, if there exists an $\epsilon > 0$ such that*

$$\sup_{P,Q\in\mathcal{Q}} |a_{PQ}|/\kappa_{QP}(\epsilon) < \infty,$$

where

$$\kappa_{QP}(\epsilon) = \left(\frac{l(P}{l(Q)}\right)^{\gamma} \left(1 + \frac{|x_Q - x_P|}{max(l(P), l(Q))}\right)^{-J-\epsilon}$$
$$\times\, min\left[\left(\frac{l(Q)}{l(P))}\right)^{(n+\epsilon)/2}, \left(\frac{l(P)}{l(Q)}\right)^{(n+\epsilon)/2+J-n}\right].$$

We note that an almost diagonal in our case is also an almost diagonal operator on the classical space $\dot{f}^{\gamma}_{p,q}$ introduced by Frazier and Jawerth [12] and is an almost diagonal operator on the general space $\dot{f}^{\gamma,\tau}_{p,q}$ introduced by Yang and Yuan[39]. Moreover, Frazier and Jawerth proved that all almost diagonal operators are bounded on $\dot{f}^{\gamma}_{p,q}$. This result is extended by Yang and Yuan [39] to the space $\dot{f}^{\gamma,\tau}_{p,q}$, see also [3, 1, 2]. In the weighted spaces, we have the following conclusion.

Theorem 2.1. *Let $\epsilon > 0$, $\gamma \in \mathbb{R}$, $0 < p, q \leq \infty, w \in A_\infty$ and $\tau \in [0, 1/p + \epsilon/(2nr_0)[$. Then, all ϵ-almost diagonal operators on $\dot{f}^{\gamma,\tau}_{p,q,w}$ are bounded on $\dot{f}^{\gamma,\tau}_{p,q,w}$.*

The proof of Theorem 2.1 is partially based on the following Lemma, which is simple consequence of Lemma 4.1 in [1] and the estimate D.1 in [12].

Lemma 2.1. *Let $i,\ j \in \mathbb{Z}$ and $Q \in \mathcal{Q}$ with $l(Q) = 2^{-j}$. Then, for any $L > n$,*

$$\sum_{l(R)=2^{-i}} \left(1 + \frac{|x_R - x_Q|}{max(l(R), l(Q))}\right)^{-L} \leq C2^{n(i-j)_+}$$

where the constant C depends only on L and n, here $(i-j)_+ = max(i-j, 0)$.

Proof of Theorem 2.1. We consider only the case $\tau - 1/p \geq 0$. If $\tau - 1/p < 0$ then, Theorem 4.2 and similar argument in [12] leads to the result. Without loss of generality, we may assume $\gamma = 0$ and $min(p/r_0, q) > 1$ (see for instance [3, 23, 40]). Let $t = \{t_Q\}_Q \in \dot{f}^{\gamma,\tau}_{p,q,w}$ and $\mathcal{A}$ is an ϵ-almost diagonal operator on $\dot{f}^{\gamma,\tau}_{p,q,w}$ with an associated matrix $\{a_{QP}\}_{Q,P\in\mathcal{Q}}$.
Since $min(p/r_0, q) > 1$, we have $J = n$ in Definition 2.1. Write $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$, with

$$(\mathcal{A}_0 t)_Q = \sum_{l(R)\geq l(Q)} a_{QR}t_R \ \text{ and } \ (\mathcal{A}_1 t)_Q = \sum_{l(R)<l(Q)} a_{QR}t_R.$$

By definition 2.1, we see that for all $Q \in \mathcal{Q}$

$$|(\mathcal{A}_0 t)_Q| \preceq \sum_{l(R)>l(Q)} \left(\frac{l(Q)}{l(R)}\right)^{(n+\epsilon)/2} |t_R| \left(1 + \frac{|x_R - x_Q|}{l(R)}\right)^{-n-\epsilon}.$$

16

Therefore

$$||\mathcal{A}_0 t||_{\dot{f}^{0,\tau}_{p,q,w}}$$
$$\preceq \sup_{P\in\mathcal{Q}} \frac{1}{[w(P)]^\tau}\left[\int_P \left(\sum_{Q\subset P}\left(\sum_{l(R)>l(Q)}\left(\frac{l(Q)}{l(R)}\right)^{(n+\epsilon)/2}|t_R|\tilde{\chi}_Q\left(1+\frac{|x_R-x_Q|}{l(R)}\right)^{-n-\epsilon}\right)^q\right)^{\frac{p}{q}} w(x)dx\right]^{\frac{1}{p}}$$
$$= J_1$$

and

$$||\mathcal{A}_1 t||_{\dot{f}^{0,\tau}_{p,q,w}}$$
$$\preceq \sup_{P\in\mathcal{Q}} \frac{1}{[w(P)]^\tau}\left[\int_P \left(\sum_{Q\subset P}\left(\sum_{l(R)\le l(Q)}\left(\frac{l(R)}{l(Q)}\right)^{(n+\epsilon)/2}|t_R|\tilde{\chi}_Q\left(1+\frac{|x_R-x_Q|}{l(Q)}\right)^{-n-\epsilon}\right)^q\right)^{\frac{p}{q}} w(x)dx\right]^{\frac{1}{p}}$$
$$= J_2.$$

Let $Q,P,R\in\mathcal{Q}$ be such that $Q\subset P$, $l(Q)=2^{-j}$, $l(P)=j_P$, $l(R)=2^{-i}$ and assume $i\le j$. Then, by $R\subset B=B\left(x_Q, c_n 2^{-i}\left(1+\frac{|x_R-x_Q|}{l(R)}\right)\right)=B\left(x_Q, c_n 2^{-j}2^{j-i}\left(1+\frac{|x_R-x_Q|}{l(R)}\right)\right)$ with $c_n>1$, we have using the r_0-doubling condition , the δ-reverse doubling condition of w and the assumption $\tau-1/p\ge 0$,

$$w(R)^{\tau-1/p}\le w(B)^{\tau-1/p}\preceq 2^{((j-i)(\tau-1/p)nr_0}\left(1+\frac{|x_R-x_Q|}{l(R)}\right)^{(\tau-1/p)nr_0} w(B(x_Q,2^{-j})^{\tau-1/p}$$
$$\preceq 2^{((j-i)(\tau-1/p)nr_0}\left(1+\frac{|x_R-x_Q|}{l(R)}\right)^{(\tau-1/p)nr_0} w(Q)^{\tau-1/p}$$
$$\preceq 2^{((j-i)(\tau-1/p)nr_0}\left(1+\frac{|x_R-x_Q|}{l(R)}\right)^{(\tau-1/p)nr_0} 2^{((j_P-j)(\tau-1/p)n\delta}w(P)^{\tau-1/p}.$$

Since $|t_R|\preceq |R|^{1/2}w(R)^{\tau-1/p}||t||_{\dot{f}^{0,\tau}_{p,q,w}}$ (see 1.3), we have,

$$|t_R|\preceq 2^{-ni/2}2^{((j-i)(\tau-1/p)nr_0}\left(1+\frac{|x_R-x_Q|}{l(R)}\right)^{(\tau-1/p)nr_0} 2^{(j_P-j)(\tau-1/p)n\delta}w(P)^{\tau-1/p}. \tag{2.2}$$

Put

$$I_1=\sum_{l(R)>l(Q)}\left(\frac{l(Q)}{l(R)}\right)^{(n+\epsilon)/2}|t_R|\tilde{\chi}_Q(x)\left(1+\frac{|x_R-x_Q|}{l(R)}\right)^{-n-\epsilon}$$

and using 2.2 to get

$$I_1\preceq \chi_Q(x)2^{(j_P-j)(\tau-1/p)n\delta}w(P)^{\tau-1/p}||t||_{\dot{f}^{0,\tau}_{p,q,w}}$$
$$\times\sum_{i=-\infty}^{j-1}\sum_{l(R)=2^{-i}}2^{(i-j)(\epsilon/2-nr_0(\tau-1/p))}\left(1+\frac{|x_R-x_Q|}{l(R)}\right)^{-n-\epsilon+(\tau-1/p)nr_0}.$$

By $L=n+\epsilon-(\tau-1/p)nr_0>n$, Lemma 2.1 implies

$$I_1\preceq \chi_Q(x)2^{(j_P-j)(\tau-1/p)n\delta}w(P)^{\tau-1/p}||t||_{\dot{f}^{0,\tau}_{p,q,w}}\sum_{i=-\infty}^{j-1}2^{(i-j)(\epsilon/2-nr_0(\tau-1/p))}$$
$$\preceq \chi_Q(x)2^{(j_P-j)(\tau-1/p)n\delta}w(P)^{\tau-1/p}||t||_{\dot{f}^{0,\tau}_{p,q,w}},$$

since by assumption, $\epsilon/2-nr_0(\tau-1/p)>0$. It follows that

$$||\mathcal{A}_0 t||_{\dot{f}^{0,\tau}_{p,q,w}}\preceq J_1\preceq ||t||_{\dot{f}^{0,\tau}_{p,q,w}}.$$

17

Some similar estimates to I_1 also yield that

$$||\mathcal{A}_1 t||_{\dot{f}^{0,\tau}_{p,q,w}} \preceq J_2 \preceq ||t||_{\dot{f}^{0,\tau}_{p,q,w}}.$$

□

3. Atomic and Molecular decomposition

3.1. **Smooth molecules.** We establish in this section, smooth atomic and molecular decomposition characterizations of $\dot{F}^{\gamma,\tau}_{p,q,w}$. As in the introduction we set $s^\star = s - \lfloor s \rfloor$ for $s \in \mathbb{R}$.

Definition 3.1. *Let $\gamma \in \mathbb{R}$, $\tau \in [0, \infty[$, $0 < p < \infty$, $0 < q \leq \infty$, $w \in A_\infty$, $J = n \times max(1, r_0/p, 1/q)$ and $N = max(\lfloor J - n - \gamma \rfloor, -1)$.*

(1) A function m_Q is called a smooth synthesis molecule for $\dot{F}^{\gamma,\tau}_{p,q,w}$ supported near dyadic cube Q if there exist $\sigma \in]max(\gamma^\star, (\gamma + n\tau)^\star), 1]$ and $M \in]J, \infty[$ such that

$$\int_Q x^\alpha m_Q(x) = 0; \quad if\ |\alpha| \leq N \tag{3.1}$$

$$|m_Q(x)| \leq |Q|^{-1/2} \left(1 + \frac{|x - x_Q|}{l(Q)}\right)^{-max(M, M-\gamma)} \tag{3.2}$$

$$|\partial^\alpha m_Q(x)| \leq |Q|^{-1/2-|\alpha|/n} \left(1 + \frac{|x - x_Q|}{l(Q)}\right)^{-M} \quad ; \quad if \quad |\alpha| \leq \lfloor \gamma + n\tau \rfloor \tag{3.3}$$

and

$$|\partial^\alpha m_Q(x) - \partial^\alpha m_Q(y)| \leq |Q|^{-1/2-|\alpha|/n-\sigma/n} |x - y|^\sigma \sup_{|z|<|x-y|} \left(1 + \frac{|x - z - x_Q|}{l(Q)}\right)^{-M} \tag{3.4}$$

if $|\alpha| = \lfloor \gamma + n\tau \rfloor$.

We say that a collection $\{m_Q\}_Q$ is a family of smooth synthesis molecules, if each m_Q is a smooth synthesis molecule supported near Q.

(2) A function b_Q is called a smooth analysis molecule for $\dot{F}^{\gamma,\tau}_{p,q,w}$ supported near dyadic cube Q if there exist an $M \in]J, \infty[$ and $\rho \in](J - \gamma)^\star, 1]$ such that

$$\int_Q x^\alpha b_Q(x) = 0; \quad if\ |\alpha| \leq \lfloor \gamma + n\tau \rfloor \tag{3.5}$$

$$|b_Q(x)| \leq |Q|^{-1/2} \left(1 + \frac{|x - x_Q|}{l(Q)}\right)^{-max(M, M+n+\gamma+n\tau-J)} \tag{3.6}$$

$$|\partial^\alpha b_Q(x)| \leq |Q|^{-1/2-\alpha/n} \left(1 + \frac{|x - x_Q|}{l(Q)}\right)^{-M} ; \quad if\ |\alpha| \leq N \tag{3.7}$$

and

$$|\partial^\alpha b_Q(x) - \partial^\alpha b_Q(y)| \leq |Q|^{-1/2-|\alpha|/n-\rho/n} |x - y|^\rho \sup_{|z|<|x-y|} \left(1 + \frac{|x - z - x_Q|}{l(Q)}\right)^{-M} ; \tag{3.8}$$

if $|\alpha| = N$.

We say that a collection $\{m_Q\}_Q$ is a family of smooth analysis molecules, if each m_Q is a smooth analysis molecule supported near Q.

Note that when $N = -1$ then, 3.1, 3.7 and 3.8 are void.

18

Lemma 3.1. *Let γ, p, q, J, M and N as in Definition 3.1, $(J-\gamma)^\star < \varrho \leq 1$ and $\sigma \in]max(\gamma^\star, (\gamma + n\tau)^\star), 1]$. Assume $\tau \in \left[0, min(\frac{1}{p} + \frac{M-J}{2nr_0}, \frac{1}{p} + \frac{\rho-(J-\gamma)^\star}{nr_0})\right]$ if $N \geq 0$ and $\tau \in \left[0, min(\frac{1}{p} + \frac{M-J}{2nr_0}, \frac{1}{p} + \frac{\gamma+n-J}{nr_0})\right]$ if $N < 0$. Suppose $\{b_Q\}_Q$ and $\{m_Q\}_Q$ are families of smooth analysis and synthesis molecules for $\dot{F}^{\gamma,\tau}_{p,q,w}$, respectively. Then, there exists, $\epsilon > 2nr_0\left(\tau - \frac{1}{p}\right)$ s.t the matrix a_{QP}, given by $a_{QP} = \langle m_Q, b_P\rangle$, is almost diagonal. More precisely, there exist $C > 0$ and $\epsilon > 2nr_0\left(\tau - \frac{1}{p}\right)$, such that*

$$|a_{QP}| \leq C\kappa_{QP}(\epsilon). \tag{3.9}$$

As an immediate consequence, we obtain the following two corollaries, see also [3, Corollaries 5.2 and 5.3].

Corollary 3.1. *Let γ, p, q, τ and ϵ be as in Lemma 3.1. Suppose $\{m_Q\}_Q$ is a family of smooth synthesis molecules for $\dot{F}^{\gamma,\tau}_{p,q,w}$ and $b \in \mathcal{S}(\mathbb{R}^n)$ with $0 \notin supp\, \hat{b}$. Then, the matrix $\{a_{QP}\}$, given by $a_{QP} = \langle m_Q, b_P\rangle$, is ϵ-almost diagonal.*

Corollary 3.2. *Let γ, p, q, τ and ϵ be as in Lemma 3.1. Suppose $\{b_Q\}_Q$ is a family of smooth analysis molecules for $\dot{F}^{\gamma,\tau}_{p,q,w}$ and $m \in \mathcal{S}(\mathbb{R}^n)$ with $0 \notin supp\, \hat{a}$. Then, the matrix $\{a_{QP}\}$, given by $a_{QP} = \langle b_Q, a_P\rangle$, is is ϵ-almost diagonal.*

We will also need the following result, which provides an approximation of smooth molecules by elements of the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. See [3, Section 5].

Lemma 3.2. *Suppose that ϕ is a smooth analysis (or synthesis) molecule supported near $Q \in \mathcal{Q}$. Then, there exists a sequence $\{\phi_k\}_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$ and $c > 0$ such that $c\phi_k$ is a smooth analysis (or synthesis) molecule supported near Q for every k, and $\phi_k \longrightarrow \phi$ uniformly on $\mathbb{R}^n$ as $k \longrightarrow \infty$.*

To prove Lemma 3.1, we need the following additional results. See [12, Appendix B, Lemma B1].

Lemma 3.3. *Let $L \in \mathbb{N}$, $R > n$, $0 < \theta \leq 1$, $S > n + L + \theta$, $i, j \in \mathbb{Z}, i \geq j$, and $x_0 \in \mathbb{R}^n$. Suppose that $g,\ h; \in C^L(\mathbb{R}^n)$ satisfy*

$$|\partial^\alpha g(x)| \leq 2^{j(\alpha+n/2)}\left(1 + 2^j|x|\right)^{-R}; \quad if\ |\alpha| \leq L, \tag{3.10}$$

$$|\partial^\alpha g(x) - \partial^\alpha g(y)| \leq 2^{j(n/2+L+\theta)}|x-y|^\theta \sup_{|z|<|x-y|}\left(1 + 2^j|x-z|\right)^{-R}; \tag{3.11}$$

if $|\alpha| = L$,

$$|h(x)| \leq 2^{ni/2}\left(1 + 2^i|x - x_0|\right)^{-max(R,S)}, \tag{3.12}$$

(3.13) *and*

$$\int_{\mathbb{R}^n} x^\alpha h(x)dx = 0;\ if\ |\alpha| \leq L. \tag{3.14}$$

Then, there exists a constant $C > 0$, which is independent of g, h, i, j, x and x_0, such that

$$g \star h(x) \leq C2^{-(i-j)(L+\theta+n/2)}\left(1 + 2^j|x - x_0|\right)^{-R}. \tag{3.15}$$

A special case of Lemma 3.3, formally corresponding to $L = -1$, where no vanishing moments on h are assumed, is the following.

Lemma 3.4. *Let $R > n$, $i, j \in \mathbb{Z}, i \geq j$, and $x_0 \in \mathbb{R}^n$. Suppose $g,\ h \in L^1(R^n)$ satisfy*

$$|g(x)| \leq 2^{jn/2}\left(1 + 2^j|x|\right)^{-R} \tag{3.16}$$

$$|h(x)| \leq 2^{ni/2}\left(1 + 2^i|x - x_0|\right)^{-R}. \tag{3.17}$$

Then,

$$g \star h(x) \leq C2^{-(i-j)n/2}\left(1 + 2^j|x - x_0|\right)^{-R} \tag{3.18}$$

for some constant $C > 0$.

Proof of Lemma 3.1. We adapt here the proof of Lemma B.3 in [12] and the proof of Lemma 5.1 in [3] .
Case 1. Reducing, $\sigma,\ \rho$, or M if necessary, we may assume that

$$\sigma - (\gamma + n\tau)^{\star} = \frac{M-J}{2} = \rho - \frac{(J-\gamma)^{\star}}{2}$$

Suppose $l(Q) \leq l(P)$ and $\gamma + n\tau \geq 0$. Let $i\ ,\ j \in \mathbb{Z}$ be such that $l(Q) = 2^{-i} \leq l(P) = 2^{-j}$. Then, it easy to check that $g(x) = m_P(x_P - x)$ and $h(x) = \overline{b_Q(x)}$ satisfy the hypotheses of Lemma 3.3 with with $R = M,\ L = \lfloor \gamma + n\tau \rfloor,\ S = M + n + \gamma + n\tau - J$ and $x_0 = x_Q$. Therefore, by Lemma 3.3 with $\theta = \rho$, we have

$$|\langle m_P, b_Q\rangle| = |g \star h(x_P)| \leq C2^{-(i-j)(L+\theta+n/2)}\left(1 + 2^j|x_P - x_Q|\right)^{-M}.$$

Set $\epsilon/2 = L + \theta - (\gamma + n\tau)$. Then, $nr_0(\tau - \frac{1}{p}) \leq \frac{\epsilon}{2} \leq \frac{M-J}{2}$ and

$$|\langle m_P, b_Q\rangle| \leq C2^{-(i-j)(\gamma+n\tau+\epsilon/2+n/2)}\left(1 + 2^j|x_P - x_Q|\right)^{-J-\epsilon}.$$

Case 2. Suppose $l(Q) \leq l(P)$ and $\gamma + n\tau < 0$. Let $i\,,\ j \in \mathbb{Z}$ be such that $l(Q) = 2^{-i} \leq l(P) = 2^{-j}$. For the same choice of g and h as in Case 1, we have by Lemma 3.4 with $R = M$

$$\begin{aligned}|\langle m_P, b_Q\rangle| &\leq C2^{-(i-j)n/2}\left(1 + 2^j|x_P - x_Q|\right)^{-M}\\ &\leq C2^{-(i-j)(\gamma r_0+\epsilon/2+n/2)}2^{(i-j)(\gamma r_0+\epsilon/2)}\left(1 + 2^j|x_P - x_Q|\right)^{-J-\epsilon}\\ &\leq C2^{-(i-j)(\gamma+\epsilon/2+n/2)}\left(1 + 2^j|x_P - x_Q|\right)^{-J-\epsilon}\end{aligned}$$

where $\epsilon/2 = min(-\gamma r_0, \frac{M-J}{2})$ satisfying $nr_0(\tau - \frac{1}{p}) \leq \frac{\epsilon}{2} \leq \frac{M-J}{2}$.

Case 3. Suppose $l(Q) > l(P)$ and $N \geq 0$. Let $i,\ j \in \mathbb{Z}$ be such that $l(Q) = 2^{-j} > l(P) = 2^{-i}$. Then, it is easy to check that $g(x) = \overline{b_Q(x_Q - x)}$ and $h(x) = m_P(x)$ satisfy the hypotheses of Lemma 3.3 with $R = M,\ L = N = \lfloor J - \gamma - n \rfloor,\ S = M - \gamma$ and $x_0 = x_P$. Therefore, by Lemma 3.3 with $\theta = \rho$, we have

$$|\langle m_P, b_Q\rangle| = |g \star h(x_Q)| \leq C2^{-(i-j)(N+\theta+n/2)}\left(1 + 2^j|x_P - x_Q|\right)^{-M}.$$

Set $\epsilon/2 = N + \theta - (J - \gamma - n)$. Then, $nr_0(\tau - \frac{1}{p}) \leq \frac{\epsilon}{2} \leq \frac{M-J}{2}$ and

$$|\langle m_P, b_Q\rangle| \leq C2^{-(i-j)(J-\gamma+\epsilon/2-n/2)}\left(1 + 2^j|x_P - x_Q|\right)^{-J-\epsilon}.$$

Case 4. Finally, Suppose $l(Q) > l(P)$, and $N = -1$. Let $i,\ j \in \mathbb{Z}$ be such that $l(Q) = 2^{-j} > l(P) = 2^{-i}$. By Lemma 3.4 with $R = M$ for the same choice of g and h as in Case 3, we have

$$\begin{aligned}|\langle m_P, b_Q\rangle| &\leq C2^{-(i-j)n/2}\left(1 + 2^j|x_P - x_Q|\right)^{-M}\\ &\leq C2^{-(i-j)(J-\gamma+\epsilon/2-n/2)}\left(1 + 2^j|x_P - x_Q|\right)^{-J-\epsilon},\end{aligned}$$

where $\epsilon/2 = min(-(J - \gamma - n), \frac{M-J}{2})$ satisfying $nr_0(\tau - \frac{1}{p}) \leq \frac{\epsilon}{2} \leq \frac{M-J}{2}$.
Combining Cases 1-4, we conclude that

$$|\langle m_P, b_Q\rangle| \leq C\kappa_{QP}(\epsilon)$$

which completes the proof of Lemma 3.1. □

3.2. **Smooth molecular decompositions.** At this stage, we are able to show generalizations of Theorem 1.1 in the situation when the usual wavelet families are replaced by families of smooth analysis and synthesis molecules. More precisely, we have

Theorem 3.1 (Smooth molecular synthesis)**.** *Suppose* $w \in A_\infty$*. There exists a constant* $C > 0$*, such that if* $f = \sum_Q t_Q\psi_Q$ *where* $\{\psi_Q\}_{Q\in\mathcal{Q}}$ *is a family of smooth synthesis molecules for* $\dot{F}^{\gamma,\tau}_{p,q,w}$*, then*

$$||f||_{\dot{F}^{\gamma,\tau}_{p,q,w}} \leq C||t||_{\dot{f}^{\gamma,\tau}_{p,q,w}}, \qquad for\ all\ t \in \dot{F}^{\gamma,\tau}_{p,q,w}.$$

20

Proof. By Lemma 0.2, we can write $\psi_P = \sum_{Q\in\mathcal{Q}} \langle \psi_P, \nu_Q\rangle \mu_Q$ with the convergence in $\mathcal{S}'_\infty(\mathbb{R}^n)$. Let $\mathcal{A}$ given by the matrix $\{a_{QP}\}_{Q,P\in\mathcal{Q}} = \{\langle \psi_P, \nu_Q\rangle\}_{Q,P\in\mathcal{Q}}$ then, By Theorem 2.1 and Corollary 3.2 $\mathcal{A}$ is a bounded operator on $\dot{f}^{\gamma,\tau}_{p,q,w}$. Since

$$T_\mu t = \sum_Q \sum_P a_{QP} t_P \mu_Q = \sum_P t_P \sum_Q a_{QP} \langle \psi_P, \nu_Q\rangle \mu_Q = \sum_P t_P \psi_P$$

then, by Theorem 1.1,

$$||f||_{\dot{F}^{\gamma,\tau}_{p,q,w}} = ||T_\mu \mathcal{A} t||_{\dot{F}^{\gamma,\tau}_{p,q,w}} \le C||\mathcal{A}t||_{\dot{f}^{\gamma,\tau}_{p,q,w}} \le C||t||_{\dot{f}^{\gamma,\tau}_{p,q,w}}.$$

□

Theorem 3.2. *(Smooth molecular analysis). Suppose $w \in A_\infty$. There exists a constant $C > 0$, such that if $\{\phi_Q\}_{Q\in\mathcal{Q}}$ is a family of smooth analysis molecules for $\dot{F}^{\gamma,\tau}_{p,q,w}$ then,*

$$||\{\langle f, \phi_Q\rangle\}_Q||_{\dot{f}^{\gamma,\tau}_{p,q,w}} \le C||f||_{\dot{F}^{\gamma,\tau}_{p,q,w}}.$$

The proof of the theorem 3.2 is easy once the significance of the pairing $\langle f, \phi_Q\rangle$ is justified, see [3, Lemma 5.7]. We omit the details of the proof. To justify the meaningfulness of the pairing $\langle f, \phi_Q\rangle$, we need the following Lemma.

Lemma 3.5. *Let γ, p, q, τ and ϵ be as in Lemma 3.1, $f \in \dot{F}^{\gamma,\tau}_{p,q,w}$ and $\{\phi_Q\}_{Q\in\mathcal{Q}}$ be a smooth analysis molecule for $\dot{F}^{\gamma,\tau}_{p,q,w}$ supported near Q. Then, $\langle f, \phi_Q\rangle$ is well defined.*
More exactly, we have for any $\mu, \nu \in \mathcal{S}(\mathbb{R}^n)$ satisfying 0.4, 0.5 and 0.6 the serie

$$\langle f, \phi_Q\rangle = \sum_{j\in\mathbb{Z}} \langle \tilde{\nu}_j \star \mu_j \star f, \phi_Q\rangle = \sum_P \langle f, \nu_P\rangle \langle \mu_P, \phi_Q\rangle \tag{3.19}$$

converges absolutely and its value is independent of the choices of μ and ν.

Proof. The proof of the Lemma 3.5 is very similar to the proof of [3, Lemma 5.7]. For the clarity, we give some few details.

We consider only the case $\tau - 1/p \ge 0$. The proof of Lemma 3.5 when $\tau - 1/p < 0$ is similar.
Assume $\tau - 1/p \ge 0$, we claim that there exists a matrix $\{a_{QP}\}_{Q,P\in\mathcal{Q}}$ such that

$$|\langle f, \nu_P\rangle||\langle \nu_P, \phi_Q\rangle| \le a_{QP} \text{ and } \sum_P a_{QP} < \infty. \tag{3.20}$$

In fact, by Lemma 3.1, there exists a positive constant C such that

$$|\langle \mu_P, \phi\rangle| \le C\kappa_{QP}(\epsilon).$$

So we can take $a_{QP} = C|\langle f, \nu_P\rangle|\kappa_{QP}(\epsilon)$. Furthermore, by Theorem 1.1, the sequence $\{\langle \mu_P, \phi\rangle\}_P \in \dot{f}^{\gamma,\tau}_{p,q,w}$, and hence by Corollary 3.1 and Theoerem 2.1 $\sum_P a_{QP} < \infty$. This shows the absolute convergence of the series 3.20.

To show independence of the choice of μ and ν, let $\{\phi_l\}_{l=1}^\infty$ be the sequence of (constant multiples of) smooth analysis molecules supported near Q and converging uniformly to ϕ_Q guaranteed by Lemma 3.2. By Theorem 1.2 there exists a sequence of polynomials $\{P_k\}_{N=1}^\infty$, with degree no more than $L = \lfloor \gamma + nr_0(\tau - 1/p)\rfloor$ such that $\sum_{j\ge -N} \tilde{\nu}_j \star \mu_j \star f + P_N$ converges in $\mathcal{S}'(\mathbb{R}^n)$. Therefore, for each l, we can define

$$\langle f, \phi_l\rangle = \left\langle \lim_{N\to\infty} \sum_{j\ge -N} \tilde{\nu}_j \star \mu_j \star f + P_N, \phi_l \right\rangle = \lim_{N\to\infty} \sum_{j\ge -N} \langle \tilde{\nu}_j \star \mu_j \star f, \phi_l\rangle$$
$$= \lim_{N\to\infty} \sum_{P\in\mathcal{Q}, l(P)\ge 2^{-N}} \langle f, \nu_P\rangle \langle \mu_P, \phi_l\rangle = \sum_P \langle f, \nu_P\rangle \langle \mu_P, \phi_l\rangle,$$

since the above series converges absolutely by 3.20. Using the similar argument given in [3, Lemma 5.7] to obtain

$$\sum_P \langle f, \nu_P\rangle\langle \mu_P, \phi_l\rangle \longrightarrow \sum_P \langle f, \nu_P\rangle\langle \mu_P, \phi_Q\rangle \quad as \quad l \longrightarrow \infty.$$

and this limit is independent of μ and ν. This shows that $\langle f, \phi_Q\rangle$ is well defined by 3.19 and completes the proof of Lemma 3.5. □

Proof of Theorem 3.2. By lemma 3.5 we have

$$\langle f, \phi_Q\rangle = \sum_P \langle f, \nu_P\rangle\langle \mu_P, \phi_Q\rangle = \sum_P \langle f, \nu_P\rangle a_{QP}$$

with $a_{QP} = \langle \mu_P, \phi_Q\rangle$. By Lemma 2.1 and Corollary 3.1 the operator $\mathcal{A}$ given by the matrix $\{a_{QP}\}_{P,Q}$ is bounded on $\dot{f}^{\gamma,\tau}_{p,q,w}$. It follows from Theorem 1.1 that,

$$||\langle f, \phi_Q\rangle||_{\dot{f}^{\gamma,\tau}_{p,q,w}} = ||\mathcal{A}S_\nu f||_{\dot{f}^{\gamma,\tau}_{p,q,w}} \preceq ||f||_{\dot{F}^{\gamma,\tau}_{p,q,w}}.$$

□

3.3. **Smooth atomic decompositions.**

Definition 3.2. *Let γ, τ, p, q, w and J and as in Definition 3.1. A function a_Q is called a smooth atom for $\dot{F}^{\gamma,\tau}_{p,q,w}$ supported near dyadic cube Q if satisfies*

$$supp\ a_Q \subset 3Q, \tag{3.21}$$

$$\int_Q x^\alpha a_Q(x) = 0; \quad if\ |\alpha|\ \leq \tilde{N}, \tag{3.22}$$

$$|\partial^\alpha a_Q(x)| \leq |Q|^{-1/2-|\alpha|/n}; \quad if\ |\alpha|\ \leq \tilde{K}, \tag{3.23}$$

where $\tilde{N} \geq max(\lfloor J-n-\gamma\rfloor, -1)$ and $\tilde{K} \geq max(\lfloor \gamma + n\tau + 1\rfloor, 0)$. We say that a collection $\{a_Q\}_Q$ is a family of smooth synthesis atoms, if each a_Q is a smooth atom supported near Q.

Remark 3.1. *Note that every smooth atom for $\dot{F}^{\gamma,\tau}_{p,q,w}$ is a multiple of a smooth synthesis molecule for $\dot{F}^{\gamma,\tau}_{p,q,w}$ supported near Q.*

Theorem 3.3 (Smooth atomic decomposition). *Let γ, τ, p, q, w and J as in Lemma 3.1. Then, for each $f \in \dot{F}^{\gamma,\tau}_{p,q,w}$ there exist smooth atoms $\{a_Q\}_Q$ and a sequence of coefficients $\{t_Q\}_Q \in \dot{f}^{\gamma,\tau}_{p,q,w}$, such that*

$$f = \sum_Q t_Q a_Q \quad and \quad ||a||_{\dot{f}^{\gamma,\tau}_{p,q,w}} \leq C||f||_{\dot{F}^{\gamma,\tau}_{p,q,w}},$$

where C is a positive constant independent of f and t. Conversely, there exists a positive constant C such that for all families of smooth atoms $\{a_Q\}_Q$,

$$||\sum_Q t_Q a_Q||_{\dot{F}^{\gamma,\tau}_{p,q,w}} \leq C||t||_{\dot{f}^{\gamma,\tau}_{p,q,w}}.$$

The proof of Theorem 3.3 uses Theorem 1.1, Theorem 3.1 and Remark 3.1 and is a verbatim copy of the corresponding result in [12, Theorem 4.1] or in [3, Theorem 5.8]. Hence, we omit the details.

Remark 3.2. *Results in the previous sections can be extended, in the natural way, to a more general Lizorkin-space defined by replacing in the definition 0.3 2^j by $\det(A)^{j/n}$, where A is an expansive matrix,i.e, A is real $n \times n$ matrix such that $min_{\lambda\in\sigma(A)}|\lambda| > 1$ where $\sigma(A)$ is the spectrum of A (the set of all eigenvalues of A), and the Euclidean metric on $\mathbb{R}^n$ can be replaced by a quasi-norm associated with the matrix A, for details see for instance* [3].

22

4. Appendix

Let $0 < p,\ q \le \infty \quad, 0 \le \tau < \infty$ and $w \in A_\infty$. The space $l^q\left(L^\tau_{p,w}\right)$ is defined to be the set of all sequences $g = \{g_j\}_{j\in\mathbb{Z}}$ of measurable functions on $\mathbb{R}^n$ s.t

$$||g||_{l^q\left(L^\tau_{p,w}\right)} = \sup_{Q\in\mathcal{Q}} \frac{1}{[w(Q)]^\tau}\left[\sum_{j=j_Q}^{\infty}\left(\int_Q |g_j(x)|^p w(x)dx\right)^{\frac{q}{p}}\right]^{\frac{1}{q}} < \infty.$$

Similarly, the space $L^\tau_{p,w}(l^q)$ with $0 < p < \infty$ is defined to be the set of all sequences $g = \{g_j\}_{j\in\mathbb{Z}}$ of measurable functions on $\mathbb{R}^n$ s.t

$$||g||_{L^\tau_{p,w}(l^q)} = \sup_{Q\in\mathcal{Q}} \frac{1}{[w(Q)]^\tau}\left[\int_Q\left(\sum_{j=j_Q}^{\infty}|g_j(x)|^q\right)^{\frac{p}{q}} w(x)dx\right]^{\frac{1}{p}} < \infty$$

where $j_Q = -log_2 l(Q)$ and $l(Q)$ is the side length of the dyadic cube Q. We need the following lemma which is a generalization of Lemma 2 in [28].

Lemma 4.1. *Let $r_0 \ge 1,\ w \in A_{r_0},\ 0 < q,\ p < \infty,\ 0 \le \tau < \infty,\ \delta > n\tau r_0$ and $g = \{g_j\}_{j\in\mathbb{Z}}$ is a collection of measurable functions and a sequence of complex numbers $a = \{a_j\}_{j\in\mathbb{Z}}$ satisfying*

$$|a_j| \le C\begin{cases} 2^j, & if \quad j \in \mathbb{Z}_- \\ 2^{-\delta j}, & if \quad j \in \mathbb{N}_0. \end{cases}$$

Then, there exists a positive constant C, independent of g s.t

$$||G||_{l^q\left(L^\tau_{p,w}\right)} \le C||g||_{l^q\left(L^\tau_{p,w}\right)}$$

and

$$||G||_{L^\tau_{p,w}(l^q)} \le C||g||_{L^\tau_{p,w}(l^q)}$$

where $G_j(x) = \sum_{m\in\mathbb{Z}} a_{j-m} g_m(x)$.

The following corollary that generalizes the result of Rychkov,V. S. [28, Lemma 2] is a direct consequence of Lemma 4.1.

Corollary 4.1. *Let $r_0 \ge 1,\ w \in A_{r_0},\ 0 < q,\ p < \infty,\ 0 \le \tau < \infty,\ \delta > n\tau r_0$ and $g = \{g_j\}_{j\in\mathbb{Z}}$ is sequences of measurable functions. Define $G_j(x) = \sum_{m\in\mathbb{Z}} 2^{-|m-j|\delta} g_m(x)$. Then, there exists a positive constant C, independent of g s.t*

$$||G||_{l^q\left(L^\tau_{p,w}\right)} \le C||g||_{l^q\left(L^\tau_{p,w}\right)}$$

and

$$||G||_{L^\tau_{p,w}(l^q)} \le C||g||_{L^\tau_{p,w}(l^q)}.$$

Lemma 4.2. *Let $0 \le \tau < \infty$, $1 < p,\ q < \infty$ and $w \in A_p$. Define*

$$||Mg||_{L^\tau_{p,w}(l^q)} = \sup_{Q} \frac{1}{[w(Q)]^\tau}\left(\int_Q\left(\sum_{j=j_Q}^{\infty}|Mg_j(x)|^q\right)^{\frac{p}{q}} w(x)dx\right)^{\frac{1}{p}}.$$

If $\tau - \frac{1}{p} < 0$ then, we have the following general weighted Fefferman-Stein inequality

$$||Mg||_{L^\tau_{p,w}(l^q)} \le C||g||_{L^\tau_{p,w}(l^q)}.$$

23

Proof

Proof of Lemma 4.1. By similarity, we only prove the first inequality.
First we note that if $k \geq j_Q$ then,

$$\frac{1}{[w(Q)]^\tau}\left(\int_Q |g_k(x)|^q w(x)dx\right)^{\frac{1}{q}} \leq \frac{1}{[w(Q)]^\tau}\left[\sum_{j=j_Q}^{\infty}\left(\int_Q |g_j(x)|^p w(x)dx\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}$$
$$\leq ||g||_{l^q\left(L^\tau_{p,w}\right)}$$

and

$$\frac{1}{[w(Q)]^\tau}\left(\int_Q |g_k(x)|^p w(x)dx\right)^{\frac{1}{p}} \leq \frac{1}{[w(Q)]^\tau}\left[\int_Q \left(\sum_{j=j_Q}^{\infty} |g_j|^q\right)^{\frac{q}{p}} w(x)dx\right]^{\frac{1}{p}}$$
$$\leq ||g||_{L^\tau_{p,w}(l^q)}.$$

Through the proof we take into the account that $a \in l^r(\mathbb{Z}),\ \forall\, r > 0$. We begin by considering the case $0 < p \leq 1$.

Fix a dyadic cube Q and use the Young's inequality

$$\forall \quad 0 < \epsilon < 1, \quad \forall \quad z_i \in \mathbb{C} : \left(\sum_{m\in\mathbb{Z}} |z_m|\right)^\epsilon \leq \sum_{m\in\mathbb{Z}} |z_m|^\epsilon \tag{4.1}$$

to obtain

$$I_Q = \frac{1}{[w(Q)]^\tau}\left[\sum_{j=j_Q}^{\infty}\left(\int_Q \left|\sum_{m\in\mathbb{Z}} a_{j-m} g_m(x)\right|^p w(x)dx\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}$$
$$\leq \frac{1}{[w(Q)]^\tau}\left[\sum_{j=j_Q}^{\infty}\left(\sum_{m\in\mathbb{Z}} |a_{j-m}|^p \int_Q |g_m(x)|^p\, w(x)dx\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}.$$

Assume $0 < q < p$. Then, 4.1 implies

$$I_Q \leq \frac{1}{[w(Q)]^\tau}\left[\sum_{j=j_Q}^{\infty}\sum_{m\in\mathbb{Z}} |a_{j-m}|^q \left(\int_Q |g_m(x)|^p\, w(x)dx\right)^{\frac{q}{p}}\right]^{\frac{1}{q}} = J_Q + H_Q$$

with

$$J_Q = \frac{1}{[w(Q)]^\tau}\left[\sum_{j=j_Q}^{\infty}\sum_{m=j_Q}^{\infty} |a_{j-m}|^q \left(\int_Q |g_m(x)|^p\, w(x)dx\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}$$
$$= \frac{1}{[w(Q)]^\tau}\left[\sum_{m=j_Q}^{\infty}\sum_{j=j_Q}^{\infty} |a_{j-m}|^q \left(\int_Q |g_m(x)|^p\, w(x)dx\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}$$
$$\leq C||g||_{l^q\left(L^\tau_{p,w}\right)}$$

24

and

$$\begin{aligned}H_Q =&\frac{1}{[w(Q)]^\tau}\left[\sum_{j=j_Q}^{\infty}\sum_{m=-\infty}^{j_Q-1}|a_{j-m}|^q\left(\int_Q|g_m(x)|^p\,w(x)dx\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}\\ \leq C&\frac{1}{[w(Q)]^\tau}\left[\sum_{j=j_Q}^{\infty}\sum_{m=-\infty}^{j_Q-1}2^{-|m-j|\delta q}\left(\int_Q|g_m(x)|^p\,w(x)dx\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}\\ \leq C&\left[\sum_{j=j_Q}^{\infty}\sum_{m=-\infty}^{j_Q-1}2^{(m-j)\delta q}\left(\frac{w(Q_m)}{w(Q)}\right)^{q\tau}\frac{1}{[w(Q_m)]^{q\tau}}\left(\int_{Q_m}|g_m(x)|^p\,w(x)dx\right)^{\frac{q}{p}}\right]^{\frac{1}{q}},\end{aligned}$$

where Q_m is a dyadic cube containing Q with side length $l(Q_m) = 2^{-m}, m \leq j_Q - 1$. From the A_{r_0} property, we have

$$\left(\frac{w(Q_m)}{w(Q)}\right)^{q\tau}\leq C\left(\frac{|Q_m|}{|Q|}\right)^{q\tau r_0}\simeq 2^{nq\tau r_0(j_Q-m)}.$$

It follows that

$$\begin{aligned}H_Q &\leq C\left[\sum_{j=j_Q}^{\infty}\sum_{m=-\infty}^{j_Q-1}2^{(m-j)\delta q}2^{nq\tau r_0(j_Q-m)}\right]^{\frac{1}{q}}||g||_{l^q\left(L^\tau_{p,w}\right)}\\ &\leq C\left[\sum_{j=j_Q}^{\infty}2^{(j_Q-j)\delta q}\sum_{m=-\infty}^{j_Q-1}2^{q(\delta-n\tau r_0)(m-j_Q)}\right]^{\frac{1}{q}}||g||_{l^q\left(L^\tau_{p,w}\right)}\\ &\leq C||g||_{l^q\left(L^\tau_{p,w}\right)}.\end{aligned}$$

Now assume that $q \geq p$ and write

$$\begin{aligned}&\frac{1}{[w(Q)]^\tau}\left[\sum_{j=j_Q}^{\infty}\left(\sum_{m\in\mathbb{Z}}|a_{m-j}|^p\int_Q|g_m(x)|^p\,w(x)dx\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}\\ &=\frac{1}{[w(Q)]^\tau}\left[\sum_{j=j_Q}^{\infty}\left(\sum_{m\in\mathbb{Z}}|a_{m-j}|^{\frac{(\delta-\epsilon)p}{\delta}}|a_{m-j}|^{\frac{\epsilon p}{\delta}}\int_Q|g_m(x)|^p\,w(x)dx\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}.\end{aligned}$$

Choose $0 < \epsilon < \delta - n\tau r_0$, arguing as before and using the following Hölder's inequality

$$\sum_{m\in\mathbb{Z}}|x_my_m|\leq\left(\sum_{m\in\mathbb{Z}}|x_m|^r\right)^{1/r}\left(\sum_{m\in\mathbb{Z}}|y_m|^{r'}\right)^{1/r'}$$

25

where $\frac{1}{r}+\frac{1}{r'}=1$, x_m, y_m are in $\mathbb{C}$ and $r=\frac{q}{p}$, we obtain

$$\begin{aligned}
I_Q \leq & \frac{1}{[w(Q)]^\tau}\left[\sum_{j=j_Q}^{\infty}\sum_{m\in\mathbb{Z}}|a_{m-j}|^{\frac{(\delta-\epsilon)q}{\delta}}\left(\int_Q |g_m(x)|^p\, w(x)dx\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}\\
\leq & \frac{1}{[w(Q)]^\tau}\\
& \times\left[\sum_{j=j_Q}^{\infty}\sum_{m=j_Q}^{\infty}|a_{m-j}|^{\frac{(\delta-\epsilon)q}{\delta}}\left(\int_Q |g_m(x)|^p\, w(x)dx\right)^{\frac{q}{p}}+\sum_{j=j_Q}^{\infty}\sum_{m=-\infty}^{j_Q-1}2^{-|m-j|(\delta-\epsilon)q}\ldots\right]^{\frac{1}{q}}\\
\leq & C||g||_{l^q\left(L^\tau_{p,w}\right)}.
\end{aligned}$$

If $1<p\leq\infty$, we use Minkowski's inequality to get

$$I_Q \leq \frac{1}{[w(Q)]^\tau}\left[\sum_{j=j_Q}^{\infty}\left(\sum_{m\in\mathbb{Z}}|a_{m-j}|\left(\int_Q |g_m(x)|^p\, w(x)dx\right)^{\frac{1}{p}}\right)^q\right]^{\frac{1}{q}}.$$

Applying Hölder's inequality if $1<q\leq\infty$ or 4.1 if $0<q\leq 1$ to conclude. □

Proof of Lemma 4.2. We adapt here the proof of [35, Lemma 2.5]. Assume $\tau-1/p=-\epsilon<0$ and denote by $\delta>0$ the reverse-doubling constant of the weight $w\in A_p$. Pick any $x_0\in\mathbb{R}^n$, and let Q be cube containing x_0 with side lenght $l(Q)=r$. Write

$$\begin{aligned}
& g_j = g_{0j}+\sum_{i=1}^{\infty}g_{ij}\\
& with\\
& g_{0j}=\chi_{B(x_0,2r)}g_j \quad and \quad g_{ij}=\chi_{B(x_0,2^{i+1}r)\setminus B(x_0,2^i r)}g_j \quad for \quad i\geq 1.
\end{aligned}$$

The Stein-Fefferman inequality implies

$$\begin{aligned}
\left(\int_Q\left(\sum_{j=j_Q}^{\infty}|Mg_{0j}(x)|^q\right)^{\frac{p}{q}}w(x)dx\right)^{\frac{1}{p}} \leq & \left(\int_{\mathbb{R}^n}\left(\sum_{j=j_Q}^{\infty}|g_{0j}(x)|^q\right)^{\frac{p}{q}}w(x)dx\right)^{\frac{1}{p}}\\
\leq & C||g||_{L^\tau_{p,w}(l^q)}\left(\int_{B(x_0,r)}w(y)dy\right)^{\tau}.
\end{aligned}$$

On the other hand for $i\geq 1$ and $x\in B(x_0,r)$, we have

$$Mg_{ij}(x)=\sup_{R>0}\frac{1}{|B(x,R)|}\int_{B(x,R)\cap\{2^i r<|y-x_0|<2^{i+1}r\}}|g_j(y)|dx \leq C(2^i r)^{-n}\int_{\mathbb{R}^n}|g_{ij}(y)|dy.$$

26

Now, the generalized Minkowski's inequality leads to

$$\left(\sum_{j=j_Q}^{\infty}|Mg_{ij}(x)|^q\right)^{\frac{1}{q}} \leq C(2^ir)^{-n}\left(\sum_{j=j_Q}^{\infty}\left(\int_{\mathbb{R}^n}|g_{ij}(y)|dy\right)^q\right)^{\frac{1}{q}}$$

$$\leq C(2^ir)^{-n}\int_{B(x_0,2^{i+1}r)}\left(\sum_{j=j_Q}^{\infty}|g_{ij}|^q\right)^{\frac{1}{q}}dy$$

$$\leq C(2^ir)^{-n}\left(\int_{B(x_0,2^{i+1}r)}\left(\sum_{j=j_Q-i-1}^{\infty}|g_{ij}|^q\right)^{\frac{p}{q}}w(y)dy\right)^{\frac{1}{p}}\left(\int_{B(x_0,2^{i+1}r)}w^{\frac{-p'}{p}}\right)^{\frac{1}{p'}}$$

$$\leq C(2^ir)^{-n}||g||_{L^{\tau}_{p,w}(l^q)}\left(\int_{B(x_0,2^{i+1}r)}w^{\frac{-p'}{p}}\right)^{\frac{1}{p'}}\left(\int_{B(x_0,2^{i+1}r)}w(y)dy\right)^{\tau}$$

$$\leq C||g||_{L^{\tau}_{p,w}(l^q)}(2^ir)^{-n}\left(\int_{B(x_0,2^{i+1}r)}w^{\frac{-p'}{p}}\right)^{\frac{1}{p'}}\left(\int_{B(x_0,2^{i+1}r)}w(y)dy\right)^{\frac{1}{p}-\epsilon}$$

$$\leq C||g||_{L^{\tau}_{p,w}(l^q)}\left(\int_{B(x_0,2^{i+1}r)}w(y)dy\right)^{-\epsilon}$$

$$\leq C||g||_{L^{\tau}_{p,w}(l^q)}\left(\int_{B(x_0,r)}w(y)dy\right)^{-\epsilon}\left(\frac{\int_{B(x_0,r)}w(y)dy}{\int_{B(x_0,2^{i+1}r)}w(y)dy}\right)^{\epsilon}$$

$$\leq C2^{-in\epsilon\delta}||g||_{L^{\tau}_{p,w}(l^q)}\left(\int_{B(x_0,r)}w(y)dy\right)^{\tau-\frac{1}{p}}.$$

Hence

$$\left(\sum_{j=j_Q}^{\infty}M^q\left(\sum_{i=1}^{\infty}g_{ij}\right)(x)\right)^{\frac{1}{q}} \leq C\left(\sum_{j=j_Q}^{\infty}\left(\sum_{i=1}^{\infty}Mg_{ij}(x)\right)^q\right)^{\frac{1}{q}}$$

$$\leq C\sum_{i=1}^{\infty}\left(\sum_{j=j_Q}^{\infty}M^qg_{ij}(x)\right)^{\frac{1}{q}}$$

$$\leq C\sum_{i=1}^{\infty}2^{-in\epsilon\delta}||g||_{L^{\tau}_{p,w}(l^q)}\left(\int_{B(x_0,r)}w(y)dy\right)^{\tau-\frac{1}{p}}$$

$$\leq C||g||_{L^{\tau}_{p,w}(l^q)}\left(\int_{B(x_0,r)}w(y)dy\right)^{\tau-\frac{1}{p}}.$$

It follows that

$$\left(\int_Q\left(\sum_{j=j_Q}^{\infty}|Mg_j(x)|^q\right)^{\frac{p}{q}}w(x)dx\right)^{\frac{1}{p}} \leq C||g||_{L^{\tau}_{p,w}(l^q)}\left(\int_{B(x_0,r)}w(y)dy\right)^{\tau}.$$

We conclude that

$$||Mg||_{L^{\tau}_{p,w}(l^q)} \leq C||g||_{L^{\tau}_{p,w}(l^q)}.$$

27

□

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WEIGHTED NORM INEQUALITIES ON ORLICZ-MORREY SPACES FOR THE MULTILINEAR FRACTIONAL INTEGRAL AND ORLICZ-FRACTIONAL MAXIMAL OPERATOR

TAKESHI IIDA

Abstract. We generalize Orlicz-Morrey spaces and Orlicz-fractional maximal operators to treat vector-valued functions, which extend to the multi-Morrey spaces and multilinear fractional maximal operators to the scale of the Orlicz spaces, respectively. In this article, we investigate the weighted norm inequalities for linear and multilinear fractional integrals and maximal operators and Orlicz-fractional maximal operators in Orlicz-Morrey spaces for multilinear version. One of the main results generalizes and improves the weighted estimate of the Adams inequality in multi-Morrey spaces. Moreover, we extend the weighted estimates to endpoint cases.

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1. Introduction

This paper describes the weighted norm inequalities of multilinear operators $\mathcal{M}_{\alpha,m}$, $I_{\alpha,m}$ and $\mathcal{M}_{\vec{B},\alpha}$ of Morrey and Orlicz-Morrey spaces for linear and multilinear version (we state the definitions of these operators and function spaces below). We use the following notation: Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. For a set $E \subset \mathbb{R}^n$, the symbols $|E|$ and χ_E denote the Lebesgue measure and characteristic function of E, respectively. Given a weight w and a measurable set E, let $w(E) := \int_E w(x)\,dx$. In this paper, we suppose that the sides of all cubes are parallel to the coordinate axes. For all cube Q and all $a > 0$, aQ denotes $\{ax : x \in Q\}$. $\mathcal{D}(\mathbb{R}^n)$ denotes the set of all dyadic cubes on $\mathbb{R}^n$ and for one dyadic cube $Q_0 \in \mathcal{D}(\mathbb{R}^n)$, let $\mathcal{D}(Q_0) := \{Q \in \mathcal{D}(\mathbb{R}^n) : Q \subset Q_0\}$.

Operators M, M_α, and I_α are fundamental tools to study Harmonic analysis and potential theory (see [5, 7, 8, 25, 29]). Recall these operators.

Definition 1.1. *Given* $0 < \alpha < n$, *define*

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy. \tag{1.1}$$

Given $0 \le \alpha < n$, *define*

$$M_\alpha f(x) := \sup_{Q:\text{cube}} \ell(Q)^\alpha \fint_Q |f(y)|dy \cdot \chi_Q(x), \tag{1.2}$$

where

$$\fint_Q F(x)dx := \frac{1}{|Q|}\int_Q F(x)dx = m_Q(F).$$

In particular, we define $Mf(x) := M_0 f(x)$, which is the Hardy-Littlewood maximal operator. Here and below a tacit understanding is that f is measurable.

In this paper, we need the multilinear versions. The symbol $\vec{f} = (f_1, \ldots, f_m)$ denotes the collection of m measurable functions. For every cube Q, and a vector valued function $\vec{f} = (f_1, \ldots, f_m)$, $m_Q\left(\vec{f}\right)$ writes $\prod_{i=1}^m \fint_Q f_i(x)dx$.

Definition 1.2. *Given $0 < \alpha < mn$, $\vec{f}$, we define*

$$I_{\alpha,m}\left(\vec{f}\right)(x) := \int_{\mathbb{R}^{mn}} \frac{f_1(y_1)\cdots f_m(y_m)}{|(x-y_1,\ldots,x-y_m)|^{mn-\alpha}} d\vec{y}. \tag{1.3}$$

Given $0 \le \alpha < mn$, $\vec{f}$, we define

$$\mathcal{M}_{\alpha,m}\left(\vec{f}\right)(x) := \sup_{Q:cube} \ell(Q)^\alpha m_Q\left(\vec{f}\right)\chi_Q(x). \tag{1.4}$$

The operator $\mathcal{M}$ denotes $\mathcal{M}_{0,m}$.

We know that many authors have investigated the boundedness of the linear and multilinear fractional integrals I_α on some Morrey type spaces; for example, [1, 2, 14, 15, 16, 17, 18, 19, 20, 21, 22, 26, 28, 30] et al. In particular, we are interested in theorem due to [1], which recovers the Hardy-Littlewood-Sobolev inequality and is the origin of many papers. Firstly, we invoke the result in [1].

Proposition 1.3. *Let $0 < \alpha < n$, $1 < p \le p_0 < \frac{n}{\alpha}$ and $0 < q \le q_0 < \infty$. If $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ and $\frac{q}{q_0} = \frac{p}{p_0}$, then*

$$\|I_\alpha f\|_{\mathcal{M}^{q_0}_q} \le C\|f\|_{\mathcal{M}^{p_0}_p},$$

here, we define the Morrey norm $\|\cdot\|_{\mathcal{M}^{p_0}_p}$ below.

Remark 1.4. *Showing Proposition 1.3 in* [1, 2], *we can verify*

$$|I_\alpha f(x)| \le \|f\|_{\mathcal{M}^{p_0}_1}^{1-\frac{p_0}{q_0}} Mf(x)^{\frac{p_0}{q_0}}. \tag{1.5}$$

Estimate (1.5) *and $M : \mathcal{M}^{p_0}_p \to \mathcal{M}^{p_0}_p$ (This result is due to* [2]*) give the sharp result as follows:*

$$\|I_\alpha f\|_{\mathcal{M}^{q_0}_q} \le \|f\|_{\mathcal{M}^{p_0}_1}^{1-\frac{p_0}{q_0}} \|f\|_{\mathcal{M}^{p_0}_p}^{\frac{p_0}{q_0}}. \tag{1.6}$$

Considering the weighted norm estimate by straightly using estimate (1.5)*, we obtain mix type norm inequalities as follows:*

$$\|(I_\alpha f)v\|_{\mathcal{M}^{q_0}_q} \le \|f\|_{\mathcal{M}^{p_0}_1}^{1-\frac{p_0}{q_0}} \left\|(Mf)v^{\frac{q}{p}}\right\|_{\mathcal{M}^{p_0}_p}^{\frac{p_0}{q_0}}. \tag{1.7}$$

However, (1.7) *fails to recover the result in* [21]*. This paper shows that under the appropriate condition of weights, some the product of weighted norms controls the following inequality*

$$\|(I_\alpha f)v\|_{\mathcal{M}^{q_0}_q} \le C\|fw\|_X^{1-\frac{p_0}{q_0}} \cdot \|fw\|_Y^{\frac{p_0}{q_0}}, \tag{1.8}$$

here, the symbols X and Y are some Morrey type spaces, which the norms of X and Y satisfy $\|\cdot\|_X \lesssim \|\cdot\|_Y$. The following is one example of problem setting in this paper for one linear version. Under the condition $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ and $\frac{q}{q_0} = \frac{p}{p_0}$, we show that

$$\|(I_\alpha f)v\|_{\mathcal{M}^{q_0}_q} \lesssim \|M_{A,\alpha}(fw)\|_{\mathcal{M}^{q_0}_q}. \tag{1.9}$$

Here, $M_{A,\alpha}$ denotes the Orlicz-fractional maximal operator which is below. Under the appropriate condition of a Young function which we define and describe in detail below, since estimate of $\|M_{A,\alpha}f\|_{\mathcal{M}_q^{q_0}}$ has similar construction of (1.6) *(see Corollary 2.5), we can show that weighted norm of estimate* (1.9) *controls* (1.8).

Secondly, we invoke the following inequality (see [21, 26]).

Proposition 1.5. *Let v be a weight on $\mathbb{R}^n$, $0<\alpha<n$, $1<p\le p_0<\infty$, $0<q\le q_0<r_0\le\infty$ and $a>1$. If $\frac{1}{q_0}=\frac{1}{p_0}+\frac{1}{r_0}-\frac{\alpha}{n}$, $\frac{q}{q_0}=\frac{p}{p_0}$ and $\|v\|_{\mathcal{M}_{aq}^{r_0}}<\infty$ then,*

$$\|(I_\alpha f)v\|_{\mathcal{M}_q^{q_0}} \lesssim \|v\|_{\mathcal{M}_{aq}^{r_0}} \|f\|_{\mathcal{M}_p^{p_0}}. \tag{1.10}$$

Remark 1.6. *By a condition of weights and the index of fractional maximal type operator, we can unify the conditions $\frac{1}{q_0}=\frac{1}{p_0}-\frac{\alpha}{n}$ and $\frac{1}{q_0}=\frac{1}{p_0}+\frac{1}{r_0}-\frac{\alpha}{n}$ in Propositions 1.3 and 1.5, respectively. In this case, if $[v,w]$ denotes one quantity of weights, we show the following type inequality.*

$$\|(I_\alpha f)v\|_{\mathcal{M}_q^{q_0}} \lesssim [v,w]\left\|M_{A,\frac{n}{p_0}-\frac{n}{q_0}}(fw)\right\|_{\mathcal{M}_q^{q_0}}. \tag{1.11}$$

This paper recovers and improves the results due to [18, 21], which also generalize Propositions 1.3 and 1.5.

Next, we introduce the multilinear operators. The multilinear maximal operator $\mathcal{M}$ acts on m Lebesgue spaces' product and is smaller than the m-fold product of the Hardy-Littlewood maximal function. This operator is used to obtain precise control of the multilinear singular integral operators of the Calderón-Zygmund type and to build a theory of weights adapted to the multilinear setting (see [24]). In [24, p.1225], there is the prototype of Orlicz maximal operators for multilinear version, and we introduce the generalized operators. Papers [13, 14] showed that the boundedness of rough multilinear fractional integrals and maximal operators; In [13], in product L^p and weighed L^p spaces, on the other hand, in [14], weighted estimates in multi-Morrey spaces. Besides, papers [9, 10, 11, 12, 15, 16] showed that the boundedness of the commutators generated by linear and multilinear fractional integrals and $\vec{b}=(b_1,\dots,b_m)$ in Morrey type spaces.

To consider the boundedness of multilinear fractional maximal and integrals, we introduce Morrey and multi-Morrey spaces (see [22]).

Definition 1.7.

(1) Let $0<p\le p_0<\infty$. One says that $f\in\mathcal{M}_p^{p_0}(\mathbb{R}^n)$ for $f\in L^p_{loc}$ if the following norm or quasi-norm is finite:

$$\|f\|_{\mathcal{M}_p^{p_0}} := \sup_{Q:cube}|Q|^{\frac{1}{p_0}}\left(\fint_Q |f(x)|^p dx\right)^{\frac{1}{p}}<\infty. \tag{1.12}$$

(2) Let $0<p_1,\dots,p_m<\infty$ and $0<p_0<\infty$. Assume that $\frac{1}{p_0}\le\frac{1}{p_1}+\cdots+\frac{1}{p_m}$. Moreover, let $\vec{P}:=(p_1,p_2,\dots,p_m)$ be the collection m indices. One says that $\vec{f}=(f_1,\dots,f_m)\in\mathcal{M}_{\vec{P}}^{p_0}(\mathbb{R}^n)$ for $f_i\in L^{p_i}_{loc}$ $(i=1,\dots,m)$ if the following quantity is finite:

$$\left\|\vec{f}\right\|_{\mathcal{M}_{\vec{P}}^{p_0}} := \sup_{Q:cube}|Q|^{\frac{1}{p_0}}\prod_{i=1}^m\left(\fint_Q |f_i(y_i)|^{p_i}dx\right)^{\frac{1}{p_i}}<\infty. \tag{1.13}$$

To state the main results precisely, we will describe the Young functions, B_p-condition, operators M_B and $M_{B,\alpha}$. As usual, one says that a function $B:[0,\infty)\to[0,\infty)$ is a Young function if it is continuous, convex and increasing satisfying $B(0)=0$ and $B(t)\to\infty$ as $t\to\infty$. Define the B-average of a function f over a cube Q employing the Luxemburg norm.

Definition 1.8. *Given a Young function B and a cube Q, define*

$$\|f\|_{B,Q} := \inf\left\{\lambda > 0 : \fint_Q B\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}. \tag{1.14}$$

A Young function B satisfies that

$$B(t) \cong tB'(t) \quad (t > 0) \tag{1.15}$$

and

$$aB(t) \le B(at) \quad \text{and} \quad B\left(\frac{t}{a}\right) \le \frac{B(t)}{a} \quad (a > 1). \tag{1.16}$$

Estimates (1.16) entail

$$\frac{B(t)}{t} \le \frac{B(s)}{s} \quad (0 < t < s). \tag{1.17}$$

Given a Young function B, we define the complementary Young function $\bar{B}$ as follows:

$$\bar{B}(t) := \sup_{s>0} (st - B(s)) \ \ (t > 0). \tag{1.18}$$

Remark 1.9. *The functions B and $\bar{B}$ satisfy the following inequalities:*

$$t \le B^{-1}(t) \cdot \bar{B}^{-1}(t) \le 2t \quad (t > 0). \tag{1.19}$$

(1.19) *shows that*

$$\overline{(\bar{B})}(t) \cong B(t).$$

We know the following as the generalized Hölder inequality to the scale of Orlicz spaces:

$$\fint_Q |f(y)g(y)|dy \le 2\, \|f\|_{B,Q}\, \|g\|_{\bar{B},Q}\,. \tag{1.20}$$

More generally, if A, B and C are Young functions such that for all $t > 0$, $A^{-1}(t)B^{-1}(t) \le C^{-1}(t)$, then

$$\|fg\|_{C,Q} \le 2\, \|f\|_{A,Q}\, \|g\|_{B,Q}\,. \tag{1.21}$$

Definition 1.10. *Given p, $1 < p < \infty$, one says that a Young function B satisfies the B_p-condition if there exists a constant $c > 0$ such that*

$$\int_c^\infty \frac{B(t)}{t^{p+1}} dt < \infty.$$

The following occurs.

Remark 1.11. *If $1 < p < q$, then,*

$$B_p \subsetneqq B_q. \tag{1.22}$$

By (1.14), we can define the Orlicz-fractional maximal operator.

Definition 1.12. *Given $0 \le \alpha < n$ and a Young function B, define the Orlicz-fractional maximal operators*

$$M_{B,\alpha}(f)(x) := \sup_{Q:cube} \ell(Q)^\alpha\, \|f\|_{B,Q} \cdot \chi_Q(x). \tag{1.23}$$

Operator M_B denotes $M_{B,0}$.

Remark 1.13. *Let $0 \le \alpha < n$. Given a Young function B, the following inequality holds (see* [3, p.108]*):*

$$M_\alpha f(x) \lesssim M_{B,\alpha}(f)(x). \tag{1.24}$$

There is the following characterization in [27, Theorem 1.7 in pp.138-139].

Proposition 1.14. *Let $1 < p < \infty$. Given a Young function B, the following statements are equivalent:*

(1) $B \in B_p$,
(2) $M_B : L^p \to L^p$.

There is the weak-type version for Proposition 1.14 in [4, Proposition 5.6 in p.100].

Proposition 1.15. *Let $1 < p < \infty$. Given a Young function B, the following statements are equivalent:*
(1) $B(t) \lesssim t^p \quad (t \geq 1)$,
(2) $M_B : L^p \to L^{p,\infty}$.

We introduce Orlicz-Morrey spaces and their multilinear version.

Definition 1.16. *Let $0 < r_0 < \infty$ and B be a Young function. One says that $f \in \mathcal{M}_B^{r_0}$ for all measurable functions f if the quasi-norm is finite:*

$$\|f\|_{\mathcal{M}_B^{r_0}} := \sup_{Q \subset \mathbb{R}^n} |Q|^{\frac{1}{r_0}} \|f\|_{B,Q}.$$

Remark 1.17. *Definition 1.16 corresponds to $\varphi(t) = t^{\frac{1}{r_0}}$ and $\Phi \equiv B$ in* [6, (2) in Definition 1.1]. *Let Φ be a Young function. Moreover, lets $1 \leq r_0 < \infty$. Then $\mathcal{M}_\Phi^{r_0}(\mathbb{R}^n) \neq \{0\}$ if and only if $\Phi(t) \lesssim t^{r_0}$ for $t \geq 1$.*

We introduce the multilinear version for Orlicz-Morrey spaces.

Definition 1.18. *Let $0 < r_0 < \infty$. Let $\vec{A} = (A_1, A_2, \ldots, A_m)$ be a collection of m Young functions. One says that $\vec{f} = (f_1, f_2, \ldots, f_m) \in \mathcal{M}_{\vec{A}}^{r_0}(\mathbb{R}^n)$ for all m measurable functions $\vec{f}$ if the quasi-norm is finite:*

$$\left\|\vec{f}\right\|_{\mathcal{M}_{\vec{A}}^{r_0}} := \sup_{Q \subset \mathbb{R}^n} |Q|^{\frac{1}{r_0}} \prod_{i=1}^m \|f_i\|_{A_i,Q}.$$

To evaluate the estimates for the multilinear fractional integrals and maximal operators, we introduce the Orlicz-fractional maximal operator for the multilinear version.

Definition 1.19. *Given $0 \leq \alpha < mn$ and A_i $(i = 1, \ldots, m)$ be Young functions, symbol $\vec{A} := (A_1, \ldots, A_m)$ denotes a collection of m Young functions. For $\vec{f} = (f_1, \ldots, f_m)$, we define*

$$\mathcal{M}_{\vec{A},\alpha}\left(\vec{f}\right)(x) := \sup_{Q:cube} \ell(Q)^\alpha \prod_{i=1}^m \|f_i\|_{A_i,Q} \cdot \chi_Q(x). \tag{1.25}$$

Operator $\mathcal{M}_{\vec{A}}\left(\vec{f}\right)(x)$ denotes $\mathcal{M}_{\vec{A},0}\left(\vec{f}\right)(x)$.

Remark 1.20. *For every cube Q_0, if $x \in Q_0$, then $\mathcal{M}_{\vec{A},\alpha}\left(\vec{f}\right)(x) \geq \ell(Q_0)^\alpha \prod_{i=1}^m \|f_i\|_{A_i,Q_0}$. This implies that the following inequalities hold: Let $0 \leq \alpha < mn$, $0 < q \leq q_0 < \infty$ and $0 < p_0 < \infty$. If $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$, then,*

$$\left\|\mathcal{M}_{\vec{A},\alpha}\left(\vec{f}\right)\right\|_{\mathcal{M}_q^{q_0}} \geq \left\|\vec{f}\right\|_{\mathcal{M}_{\vec{A}}^{p_0}}. \tag{1.26}$$

For each $1 \leq p \leq \infty$, p' will denote the dual exponent of p, i.e., $p' = \frac{p}{p-1}$ with the usual modifications $1' = \infty$ and $\infty' = 1$.

We organize the rest of this paper as follows: In Section 2, we formulate the main results, in Section 3, we list some lemmas, and in Section 4, we prove the main results.

2. Main results

We establish the boundedness of the Orlicz-fractional maximal operator for the multilinear version for unweighted version.

Theorem 2.1. *Let $0 \le \alpha < mn$, $1 \le p_1, \dots, p_m < \infty$, $0 < p \le p_0 < \infty$, $0 < q \le q_0 < \infty$ and A_i $(i = 1, 2, \dots, m)$ be Young functions. If $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ and $\frac{q}{q_0} = \frac{p}{p_0}$, then,*

$$\left\| \mathcal{M}_{\vec{A},\alpha}\left(\vec{f}\right) \right\|_{\mathcal{M}_q^{q_0}} \lesssim \left\| \vec{f} \right\|_{\mathcal{M}_{\vec{A}}^{p_0}}^{1-\frac{p_0}{q_0}} \left\| \mathcal{M}_{\vec{A}}\left(\vec{f}\right) \right\|_{\mathcal{M}_p^{p_0}}^{\frac{p_0}{q_0}}. \tag{2.1}$$

Remark 2.2. *The proof of Theorem 2.1 originates from* [3]*. By Theorem 2.1 and Lemma 3.3 below, we obtain the following inequalities:*

Theorem 2.3. *Under the condition of Theorem 2.1, we have the followings:*
(1) If $p_i > 1$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m}$ and $A_i \in B_{p_i}$ $(i = 1, 2, \dots, m)$, then,

$$\left\| \mathcal{M}_{\vec{A},\alpha}\left(\vec{f}\right) \right\|_{\mathcal{M}_q^{q_0}} \lesssim \left\| \vec{f} \right\|_{\mathcal{M}_{\vec{A}}^{p_0}}^{1-\frac{p_0}{q_0}} \left\| \vec{f} \right\|_{\mathcal{M}_{\vec{P}}^{p_0}}^{\frac{p_0}{q_0}}. \tag{2.2}$$

(2) If $p_1 = p_2 = \dots = p_m = 1$, then,

$$\left\| \mathcal{M}_{\vec{A},\alpha}\left(\vec{f}\right) \right\|_{\mathcal{M}_q^{q_0}} \lesssim \left\| \vec{f} \right\|_{\mathcal{M}_{\vec{A}}^{p_0}}^{1-\frac{p_0}{q_0}} \left\| \vec{f} \right\|_{\mathcal{M}_{\vec{D}}^{p_0}}^{\frac{p_0}{q_0}}. \tag{2.3}$$

Here, $\vec{D}(t) := (A_1(t) \log^+(t), \dots, A_m(t) \log^+(t))$.
(3) If $p < \frac{1}{m}$, then,

$$\left\| \mathcal{M}_{\vec{A},\alpha}\left(\vec{f}\right) \right\|_{\mathcal{M}_q^{q_0}} \lesssim \left\| \vec{f} \right\|_{\mathcal{M}_{\vec{A}}^{p_0}}. \tag{2.4}$$

Remark 2.4. *Under the condition of (1) in Theorem 2.3, by Propositions 1.14 and 1.15, $\left\| \vec{f} \right\|_{\mathcal{M}_{\vec{A}}^{p_0}} \lesssim \left\| \vec{f} \right\|_{\mathcal{M}_{\vec{P}}^{p_0}}$ holds. Theorem 2.3 partially extend the result in* [18, Theorem 3] *to multilinear version.*

Corollary 2.5. *Let $0 \le \alpha < n$, $0 < p \le p_0 < \frac{n}{\alpha}$, $0 < q \le q_0 < \infty$, $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$, $\frac{q}{q_0} = \frac{p}{p_0}$ and A be Young function.*
(1) If $p > 1$ and $A \in B_p$, then,

$$\|M_{A,\alpha} f\|_{\mathcal{M}_q^{q_0}} \lesssim \|f\|_{\mathcal{M}_A^{p_0}}^{1-\frac{p_0}{q_0}} \|f\|_{\mathcal{M}_p^{p_0}}^{\frac{p_0}{q_0}}. \tag{2.5}$$

(2) If $p = 1$, then,

$$\|M_{A,\alpha} f\|_{\mathcal{M}_q^{q_0}} \lesssim \|f\|_{\mathcal{M}_A^{p_0}}^{1-\frac{p_0}{q_0}} \|f\|_{\mathcal{M}_D^{p_0}}^{\frac{p_0}{q_0}}. \tag{2.6}$$

Here, $D(t) = A(t) \log^+ t$.
(3) If $p < 1$, then,

$$\|M_{A,\alpha} f\|_{\mathcal{M}_q^{q_0}} \lesssim \|f\|_{\mathcal{M}_A^{p_0}}. \tag{2.7}$$

Secondly, we investigate the weighted estimates for the multilinear fractional integrals and maximal operators in multi-Morrey spaces. To simplify the notation, we introduce multiple weights constants: Let v be a weight and $\vec{w} = (w_1, \dots, w_m)$ be a collection of m weights. Let A_i $(i = 1, \dots, m)$ and B be Young functions. Let $0 \le \alpha < mn$, $0 < p_0 < q_0 < \infty$ and $0 < r_0 < \infty$. We define the following quantity

$$[v, \vec{w}]_{p_0,q_0,r_0,\alpha,B,\vec{A}} := \sup_{Q \subset Q'} \left(\frac{|Q|}{|Q'|} \right)^{\frac{1}{r_0}} |Q'|^{\frac{1}{q_0} - \frac{1}{p_0} + \frac{\alpha}{n}} \|v\|_{B,Q} \prod_{i=1}^{m} \left\| w_i^{-1} \right\|_{\overline{A_i},Q'}. \tag{2.8}$$

In particular, quantity $[v,\vec{w}]_{p_0,q_0,r_0,\alpha,q,\vec{A}}$ denotes $[v,\vec{w}]_{p_0,q_0,r_0,\alpha,B,\vec{A}}$ for the case of $B(t)\cong t^q$ (see the 5th index in (2.8)).

Remark 2.6. *In this paper, for $a>1$, we choose $r_0\in\{q_0,aq_0\}$. Quantity* (2.8) *generalizes several quantities in* [21, pp.152-153]*. For example, we consider the following case:*

$$[v,\vec{w}]^{s_0,q_0}_{q,\vec{P}}=\sup_{Q\subset Q'}\left(\frac{|Q|}{|Q'|}\right)^{\frac{1}{q_0}}|Q'|^{\frac{1}{s_0}}\left(\fint_Q v(x)^q dx\right)^{\frac{1}{q}}\prod_{i=1}^m\left(\fint_{Q'} w_i(x)^{-p_i'}dx\right)^{\frac{1}{p_i'}}. \tag{2.9}$$

Taking $r_0=q_0$, $\frac{1}{q_0}=\frac{1}{p_0}+\frac{1}{s_0}-\frac{\alpha}{n}$, $v(t)=t^q$ and $A_i(t)=t^{p_i}$ $(i=1,2,\ldots,m)$, (2.8) *corresponds to* (2.9).

Theorem 2.7. *Let $0\le\alpha<mn$, $0<p_0<\infty$, $0<q\le q_0<\infty$, $p_0<q_0$ and $a>1$. Let A_i $(i=1,\ldots,m)$ and B be Young functions.*

(1) If $[v,\vec{w}]_{p_0,q_0,q_0,\alpha,q,\vec{A}}<\infty$, then,

$$\left\|\mathcal{M}_{\alpha,m}\left(\vec{f}\right)v\right\|_{\mathcal{M}^{q_0}_q}\lesssim[v,\vec{w}]_{p_0,q_0,q_0,\alpha,q,\vec{A}}\left\|\mathcal{M}_{\vec{A},\frac{n}{p_0}-\frac{n}{q_0}}\left(\vec{f_w}\right)\right\|_{\mathcal{M}^{q_0}_q}. \tag{2.10}$$

Here, $\vec{f_w}:=(f_1w_1,f_2w_2,\ldots,f_mw_m)$.

(2) If $\alpha>0$, $0<q\le1$ and $[v,\vec{w}]_{p_0,q_0,aq_0,\alpha,q,\vec{A}}<\infty$, then,

$$\left\|I_{\alpha,m}\left(\vec{f}\right)v\right\|_{\mathcal{M}^{q_0}_q}\lesssim[v,\vec{w}]_{p_0,q_0,aq_0,\alpha,q,\vec{A}}\left\|\mathcal{M}_{\vec{A},\frac{n}{p_0}-\frac{n}{q_0}}\left(\vec{f_w}\right)\right\|_{\mathcal{M}^{q_0}_q}. \tag{2.11}$$

(3) If $\alpha>0$, $1<q<\infty$, $\overline{B}\in B_{q'}$ and $[v,\vec{w}]_{p_0,q_0,aq_0,\alpha,B,\vec{A}}<\infty$, then,

$$\left\|I_{\alpha,m}\left(\vec{f}\right)v\right\|_{\mathcal{M}^{q_0}_q}\lesssim[v,\vec{w}]_{p_0,q_0,aq_0,\alpha,B,\vec{A}}\left\|\mathcal{M}_{\vec{A},\frac{n}{p_0}-\frac{n}{q_0}}\left(\vec{f_w}\right)\right\|_{\mathcal{M}^{q_0}_q}. \tag{2.12}$$

Remark 2.8. *Theorems 2.3 and 2.7 improve the results in* [21, Theorem 3.3 in p.152]*: For $a>1$ sufficiently small, $p_i>1$ $(i=1,2,\ldots,m)$ and $q>1$, if $A_i(t)=t^{p_i/a}$ and $B(t)=t^{aq}$, then $\bar{A}_i(t)\cong t^{(p_i/a)'}$ and $\bar{B}(t)\in B_{q'}$ hold, respectively. Theorem 2.7 implies that under the appropriate condition of weights, Orlicz-fractional maximal operator controls weighted norms of fractional integrals and maximal operators.*

We invoke the results in [21, Theorem 3.3 in p.152] which Theorems 2.3 and 2.7 enhance.

Corollary 2.9. *Let v be a weight on $\mathbb{R}^n$ and $\vec{w}=(w_1,\ldots,w_m)$ be a collection of m weights on $\mathbb{R}^n$. Let $0\le\alpha<mn$, $0<p\le p_0<\infty$, $0<q\le q_0<r_0\le\infty$ and $1<a<\min\left\{\frac{r_0}{q_0},p_1,\ldots,p_m\right\}$. Suppose that $\frac{1}{q_0}=\frac{1}{p_0}+\frac{1}{r_0}-\frac{\alpha}{n}$ and $\frac{q}{q_0}=\frac{p}{p_0}$. Taking $\vec{A}(t)=\left(t^{\frac{p_1}{a}},\ldots,t^{\frac{p_m}{a}}\right)$ and $B(t)=t^{aq}$, we have the followings:*

(1) If $[v,\vec{w}]_{p_0,q_0,q_0,\alpha,q,\vec{A}}<\infty$, then,

$$\left\|\mathcal{M}_{\alpha,m}\left(\vec{f}\right)v\right\|_{\mathcal{M}^{q_0}_q}\lesssim[v,\vec{w}]_{p_0,q_0,q_0,\alpha,q,\vec{A}}\left\|\vec{f_w}\right\|_{\mathcal{M}^{p_0}_{\vec{P}}}.$$

(2) Let $\alpha>0$ and $0<q\le1$. If $[v,\vec{w}]_{p_0,q_0,aq_0,\alpha,q,\vec{A}}<\infty$, then

$$\left\|I_{\alpha,m}\left(\vec{f}\right)v\right\|_{\mathcal{M}^{q_0}_q}\lesssim[v,\vec{w}]_{p_0,q_0,aq_0,\alpha,q,\vec{A}}\left\|\vec{f_w}\right\|_{\mathcal{M}^{p_0}_{\vec{P}}}.$$

(3) For $\alpha>0$ and $q>1$, if $[v,\vec{w}]_{p_0,q_0,aq_0,\alpha,B,\vec{A}}<\infty$, then

$$\left\|I_{\alpha,m}\left(\vec{f}\right)v\right\|_{\mathcal{M}^{q_0}_q}\lesssim[v,\vec{w}]_{p_0,q_0,aq_0,\alpha,B,\vec{A}}\left\|\vec{f_w}\right\|_{\mathcal{M}^{p_0}_{\vec{P}}}.$$

Thirdly, we can generalize (1) in Theorem 2.7.

Theorem 2.10. *Let $0 \le \alpha < mn$, $0 < p_0 < \infty$, $0 < q \le q_0 < \infty$ and $p_0 < q_0$. Let A_i, B_i and C_i $(i = 1, 2, \ldots, m)$ be Young functions. For each A_i, B_i and C_i, we assume that $A_i^{-1}(t)\overline{B_i}^{-1}(t) \le C_i^{-1}(t)$ $(i = 1, 2, \ldots, m)$. Let $\vec{A} = (A_1, \ldots, A_m)$, $\overline{\vec{B}} = \left(\overline{B_1}, \ldots, \overline{B_m}\right)$ and $\vec{C} = (C_1, \ldots, C_m)$. If $[v, \vec{w}]_{p_0,q_0,q_0,\alpha,q,\overline{\vec{B}}} < \infty$, then we have*

$$\left\| \mathcal{M}_{\vec{C},\alpha}\left(\vec{f}\right) v \right\|_{\mathcal{M}^{q_0}_q} \lesssim [v, \vec{w}]_{p_0,q_0,q_0,\alpha,q,\overline{\vec{B}}} \left\| \mathcal{M}_{\vec{A},\frac{n}{p_0}-\frac{n}{q_0}}\left(\vec{f}_w\right) \right\|_{\mathcal{M}^{q_0}_q}. \tag{2.13}$$

Theorems 2.1 and 2.10 give the following estimates:

Corollary 2.11. *Under the condition of Theorem 2.10, add the assumption $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ and $\frac{q}{q_0} = \frac{p}{p_0}$.*
(1) $p_i > 1$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $A_i \in B_{p_i}$ $(i = 1, 2, \ldots, m)$, then

$$\left\| \mathcal{M}_{\vec{C},\alpha}\left(\vec{f}\right) v \right\|_{\mathcal{M}^{q_0}_q} \lesssim [v, \vec{w}]_{p_0,q_0,q_0,\alpha,q,\overline{\vec{B}}} \left\| \vec{f}_w \right\|^{1-\frac{p_0}{q_0}}_{\mathcal{M}^{p_0}_{\vec{A}}} \left\| \vec{f}_w \right\|^{\frac{p_0}{q_0}}_{\mathcal{M}^{p_0}_{\vec{P}}}. \tag{2.14}$$

(2) If $p_1 = p_2 = \cdots = p_m = 1$, then,

$$\left\| \mathcal{M}_{\vec{C},\alpha}\left(\vec{f}\right) v \right\|_{\mathcal{M}^{q_0}_q} \lesssim [v, \vec{w}]_{p_0,q_0,q_0,\alpha,q,\overline{\vec{B}}} \left\| \vec{f}_w \right\|^{1-\frac{p_0}{q_0}}_{\mathcal{M}^{p_0}_{\vec{A}}} \left\| \vec{f}_w \right\|^{\frac{p_0}{q_0}}_{\mathcal{M}^{p_0}_{\vec{D}}}. \tag{2.15}$$

(3) If $p < \frac{1}{m}$, then,

$$\left\| \mathcal{M}_{\vec{C},\alpha}\left(\vec{f}\right) v \right\|_{\mathcal{M}^{q_0}_q} \lesssim [v, \vec{w}]_{p_0,q_0,q_0,\alpha,q,\overline{\vec{B}}} \left\| \vec{f}_w \right\|_{\mathcal{M}^{p_0}_{\vec{A}}}. \tag{2.16}$$

3. Some lemmas

In Sections 3 and 4, we assume that $f_i(x) \ge 0$ a.e. $x \in \mathbb{R}^n$ $(i = 1, \ldots, m)$. Firstly, to show (3) in Lemma 3.3 and (4) in Lemma 3.12 below, we invoke the next lemma (see [27, Lemma 4.1 in p.146]):

Lemma 3.1. *Suppose that B is a Young function and that f is a non-negative bounded function with compact support. For each $\lambda > 0$,*

$$|\{x \in \mathbb{R}^n : M_B f(x) > \lambda\}| \le C_0 \int_{\{x \in \mathbb{R}^n : 2f(x) > \lambda\}} B\left(\frac{f(x)}{\lambda}\right) dx.$$

Remark 3.2. *We use constant C_0 in Lemma 3.1 below.*

Lemma 3.3. *Let $1 \le p_1, \ldots, p_m < \infty$, $0 < p \le p_0 < \infty$ and A_i $(i = 1, 2, \ldots, m)$ be Young functions.*
(1) If $p_i > 1$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $A_i \in B_{p_i}$ $(i = 1, 2, \ldots, m)$, then,

$$\left\| \mathcal{M}_{\vec{A}}\left(\vec{f}\right) \right\|_{\mathcal{M}^{p_0}_p} \lesssim \left\| \vec{f} \right\|_{\mathcal{M}^{p_0}_{\vec{P}}}. \tag{3.1}$$

(2) If $p = \frac{1}{m}$, then

$$\left\| \mathcal{M}_{\vec{A}}\left(\vec{f}\right) \right\|_{\mathcal{M}^{p_0}_p} \lesssim \left\| \vec{f} \right\|_{\mathcal{M}^{p_0}_{\vec{D}}}. \tag{3.2}$$

(3) If $p < \frac{1}{m}$, then

$$\left\| \mathcal{M}_{\vec{A}}\left(\vec{f}\right) \right\|_{\mathcal{M}^{p_0}_p} \lesssim \left\| \vec{f} \right\|_{\mathcal{M}^{p_0}_{\vec{A}}}. \tag{3.3}$$

Proof. (1) Fix $Q_0 \subset \mathbb{R}^n$ a cube. $f_j = f_j \chi_{3Q_0} + f_j \chi_{(3Q_0)^C} = f_j^0 + f_j^\infty$ $(j = 1, 2, \ldots, m)$. Then,

$$\mathcal{M}_{\vec{A}}\left(\vec{f}\right)(x) \le \mathcal{M}_{\vec{A}}\left(\vec{f_0}\right)(x) + \sum_{\vec{\ell} \ne \vec{0}} \mathcal{M}_{\vec{A}}\left(\vec{f_\ell}\right)(x),$$

where $\vec{f}_0 = (f_1^0, \dots, f_m^0)$, $\vec{f}_\ell = \left(f_1^{\ell_1}, \dots, f_m^{\ell_m}\right)$ and $\vec{\ell} = (\ell_1, \dots, \ell_m) \in \{0, \infty\}^m$. We take a cube $Q \subset \mathbb{R}^n$ that the cube Q satisfies $Q \cap Q_0 \neq \emptyset$ and assume that $x \in Q \cap Q_0$. In the case $\vec{\ell} \neq \vec{0}$, there exists at least $i \in \{1, 2, \dots, m\}$ such that $\ell_i = \infty$. If $Q \cap (3Q_0)^C = \emptyset$, for this index i, $\left\| f_i^{\ell_i} \right\|_{A_i, Q} = 0$ holds. This shows that $\prod_{j=1}^m \left\| f_j^{\ell_j} \right\|_{A_j, Q} = 0$ holds. Therefore, we may assume that $Q \cap (3Q_0)^C \neq \emptyset$. In this situation, note that $Q_0 \subset 3Q$.
Keeping this in mind, we obtain

$$\mathcal{M}_{\vec{A}}\left(\vec{f}_\ell\right)(x) \lesssim \sup_{Q_0 \subset 3Q} \prod_{i=1}^m \|f_i\|_{A_i, 3Q} = \sup_{Q_0 \subset Q'} \prod_{i=1}^m \|f_i\|_{A_i, Q'}.$$

Since $A_i \in B_{p_i}$ $(i = 1, 2, \dots, m)$, Proposition 1.15 gives the inequality:

$$\|f_i\|_{A_i, Q'} \lesssim \left(\fint_{Q'} |f_i(y_i)|^{p_i} \, dy_i \right)^{\frac{1}{p_i}}. \tag{3.4}$$

Therefore, we have

$$\sup_{Q_0 \subset Q'} \prod_{i=1}^m \|f_i\|_{A_i, Q'} \lesssim \sup_{Q_0 \subset Q'} \prod_{i=1}^m \left(\fint_{Q'} |f_i(y_i)|^{p_i} \, dy_i \right)^{\frac{1}{p_i}}.$$

By (1.13),

$$\mathcal{M}_{\vec{A}}\left(\vec{f}_\ell\right)(x) \lesssim \left\|\vec{f}\right\|_{\mathcal{M}^{p_0}_{\vec{P}}} \sup_{Q_0 \subset Q'} |Q'|^{-\frac{1}{p}} = |Q_0|^{-\frac{1}{p}} \left\|\vec{f}\right\|_{\mathcal{M}^{p_0}_{\vec{P}}}.$$

Therefore, we get the following inequality:

$$|Q_0|^{\frac{1}{p_0}} \left(\fint_{Q_0} \mathcal{M}_{\vec{A}}\left(\vec{f}_\ell\right)(x)^p dx \right)^{\frac{1}{p}} \lesssim \left\|\vec{f}\right\|_{\mathcal{M}^{p_0}_{\vec{P}}}.$$

On the other hand, we evaluate $\mathcal{M}_{\vec{A}}\left(\vec{f}_0\right)(x)$. Changing the order of 'sup' and '$\prod$', we obtain

$$\mathcal{M}_{\vec{A}}\left(\vec{f}_0\right)(x) \leq \prod_{i=1}^m M_{A_i}\left(f_i^0\right)(x).$$

By Hölder's inequality for $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$,

$$\left(\fint_{Q_0} \left(\prod_{i=1}^m M_{A_i}\left(f_i^0\right)(x) \right)^p dx \right)^{\frac{1}{p}} \leq \prod_{i=1}^m \left(\fint_{Q_0} \left(M_{A_i}\left(f_i^0\right)(x) \right)^{p_i} dx \right)^{\frac{1}{p_i}}.$$

Since $A_i \in B_{p_i}$ $(i = 1, 2, \dots, m)$, Proposition 1.14 implies that

$$\prod_{i=1}^m \left(\fint_{Q_0} \left(M_{A_i}\left(f_i^0\right)(x) \right)^{p_i} dx \right)^{\frac{1}{p_i}} \lesssim \prod_{i=1}^m \left(\fint_{3Q_0} |f_i(x)|^{p_i} \, dx \right)^{\frac{1}{p_i}}.$$

Therefore, we obtain

$$|Q_0|^{\frac{1}{p_0}} \left(\fint_{Q_0} \mathcal{M}_{\vec{A}}\left(\vec{f}_0\right)(x)^p dx \right)^{\frac{1}{p}} \lesssim \left\|\vec{f}\right\|_{\mathcal{M}^{p_0}_{\vec{P}}}.$$

Hence, we get the desired result.

(2) Changing the order 'sup' and '$\prod$' again, we obtain

$$\mathcal{M}_{\vec{A}}\left(\vec{f}\right)(x) \leq \prod_{i=1}^m M_{A_i}\left(f_i\right)(x). \tag{3.5}$$

Moreover, since $m = \underbrace{1 + \cdots + 1}_{m}$, by Hölder's inequality, we obtain for every cube Q_0,

$$\left(\fint_{Q_0} \mathcal{M}_{\vec{A}}\left(\vec{f}\right)(x)^{\frac{1}{m}} dx\right)^m \le \left(\fint_{Q_0} \prod_{i=1}^m M_{A_i}\left(f_i\right)(x)^{\frac{1}{m}} dx\right)^m \le \prod_{i=1}^m \fint_{Q_0} M_{A_i} f_i(x) dx. \tag{3.6}$$

In [23, p. 371 in Theorem 1.5], we showed the following inequalities: For every cube Q_0,

$$\fint_{Q_0} M_A f(x) dx \lesssim \|f\|_{D,Q_0}. \tag{3.7}$$

Estimates (3.6) and (3.7) imply that

$$|Q_0|^{\frac{1}{p_0}} \left(\fint_{Q_0} \mathcal{M}_{\vec{A}}\left(\vec{f}\right)(x)^{\frac{1}{m}} dx\right)^m \lesssim \left\|\vec{f}\right\|_{\mathcal{M}_{\vec{D}}^{p_0}}. \tag{3.8}$$

(3) Note that

$$\begin{aligned} &\int_{Q_0} \mathcal{M}_{\vec{A}}\left(\vec{f}\right)(x)^p dx \\ &= p\int_0^\infty \lambda^{p-1} \left|\left\{x \in Q_0 : \mathcal{M}_{\vec{A}}\left(\vec{f}\right)(x) > \lambda\right\}\right| d\lambda \\ &= p\left(\int_0^{\prod_{i=1}^m \|f_i\|_{A_i,Q_0}} + \int_{\prod_{i=1}^m \|f_i\|_{A_i,Q_0}}^\infty\right) \lambda^{p-1} \left|\left\{x \in Q_0 : \mathcal{M}_{\vec{A}}\left(\vec{f}\right)(x) > \lambda\right\}\right| d\lambda \\ &= p(I + II). \end{aligned} \tag{3.9}$$

We evaluate I:

$$I \le \int_0^{\prod_{i=1}^m \|f_i\|_{A_i,Q_0}} \lambda^{p-1}|Q_0| dx = |Q_0| \left[\frac{\lambda^p}{p}\right]_0^{\prod_{i=1}^m \|f_i\|_{A_i,Q_0}} = \frac{1}{p}|Q_0| \prod_{i=1}^m \|f_i\|_{A_i,Q_0}^p. \tag{3.10}$$

Next, we evaluate II. By (3.5),

$$II \le \int_{\prod_{i=1}^m \|f_i\|_{A_i,Q_0}}^\infty \lambda^{p-1} \left|\left\{x \in Q_0 : \prod_{i=1}^m M_{A_i}\left(f_i\right)(x) > \lambda\right\}\right| d\lambda. \tag{3.11}$$

For $\lambda > 0$, we take λ_i $(i = 1, 2, \ldots, m)$ as follows:

$$\lambda_i = \left(\frac{\lambda}{\prod_{j=1}^m \|f_j\|_{A_j,Q_0}}\right)^{\frac{1}{m}} \cdot \|f_i\|_{A_i,Q_0}. \tag{3.12}$$

Then, arithmetic shows $\prod_{i=1}^m \lambda_i = \lambda$ holds. Hence, we have

$$II \le \int_{\prod_{i=1}^m \|f_i\|_{A_i,Q_0}}^\infty \lambda^{p-1} \left|\left\{x \in Q_0 : \prod_{i=1}^m M_{A_i}\left(f_i\right)(x) > \prod_{i=1}^m \lambda_i\right\}\right| d\lambda. \tag{3.13}$$

Moreover, considering the contraposition,

$$\left\{x \in Q_0 : \prod_{i=1}^m M_{A_i}\left(f_i\right)(x) > \prod_{i=1}^m \lambda_i\right\} \subset \bigcup_{i=1}^m \left\{x \in Q_0 : M_{A_i}\left(f_i\right)(x) > \lambda_i\right\}. \tag{3.14}$$

By (3.14),

$$II \le \sum_{j=1}^m \int_{\prod_{i=1}^m \|f_i\|_{A_i,Q_0}}^\infty \lambda^{p-1} \left|\left\{x \in Q_0 : M_{A_j}\left(f_j\right)(x) > \lambda_j\right\}\right| d\lambda. \tag{3.15}$$

By Lemma 3.1 and the definition of λ_j,

$$\begin{aligned} II &\leq C_0 \sum_{j=1}^{m} \int_{\prod_{i=1}^{m}\|f_i\|_{A_i,Q_0}}^{\infty} \lambda^{p-1} \left(\int_{Q_0} A_j \left(\frac{f_j(x)}{\lambda_j} \right) dx \right) d\lambda \\ &= C_0 \sum_{j=1}^{m} \int_{\prod_{i=1}^{m}\|f_i\|_{A_i,Q_0}}^{\infty} \int_{Q_0} \lambda^{p-1} A_j \left(\frac{f_j(x)}{\|f_j\|_{A_j,,Q_0}} \cdot \left(\frac{\prod_{k=1}^{m}\|f_k\|_{A_k,Q_0}}{\lambda} \right)^{\frac{1}{m}} \right) dx d\lambda. \end{aligned} \tag{3.16}$$

Since $\prod_{i=1}^{m}\|f_i\|_{A_i,Q_0} < \lambda$, (1.16) gives

$$\begin{aligned} II &\leq C_0 \left(\prod_{k=1}^{m} \|f_k\|_{A_k,Q_0} \right)^{\frac{1}{m}} \left(\int_{\prod_{i=1}^{m}\|f_i\|_{A_i,Q_0}}^{\infty} \lambda^{p-1-\frac{1}{m}} d\lambda \right) \\ &\quad \times \sum_{j=1}^{m} \left(\fint_{Q_0} A_j \left(\frac{f_j(x)}{\|f_j\|_{A_j,,Q_0}} \right) dx \right) |Q_0| \\ &= \frac{C_0}{\frac{1}{m} - p} \left(\prod_{i=1}^{m} \|f_i\|_{A_i,Q_0} \right)^{p} \sum_{j=1}^{m} \left(\fint_{Q_0} A_j \left(\frac{f_j(x)}{\|f_j\|_{A_j,,Q_0}} \right) dx \right) |Q_0|. \end{aligned} \tag{3.17}$$

By (1.14),

$$II \leq C_0 \frac{m^2}{1-mp} |Q_0| \left(\prod_{i=1}^{m} \|f_i\|_{A_i,Q_0} \right)^{p}. \tag{3.18}$$

(3.10) and (3.18) imply that

$$|Q_0|^{\frac{1}{p_0}} \left(\fint_{Q_0} \mathcal{M}_{\vec{A}} \left(\vec{f} \right)(x)^p dx \right)^{\frac{1}{p}} \leq \left(1 + C_0 \frac{m^2 p}{1-mp} \right)^{\frac{1}{p}} \left\| \vec{f} \right\|_{\mathcal{M}_{\vec{A}}^{p_0}}. \tag{3.19}$$

□

To analyze $I_{\alpha,m}\left(\vec{f}\right)(x)$, the following is an essential lemma (see [21, p.157]).

Lemma 3.4. *For a dyadic cube Q_0, fix $x \in Q_0$. Let $\gamma_0 = m_{3Q_0}\left(\vec{f}\right)$ and $A_0 = \left(2^{n+1}3^{2n}m\right)^m$. Set, for $k = 1, 2, \ldots,$*

$$D_k = \bigcup \left\{ Q \in \mathcal{D}(Q_0), Q \ni x, m_{3Q}\left(\vec{f}\right) > \gamma_0 A_0^k \right\}.$$

Considering the maximal cubes concerning inclusion, we can write

$$D_k = \bigcup_j Q_{k,j}.$$

Here, the cubes $\{Q_{k,j}\}$ has the following properties:
(1) $Q_{k,j} \in \mathcal{D}(Q_0)$ are nonoverlapping.
(2) The following inequalities hold:

$$\gamma_0 A_0^k < m_{3Q_{k,j}}\left(\vec{f}\right) \leq 2^{mn}\gamma_0 A_0^k. \tag{3.20}$$

Moreover, let $E_0 = Q_0 \backslash D_1$ and $E_{k,j} = Q_{k,j} \backslash D_{k+1}$. Then, the following properties hold:
(3) $\{E_0\}$ and $\{E_{k,j}\}$ are a disjoint family of sets which decomposes Q_0.
(4) The sets E_0 and $E_{k,j}$ satisfy

$$|E_0| \leq |Q_0| \leq 2|E_0| \quad \text{and} \quad |E_{k,j}| \leq |Q_{k,j}| \leq 2|E_{k,j}|. \tag{3.21}$$

(5) Let

$$\mathcal{D}_0(Q_0) := \left\{Q \in \mathcal{D}(Q_0) : m_{3Q}\left(\vec{f}\right) > \gamma_0 A_0\right\},$$
$$\mathcal{D}_{k,j}(Q_0) := \left\{Q \in \mathcal{D}(Q_0) : Q \subset Q_{k,j}, \gamma_0 A_0^k < m_{3Q}\left(\vec{f}\right) \le \gamma_0 A_0^{k+1}\right\}. \tag{3.22}$$

Then, $\mathcal{D}(Q_0) = \mathcal{D}_0(Q_0) \cup \left(\bigcup_{k,j} \mathcal{D}_{k,j}(Q_0)\right)$ *holds and we have*

$$\int_{Q_0} I_{\alpha,m}\left(\vec{f_0}\right)(x)v(x)g(x)dx$$
$$\lesssim \ell(Q_0)^\alpha m_{3Q_0}\left(\vec{f}\right) m_{Q_0}(vg)\,|Q_0| + \sum_{k,j} \ell(Q_{k,j})^\alpha m_{3Q_{k,j}}\left(\vec{f}\right) m_{Q_{k,j}}(vg)\,|Q_{k,j}|. \tag{3.23}$$

Next, in the proof of Theorem 2.7, we use the following principal lemma (see [21, p.156]). To simplify the notation, let $\vec{I}_m(t) := \underbrace{(t, t, \dots, t)}_{m}$.

Lemma 3.5. *Let v be a weight on $\mathbb{R}^n$. For a dyadic cube Q_0, fix $\vec{f_0} = (f_1\chi_{3Q_0}, \dots, f_m\chi_{3Q_0})$. Adding t^q to $\vec{I}_m(t)$, let $\vec{I}_{m,q}(t) = \left(\vec{I}_m(t), t^q\right)$ be the collection $m+1$ Young functions. Then, there exists a constant C independent of v, $\vec{f}$ and Q_0 such that the following inequalities hold:*
(1) Let $0 \le \alpha < mn$. If $0 < q < \infty$, then

$$\left\|\mathcal{M}_{\alpha,m}\left(\vec{f_0}\right)v\right\|_{L^q(Q_0)} \le C\left\|\tilde{\mathcal{M}}_{\vec{I}_{m,q},\alpha}\left(\vec{f_0}, v\right)\right\|_{L^q(Q_0)}. \tag{3.24}$$

Here $\left(\vec{f}, v\right) := (f_1, \dots, f_m, v)$ and

$$\tilde{\mathcal{M}}_{\vec{I}_{m,q},\alpha}\left(\vec{f}, v\right)(x) := \sup_{Q:\text{cube}} \ell(Q)^\alpha m_{3Q}\left(\vec{f}\right)\left(⨍_Q v(y)^q dy\right)^{\frac{1}{q}} \chi_Q(x).$$

(2) Let $0 < \alpha < mn$. If $0 < q \le 1$, then

$$\left\|I_{\alpha,m}\left(\vec{f_0}\right)v\right\|_{L^q(Q_0)} \le C\left\|\tilde{\mathcal{M}}_{\vec{I}_{m,q},\alpha}\left(\vec{f_0}, v\right)\right\|_{L^q(Q_0)}. \tag{3.25}$$

(3) Let $0 < \alpha < mn$. If $q > 1$, then

$$\left\|I_{\alpha,m}\left(\vec{f_0}\right)v\right\|_{L^q(Q_0)} \le C\left\|\tilde{\mathcal{M}}_{\vec{I}_{m,q},\alpha}\left(\vec{f_0}, v\right)\right\|_{L^q(Q_0)}. \tag{3.26}$$

Remark 3.6. *Lemma 3.5 implies that the norm of $m+1$-fold multilinear type operator controls weighted norm of m-fold multilinear fractional integrals and maximal operator. Even one linear case, we need to consider the estimate of bilinear type maximal operator. To show (3) of Theorem 2.7, we need to modify* (3.26) *to Orlicz-fractional type maximal operator.*

Lemma 3.7. *Let v be a weight on $\mathbb{R}^n$ and B be a Young function. Adding $B(t)$ to $\vec{I}_m(t)$, let $\vec{I}_{m,B}(t) = \left(\vec{I}_m(t), B(t)\right)$ be the collection of $m+1$ Young functions. For a dyadic cube Q_0, fix $\vec{f_0} = (f_1\chi_{3Q_0}, \dots, f_m\chi_{3Q_0})$. If $q > 1$ and $\overline{B} \in B_{q'}$, then,*

$$\left\|I_{\alpha,m}\left(\vec{f_0}\right)v\right\|_{L^q(Q_0)} \lesssim \left\|\tilde{\mathcal{M}}_{\vec{I}_{m,B},\alpha}\left(\vec{f_0}, v\right)\right\|_{L^q(Q_0)}. \tag{3.27}$$

Here,

$$\tilde{\mathcal{M}}_{\vec{I}_{m,B},\alpha}\left(\vec{f}, v\right)(x) := \sup_{Q:\text{cube}} \ell(Q)^\alpha m_{3Q}\left(\vec{f}\right)\|v\|_{B,Q}\,\chi_Q(x).$$

Proof. Let $g \in L^{q'}(Q_0)$ such that $\|g\|_{L^{q'}(Q_0)} = 1$, supp $(g) \subset Q_0$, $g(x) \geq 0$ a.e. $x \in Q_0$. By duality argument, we analyze

$$\int_{Q_0} I_{\alpha,m}\left(\vec{f_0}\right)(x)v(x)g(x)dx.$$

By (3.23),

$$\begin{aligned}(3.28)\qquad &\int_{Q_0} I_{\alpha,m}\left(\vec{f_0}\right)(x)v(x)g(x)dx \\ &\lesssim \ell(Q_0)^{\alpha} m_{3Q_0}\left(\vec{f}\right) m_{Q_0}(vg)\,|Q_0| + \sum_{k,j} \ell(Q_{k,j})^{\alpha} m_{3Q_{k,j}}\left(\vec{f}\right) m_{Q_{k,j}}(vg)\,|Q_{k,j}|.\end{aligned}$$

By (1.20) and (3.21),

$$(3.29)\qquad \ell(Q_{k,j})^{\alpha} m_{3Q_{k,j}}\left(\vec{f}\right) m_{Q_{k,j}}(vg)\,|Q_{k,j}| \leq 4\ell(Q_{k,j})^{\alpha} m_{3Q_{k,j}}\left(\vec{f}\right) \|v\|_{B,Q_{k,j}} \|g\|_{\overline{B},Q_{k,j}} |E_{k,j}|.$$

Since $|E_{k,j}| = \int_{E_{k,j}} dx$,

$$\begin{aligned}(3.30)\qquad &\ell(Q_{k,j})^{\alpha} m_{3Q_{k,j}}\left(\vec{f}\right) \|v\|_{B,Q_{k,j}} \|g\|_{\overline{B},Q_{k,j}} |E_{k,j}| \\ &= \int_{E_{k,j}} \ell(Q_{k,j})^{\alpha} m_{3Q_{k,j}}\left(\vec{f}\right) \|v\|_{B,Q_{k,j}} \|g\|_{\overline{B},Q_{k,j}}\, dx \\ &\leq \int_{E_{k,j}} \tilde{\mathcal{M}}_{\vec{I}_{m,B},\alpha}\left(\vec{f_0}, v\right)(x) M_{\bar{B}} g(x)dx,\end{aligned}$$

A similar argument to (3.29) and (3.30) gives

$$\begin{aligned}(3.31)\qquad &\ell(Q_0)^{\alpha}\left(\prod_{i=1}^{m} \fint_{3Q_0} f_i(y_i)dy_i\right) \fint_{Q_0} v(x)g(x)dx|Q_0| \\ &\lesssim \int_{E_0} \tilde{\mathcal{M}}_{\vec{I}_{m,B},\alpha}\left(\vec{f_0}, v\right) M_{\bar{B}} g(x)dx.\end{aligned}$$

Estimates (3.28)-(3.31) imply that

$$(3.32)\qquad \int_{Q_0} I_{\alpha,m}\left(\vec{f_0}\right)(x)v(x)g(x)dx \lesssim \int_{Q_0} \tilde{\mathcal{M}}_{\vec{I}_{m,B},\alpha}\left(\vec{f_0}, v\right)(x) M_{\bar{B}} g(x)dx.$$

By the Hölder inequality for $q > 1$,

$$\begin{aligned}(3.33)\qquad &\int_{Q_0} \tilde{\mathcal{M}}_{\vec{I}_{m,B},\alpha}\left(\vec{f_0}, v\right)(x) M_{\bar{B}} g(x)dx \\ &\lesssim \left(\int_{Q_0} \tilde{\mathcal{M}}_{\vec{I}_{m,B},\alpha}\left(\vec{f_0}, v\right)(x)^q dx\right)^{\frac{1}{q}} \left(\int_{Q_0} M_{\bar{B}} g(x)^{q'} dx\right)^{\frac{1}{q'}}.\end{aligned}$$

Since $\overline{B} \in B_{q'}$, by Proposition 1.14,

$$(3.34)\qquad \left(\int_{Q_0} M_{\bar{B}} g(x)^{q'} dx\right)^{\frac{1}{q'}} \lesssim \|g\|_{L^{q'}(Q_0)} = 1.$$

Estimates (3.32)-(3.34) imply that

$$(3.35)\qquad \int_{Q_0} I_{\alpha,m}\left(\vec{f_0}\right)(x)v(x)g(x)dx \lesssim \left(\int_{Q_0} \tilde{\mathcal{M}}_{\vec{I}_{m,B},\alpha}\left(\vec{f_0}, v\right)(x)^q dx\right)^{\frac{1}{q}}.$$

□

By [4, Remark 5.12 in p.102], Lemma 3.8 holds:

Lemma 3.8. *If* $1 < q < \infty$ *and* $\overline{B}(t) \lesssim t^{q'}$ $(t \geq 1)$*, then,* $t^q \lesssim B(t)$.

Lemma 3.9. *Given* $0 \leq \alpha < n$*, the followings hold:*
(1) For all $x \in \mathbb{R}^n$,

$$\mathcal{M}_{\alpha,m}\left(\vec{f}\right)(x) \cong \mathcal{M}_{\alpha,m,3\mathcal{D}}\left(\vec{f}\right)(x). \tag{3.36}$$

Here, $\mathcal{M}_{\alpha,m,3\mathcal{D}}\left(\vec{f}\right)(x) := \sup\limits_{x \in Q \in \mathcal{D}(\mathbb{R}^n)} \ell(3Q)^\alpha ⨍_{3Q} |f(y)|dy.$

(2) Let $\vec{A} = (A_1, \dots, A_m)$ *be collection of* m *Young functions. For all* $x \in \mathbb{R}^n$,

$$\mathcal{M}_{\vec{A},\alpha}\left(\vec{f}\right)(x) \cong \mathcal{M}_{\vec{A},\alpha,3\mathcal{D}}\left(\vec{f}\right)(x). \tag{3.37}$$

Here, $\mathcal{M}_{\vec{A},\alpha,3\mathcal{D}}\left(\vec{f}\right)(x) := \sup\limits_{x \in Q \in \mathcal{D}(\mathbb{R}^n)} \ell(3Q)^\alpha \prod\limits_{i=1}^m \|f_i\|_{A_i,3Q}.$

The proof of Lemma 3.9 originates from [23] (see also [27, proof of Lemma 4.1]).

Proof. Fix a point $x \in \mathbb{R}^n$. It suffices to verify $\mathcal{M}_{\vec{A},\alpha}\left(\vec{f}\right)(x) \lesssim \mathcal{M}_{\vec{A},\alpha,3\mathcal{D}}\left(\vec{f}\right)(x)$. For every cube $Q \subset \mathbb{R}^n$ such that $Q \ni x$, there exists a unique integer $k \in \mathbb{Z}$ such that $2^{-(k+1)n} \leq |Q| < 2^{-kn}$. Then, we can choose dyadic cubes J_i $(i = 1, 2, \dots, 2^n)$ such that $|J_i| = 2^{-kn}$ and the dyadic cubes J_i $(i = 1, 2, \dots, 2^n)$ cover Q. That is,

$$Q \subset \bigcup_{i=1}^{2^n} J_i \tag{3.38}$$

and

$$|Q| < |J_i| \leq 2^n |Q|. \tag{3.39}$$

Hence,

$$\ell(Q)^\alpha \prod_{j=1}^m \|f_j\|_{A_j,Q} = \ell(Q)^\alpha \prod_{j=1}^m \|f_j \chi_J\|_{A_j,Q}, \tag{3.40}$$

where $J := \bigcup\limits_{i=1}^{2^n} J_i$. Obviously, for $i = 1, 2, \dots, 2^n$,

$$|J| = 2^n |J_i|. \tag{3.41}$$

By (3.38),

$$\ell(Q)^\alpha \prod_{j=1}^m \|f_j \chi_J\|_{A_j,Q} \leq \ell(Q)^\alpha \prod_{j=1}^m \sum_{i=1}^{2^n} \|f_j \chi_{J_i}\|_{A_j,Q}. \tag{3.42}$$

By (3.39), for $j = 1, 2, \dots, m$,

$$\|f_j \chi_{J_i}\|_{A_j,Q} \leq \inf\left\{\lambda_j > 0 : \frac{2^n}{|J_i|} \int_{J_i} A_j\left(\frac{f_j(x)}{\lambda_j}\right) dx \leq 1\right\}. \tag{3.43}$$

Since $J_i \subset 3J_1$,

$$\begin{aligned} &\inf\left\{\lambda_j > 0 : \frac{2^n}{|J_i|} \int_{J_i} A_j\left(\frac{f_j(x)}{\lambda_j}\right) dx \leq 1\right\} \\ &\leq \inf\left\{\lambda_j > 0 : \frac{6^n}{|3J_1|} \int_{3J_1} A_j\left(\frac{f_j(x)}{\lambda_j}\right) dx \leq 1\right\}. \end{aligned} \tag{3.44}$$

By (1.16),

$$\inf\left\{\lambda_j>0:\frac{6^n}{|3J_1|}\int_{3J_1}A_j\left(\frac{f_j(x)}{\lambda_j}\right)dx\le 1\right\}\le 6^n\|f_j\|_{A_j,3J_1}. \tag{3.45}$$

Estimates (3.40)–(3.45) give

$$\ell(Q)^\alpha\prod_{j=1}^m\|f_j\|_{A_j,Q}\le 6^{mn}\ell(3J_1)^\alpha\prod_{j=1}^m\|f_j\|_{A_j,3J_1}. \tag{3.46}$$

Since the cube $J_1\ni x$ is one dyadic cube, we obtain the desired equivalent. □

Similarly to the proof of Lemma 3.9, we can show that the ordinary Morrey spaces are the equivalence of the dyadic Morrey spaces:

Lemma 3.10. *For $0<p\le p_0<\infty$ and $F\in L^p_{\rm loc}$, we have*

$$\|F\|_{\mathcal{M}^{p_0}_p}\cong\|F\|_{\mathcal{M}^{p_0}_{p,\mathcal{D}}}, \tag{3.47}$$

where,

$$\|F\|_{\mathcal{M}^{p_0}_{p,\mathcal{D}}}:=\sup_{Q\in\mathcal{D}(\mathbb{R}^n)}|Q|^{\frac{1}{p_0}}\left(\fint_Q|F(x)|^pdx\right)^{\frac{1}{p}}.$$

Lemma 3.11. *For $x\in Q_0$ and $\vec{f}=(f_1,\dots,f_m)$,*

$$I_{\alpha,m}\left(\vec{f_0}\right)(x)\lesssim\sum_{Q\in\mathcal{D}(Q_0)}\ell(Q)^\alpha m_{3Q}\left(\vec{f}\right)\chi_Q(x). \tag{3.48}$$

To analyze $\mathcal{M}_{\vec{C},\alpha,3\mathcal{D}}\left(f_1\chi_{3Q_0},\dots,f_m\chi_{3Q_0}\right)(x)$ for $Q_0\in\mathcal{D}(\mathbb{R}^n)$, the following is an essential Lemma.

Lemma 3.12. *For a dyadic cube Q_0, fix $x\in Q_0$. Let $\gamma_1:=\ell(3Q_0)^\alpha\prod_{i=1}^m\|f_i\|_{C_i,3Q_0}$ and $A_1>\max\{(2\cdot 3^nC_0m)^m,2^{mn}\}$. Here, the constant C_0 is in Lemma 3.1. Set, for $k=1,2,\dots,$*

$$D_k=\bigcup\left\{Q\in\mathcal{D}(Q_0),Q\ni x,\ell(3Q)^\alpha\prod_{i=1}^m\|f_i\|_{C_i,3Q}>\gamma_1A_1^k\right\}.$$

Considering the maximal cubes concerning inclusion, we can write

$$D_k:=\bigcup_j Q_{k,j}.$$

Then, the cubes $\{Q_{k,j}\}$ has the following properties:
(1) $Q_{k,j}\in\mathcal{D}(Q_0)$ are nonoverlapping.
(2) The following inequalities hold:

$$\gamma_1A_1^k<\ell(3Q_{k,j})^\alpha\prod_{i=1}^m\|f_i\|_{C_i,3Q_{k,j}}\le 2^{mn}\gamma_1A_1^k. \tag{3.49}$$

Let $E_0:=Q_0\backslash D_1$ and $E_{k,j}=Q_{k,j}\backslash D_{k+1}$. Then, the sets E_0 and $E_{k,j}$ have the following properties:
(3) $\{E_0\}$ and $\{E_{k,j}\}$ are a disjoint family of sets which decomposes Q_0.
(4) The sets E_0 and $E_{k,j}$ satisfy that

$$|E_0|\le|Q_0|\le 2|E_0|\quad\text{and}\quad|E_{k,j}|\le|Q_{k,j}|\le 2|E_{k,j}|. \tag{3.50}$$

Lemma 3.1 gives the proof of Lemma 3.12, which originates from [21, p.158].

Proof of Lemma 3.12. Note that

$$\begin{aligned}
&Q_{k,j} \cap D_{k+1} \\
&\subset \left\{ x \in Q_{k,j} : Q \ni x, Q \in \mathcal{D}(Q_0), Q \subset Q_{k,j}, \ell(3Q)^{\alpha} \prod_{i=1}^{m} \|f_i\|_{C_i,3Q} > \gamma_1 A_1^{k+1} \right\} \\
&\subset \left\{ x \in Q_{k,j} : \mathcal{M}_{\vec{C}} \left(f_1\chi_{3Q_{k,j}}, \ldots, f_m\chi_{3Q_{k,j}} \right)(x) > \frac{\gamma_1 A_1^{k+1}}{\ell(3Q_{k,j})^{\alpha}} \right\}.
\end{aligned} \tag{3.51}$$

Changing the order of 'sup' and '$\prod$', we obtain

$$\begin{aligned}
&\left\{ x \in Q_{k,j} : \mathcal{M}_{\vec{C}} \left(f_1\chi_{3Q_{k,j}}, \ldots, f_m\chi_{3Q_{k,j}} \right)(x) > \frac{\gamma_1 A_1^{k+1}}{\ell(3Q_{k,j})^{\alpha}} \right\} \\
&\subset \left\{ x \in Q_{k,j} : \prod_{i=1}^{m} M_{C_i} \left(f_i\chi_{3Q_{k,j}} \right)(x) > \frac{\gamma_1 A_1^{k+1}}{\ell(3Q_{k,j})^{\alpha}} \right\}.
\end{aligned} \tag{3.52}$$

Letting $\Gamma_i := \left(\frac{\gamma_1 A_1^{k+1}}{\ell(3Q_{k,j})^{\alpha}} \right)^{\frac{1}{m}} \frac{\|f_i\|_{C_i,3Q_{k,j}}}{\left(\|f_1\|_{C_1,3Q_{k,j}} \cdots \|f_m\|_{C_m,3Q_{k,j}} \right)^{\frac{1}{m}}}$, we have $\prod_{i=1}^{m} \Gamma_i = \frac{\gamma_1 A_1^{k+1}}{\ell(3Q_{k,j})^{\alpha}}$. Considering the contraposition of (3.52), we can show that

$$\begin{aligned}
&\left\{ x \in Q_{k,j} : \prod_{i=1}^{m} M_{C_i} \left(f_i\chi_{3Q_{k,j}} \right)(x) > \frac{\gamma_1 A_1^{k+1}}{\ell(3Q_{k,j})^{\alpha}} \right\} \\
&\subset \bigcup_{i=1}^{m} \left\{ x \in Q_{k,j} : M_{C_i} \left(f_i\chi_{3Q_{k,j}} \right)(x) > \Gamma_i \right\}.
\end{aligned} \tag{3.53}$$

By Lemma 3.1, for $i = 1, 2, \ldots, m$,

$$\left| \left\{ x \in Q_{k,j} : M_{C_i} \left(f_i\chi_{3Q_{k,j}} \right)(x) > \Gamma_i \right\} \right| \le C_0 \int_{3Q_{k,j}} C_i \left(\frac{f_i(x)}{\Gamma_i} \right) dx. \tag{3.54}$$

By the definition of Γ_i,

$$\begin{aligned}
&\int_{3Q_{k,j}} C_i \left(\frac{f_i(x)}{\Gamma_i} \right) dx \\
&= \int_{3Q_{k,j}} C_i \left(\frac{2^n}{A_1^{\frac{1}{m}}} \left(\frac{\ell(3Q_{k,j})^{\alpha} \|f_1\|_{C_1,3Q_{k,j}} \cdots \|f_m\|_{C_m,3Q_{k,j}}}{2^{mn}\gamma_1 A_1^k} \right)^{\frac{1}{m}} \cdot \frac{f_i(x)}{\|f_i\|_{C_i,3Q_{k,j}}} \right) dx
\end{aligned} \tag{3.55}$$

Since $A_1 > 2^{mn}$ and (3.49), applying (1.16),

$$\begin{aligned}
&\int_{3Q_{k,j}} C_i \left(\frac{2^n}{A_1^{\frac{1}{m}}} \left(\frac{\ell(3Q_{k,j})^{\alpha} \|f_1\|_{C_1,3Q_{k,j}} \cdots \|f_m\|_{C_m,3Q_{k,j}}}{2^{mn}\gamma_1 A_1^k} \right)^{\frac{1}{m}} \cdot \frac{f_i(x)}{\|f_i\|_{C_i,3Q_{k,j}}} \right) dx \\
&\le \frac{2^n}{A_1^{\frac{1}{m}}} \left(\frac{\ell(3Q_{k,j})^{\alpha} \|f_1\|_{C_1,3Q_{k,j}} \cdots \|f_m\|_{C_m,3Q_{k,j}}}{2^{mn}\gamma_1 A_1^k} \right)^{\frac{1}{m}} \int_{3Q_{k,j}} C_i \left(\frac{f_i(x)}{\|f_i\|_{C_i,3Q_{k,j}}} \right) dx.
\end{aligned} \tag{3.56}$$

By (1.14),

$$\int_{3Q_{k,j}} C_i \left(\frac{f_i(x)}{\|f_i\|_{C_i,3Q_{k,j}}} \right) dx \le 3^n |Q_{k,j}|. \tag{3.57}$$

Estimates (3.51)-(3.57) give

$$|Q_{k,j} \cap D_{k+1}| \le \left(\frac{(3^n C_0 m)^m}{A_1} \right)^{\frac{1}{m}} |Q_{k,j}|. \tag{3.58}$$

Since $A_1 > (2 \cdot 3^n C_0 m)^m$,

$$|Q_{k,j} \cap D_{k+1}| \le \frac{1}{2} |Q_{k,j}|. \tag{3.59}$$

A similar argument to (3.51)-(3.59) gives

$$|Q_0 \cap D_1| \le \frac{1}{2} |Q_0|. \tag{3.60}$$

By (3.59) and (3.60), we obtain the desired result. □

Remark 3.13. *In Lemmas 3.4 and 3.12, the sets D_k, $Q_{k,j}$ and $E_{k,j}$ are different, respectively. In the context of this paper, we can distinguish these sets explicitly. So, we use these symbols without distinction, respectively.*

(2) in Lemma 3.9 and Lemma 3.12 give the following inequality:

Lemma 3.14. *Let $0 \le \alpha < mn$, $0 < q < \infty$, $\vec{C} = (C_1, C_2, \dots, C_m)$ be a collection of m Young functions. Adding a function t^q to $\vec{C}$, we let $\vec{C}_q = \left(\vec{C}, t^q\right)$ be a collection of $m+1$ Young functions. For a dyadic cube Q_0, fix $\vec{f}_0 = (f_1 \chi_{3Q_0}, \dots, f_m \chi_{3Q_0})$. Then,*

$$\int_{Q_0} \mathcal{M}_{\vec{C},\alpha} \left(\vec{f}_0\right)(x)^q v(x)^q dx \lesssim \int_{Q_0} \tilde{\mathcal{M}}_{\vec{C}_q,\alpha} \left(\vec{f}_0, v\right)(x)^q dx, \tag{3.61}$$

Here,

$$\tilde{\mathcal{M}}_{\vec{C}_q,\alpha} \left(\vec{f}, v\right)(x) := \sup_{Q:\text{cube}} \ell(Q)^\alpha \prod_{i=1}^m \|f_i\|_{C_i,3Q} \left(⨍_Q v(y)^q dy \right)^{\frac{1}{q}} \chi_Q(x).$$

Proof. By Lemma 3.9, we may verify

$$\int_{Q_0} \mathcal{M}_{\vec{C},\alpha,3\mathcal{D}} \left(\vec{f}_0\right)(x)^q v(x)^q dx \lesssim \int_{Q_0} \tilde{\mathcal{M}}_{\vec{C}_q,\alpha} \left(\vec{f}, v\right)(x)^q dx. \tag{3.62}$$

Using E_0 and $E_{k,j}$ in Lemma 3.12, we can decompose $Q_0 = E_0 \cup \left(\bigcup_{k,j} E_{k,j} \right)$. Then,

$$\begin{aligned} &\int_{Q_0} \mathcal{M}_{\vec{C},\alpha,3\mathcal{D}} \left(\vec{f}_0\right)(x)^q v(x)^q dx \\ &= \left(\int_{E_0} + \sum_{k,j} \int_{E_{k,j}} \right) \tilde{\mathcal{M}}_{\vec{C},\alpha,3\mathcal{D}} \left(\vec{f}_0\right)(x)^q v(x)^q dx = I_0 + \sum_{k,j} II_{k,j}. \end{aligned} \tag{3.63}$$

By definitions of sets $E_{k,j}$ and $Q_{k,j}$,

$$II_{k,j} \lesssim \left(\ell(Q_{k,j})^\alpha \prod_{i=1}^m \|f_i\|_{C_i, 3Q_{k,j}} \right)^q \cdot \int_{E_{k,j}} v(x)^q dx. \tag{3.64}$$

By (3.50),

$$II_{k,j} \lesssim \int_{E_{k,j}} \tilde{\mathcal{M}}_{\vec{C}_q,\alpha} \left(\vec{f}_0, v\right)(x)^q dx. \tag{3.65}$$

By the definition of set E_0, a similar argument to (3.64) and (3.50) gives

$$I_0 \lesssim \int_{E_0} \tilde{\mathcal{M}}_{\vec{C}_q,\alpha}\left(\vec{f}_0, v\right)(x)^q dx. \tag{3.66}$$

Estimates (3.63)-(3.66) give (3.62). □

4. Proofs of the main theorems

Proof of Theorem 2.1. Note that

$$\mathcal{M}_{\vec{A},\alpha}\left(\vec{f}\right)(x) = \sup_{Q\ni x}\left(|Q|^{\frac{\alpha}{n}\cdot\frac{q_0}{q_0-p_0}}\prod_{i=1}^m \|f_i\|_{A_i,Q}\right)^{1-\frac{p_0}{q_0}}\left(\prod_{i=1}^m \|f_i\|_{A_i,Q}\right)^{\frac{p_0}{q_0}}. \tag{4.1}$$

Since $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$,

$$\mathcal{M}_{\vec{A},\alpha}\left(\vec{f}\right)(x) = \sup_{Q\ni x}\left(|Q|^{\frac{1}{p_0}}\prod_{i=1}^m \|f_i\|_{A_i,Q}\right)^{\frac{q_0-p_0}{q_0}}\left(\prod_{i=1}^m \|f_i\|_{A_i,Q}\right)^{\frac{p_0}{q_0}}. \tag{4.2}$$

By (1.18) and (1.25),

$$\sup_{Q\ni x}\left(|Q|^{\frac{1}{p_0}}\prod_{i=1}^m \|f_i\|_{A_i,Q}\right)^{\frac{q_0-p_0}{q_0}}\left(\prod_{i=1}^m \|f_i\|_{A_i,Q}\right)^{\frac{p_0}{q_0}} \leq \left\|\vec{f}\right\|_{\mathcal{M}^{p_0}_{\vec{A}}}^{\frac{q_0-p_0}{q_0}}\mathcal{M}_{\vec{A}}\left(\vec{f}\right)(x)^{\frac{p_0}{q_0}}. \tag{4.3}$$

Estimates (4.1)-(4.3) give

$$\left\|\mathcal{M}_{\vec{A},\alpha}\left(\vec{f}\right)(x)\right\|_{\mathcal{M}^{q_0}_q} \leq \left\|\vec{f}\right\|_{\mathcal{M}^{p_0}_{\vec{A}}}^{\frac{q_0-p_0}{q_0}}\cdot\left\|\left(\mathcal{M}_{\vec{A}}\left(\vec{f}\right)\right)^{\frac{p_0}{q_0}}\right\|_{\mathcal{M}^{q_0}_q}. \tag{4.4}$$

Since $\frac{q}{q_0} = \frac{p}{p_0}$,

$$\left\|\left(\mathcal{M}_{\vec{A}}\left(\vec{f}\right)\right)^{\frac{p_0}{q_0}}\right\|_{\mathcal{M}^{q_0}_q} = \left\|\mathcal{M}_{\vec{A}}\left(\vec{f}\right)\right\|_{\mathcal{M}^{p_0}_p}^{\frac{p_0}{q_0}}. \tag{4.5}$$

Estimates (4.4) and (4.5) imply that

$$\left\|\mathcal{M}_{\vec{A},\alpha}\left(\vec{f}\right)\right\|_{\mathcal{M}^{q_0}_q} \leq \left\|\vec{f}\right\|_{\mathcal{M}^{p_0}_{\vec{A}}}^{\frac{q_0-p_0}{q_0}}\left\|\mathcal{M}_{\vec{A}}\left(\vec{f}\right)\right\|_{\mathcal{M}^{p_0}_p}^{\frac{p_0}{q_0}}. \tag{4.6}$$

□

Proof of Theorem 2.7. By Lemma 3.10, we may analyze the weighted estimate of the operator $\mathcal{M}_{\alpha,m}\left(\vec{f}\right)(x)$ in $\mathcal{M}^{q_0}_{q,\mathcal{D}}$. For a dyadic cube Q_0, fix $x \in Q_0$. then, let $f_j = f_j\chi_{3Q_0} + f_j\chi_{(3Q_0)^C} = f_j^0 + f_j^\infty$ $(j = 1, 2, \ldots, m)$. Then, we decompose $\vec{f} = \vec{f}_0 + \sum_{\vec{\ell}\neq\vec{0}}\vec{f}_\ell$, where $\vec{f}_\ell = \left(f_1^{\ell_1}, \ldots, f_m^{\ell_m}\right)$ and $(\ell_1, \ldots, \ell_m) \in \{0, \infty\}^m$. Since $\vec{f} = \vec{f}_0 + \sum_{\vec{\ell}\neq\vec{0}}\vec{f}_\ell$,

$$\mathcal{M}_{\alpha,m}\left(\vec{f}\right)(x) \leq \mathcal{M}_{\alpha,m}\left(\vec{f}_0\right)(x) + \sum_{\vec{\ell}\neq\vec{0}}\mathcal{M}_{\alpha,m}\left(\vec{f}_\ell\right)(x).$$

and

$$I_{\alpha,m}\left(\vec{f}\right)(x) = I_{\alpha,m}\left(\vec{f}_0\right)(x) + \sum_{\vec{\ell}\neq\vec{0}}I_{\alpha,m}\left(\vec{f}_\ell\right)(x).$$

(1) Firstly, we evaluate $\mathcal{M}_{\alpha,m}\left(\vec{f_\ell}\right)(x)$. By a similar argument in the proof of Lemma 3.3, if $x \in Q_0$, then

$$\mathcal{M}_{\alpha,m}\left(\vec{f_\ell}\right)(x) \lesssim \sup_{Q_0\subset Q} \ell(Q)^\alpha m_Q\left(\vec{f}\right). \tag{4.7}$$

By (1.20),

$$\begin{aligned}
&\sup_{Q_0\subset Q} \ell(Q)^\alpha m_Q\left(\vec{f}\right) \lesssim \sup_{Q_0\subset Q} \ell(Q)^\alpha \prod_{i=1}^m \|f_i w_i\|_{A_i,Q} \left\|w_i^{-1}\right\|_{\overline{A_i},Q}. \\
&= \sup_{Q_0\subset Q} |Q|^{\frac{1}{q_0}} \left(\ell(Q)^{\frac{n}{p_0}-\frac{n}{q_0}} \prod_{i=1}^m \|f_i w_i\|_{A_i,Q}\right) |Q|^{\frac{\alpha}{n}-\frac{1}{p_0}} \prod_{i=1}^m \left\|w_i^{-1}\right\|_{\overline{A_i},Q}. \\
&= \sup_{Q_0\subset Q} |Q|^{\frac{1}{q_0}} \left(\fint_Q \left(\ell(Q)^{\frac{n}{p_0}-\frac{n}{q_0}} \prod_{i=1}^m \|f_i w_i\|_{A_i,Q}\right)^q dx\right)^{\frac{1}{q}} |Q|^{\frac{\alpha}{n}-\frac{1}{p_0}} \prod_{i=1}^m \left\|w_i^{-1}\right\|_{\overline{A_i},Q}.
\end{aligned} \tag{4.8}$$

By (1.25), we have

$$\begin{aligned}
&|Q|^{\frac{1}{q_0}} \left(\fint_Q \left(\ell(Q)^{\frac{n}{p_0}-\frac{n}{q_0}} \prod_{i=1}^m \|f_i w_i\|_{A_i,Q}\right)^q dx\right)^{\frac{1}{q}} |Q|^{\frac{\alpha}{n}-\frac{1}{p_0}} \prod_{i=1}^m \left\|w_i^{-1}\right\|_{\overline{A_i},Q} \\
&\le |Q|^{\frac{1}{q_0}} \left(\fint_Q \mathcal{M}_{\vec{A},\frac{n}{p_0}-\frac{n}{q_0}}\left(\vec{f_w}\right)(x)^q dx\right)^{\frac{1}{q}} |Q|^{\frac{\alpha}{n}-\frac{1}{p_0}} \prod_{i=1}^m \left\|w_i^{-1}\right\|_{\overline{A_i},Q}.
\end{aligned} \tag{4.9}$$

By (1.12),

$$\begin{aligned}
&|Q|^{\frac{1}{q_0}} \left(\fint_Q \mathcal{M}_{\vec{A},\frac{n}{p_0}-\frac{n}{q_0}}\left(\vec{f_w}\right)(x)^q dx\right)^{\frac{1}{q}} |Q|^{\frac{\alpha}{n}-\frac{1}{p_0}} \prod_{i=1}^m \left\|w_i^{-1}\right\|_{\overline{A_i},Q} \\
&\le \left\|\mathcal{M}_{\vec{A},\frac{n}{p_0}-\frac{n}{q_0}}\left(\vec{f_w}\right)\right\|_{\mathcal{M}^{q_0}_q} |Q|^{\frac{\alpha}{n}-\frac{1}{p_0}} \prod_{i=1}^m \left\|w_i^{-1}\right\|_{\overline{A_i},Q}.
\end{aligned} \tag{4.10}$$

Estimates (4.7)-(4.10) imply that

$$\begin{aligned}
&|Q_0|^{\frac{1}{q_0}} \left(\fint_{Q_0} \mathcal{M}_{\alpha,m}\left(\vec{f_\ell}\right)(x)^q v(x)^q dx\right)^{\frac{1}{q}} \\
&\lesssim \left\|\mathcal{M}_{\vec{A},\frac{n}{p_0}-\frac{n}{q_0}}\left(\vec{f_w}\right)\right\|_{\mathcal{M}^{q_0}_q} \\
&\times \sup_{Q_0\subset Q} \left(\frac{|Q_0|}{|Q|}\right)^{\frac{1}{q_0}} |Q|^{\frac{1}{q_0}+\frac{\alpha}{n}-\frac{1}{p_0}} \left(\fint_{Q_0} v(x)^q dx\right)^{\frac{1}{q}} \prod_{i=1}^m \left\|w_i^{-1}\right\|_{\overline{A_i},Q}.
\end{aligned} \tag{4.11}$$

By using $[v,\vec{w}]_{p_0,q_0,q_0,\alpha,q,\vec{A}}$, we obtain

$$\sup_{Q_0\subset Q} \left(\frac{|Q_0|}{|Q|}\right)^{\frac{1}{q_0}} |Q|^{\frac{1}{q_0}-\frac{1}{p_0}+\frac{\alpha}{n}} \left(\fint_{Q_0} v(x)^q dx\right)^{\frac{1}{q}} \prod_{i=1}^m \left\|w_i^{-1}\right\|_{\overline{A_i},Q} \le [v,\vec{w}]_{p_0,q_0,q_0,\alpha,q,\vec{A}}. \tag{4.12}$$

Hence, we have

$$\begin{aligned}
&|Q_0|^{\frac{1}{q_0}} \left(\fint_{Q_0} \mathcal{M}_{\alpha,m}\left(\vec{f_\ell}\right)(x)^q v(x)^q dx\right)^{\frac{1}{q}} \\
&\le [v,\vec{w}]_{p_0,q_0,q_0,\alpha,q,\vec{A}} \left\|\mathcal{M}_{\vec{A},\frac{n}{p_0}-\frac{n}{q_0}}\left(\vec{f_w}\right)\right\|_{\mathcal{M}^{q_0}_q}.
\end{aligned} \tag{4.13}$$

Secondly, we evaluate $\mathcal{M}_{\alpha,m}\left(\vec{f_0}\right)(x)$. By (1) in Lemma 3.9, we may replace $\mathcal{M}_{\alpha,m}$ with $\mathcal{M}_{\alpha,m,3\mathcal{D}}$. By Lemma 3.5, we have

$$\int_{Q_0} \mathcal{M}_{\alpha,m,3\mathcal{D}}\left(\vec{f_0}\right)(x)^q v(x)^q dx \lesssim \int_{Q_0} \tilde{\mathcal{M}}_{\vec{I}_{m,q},\alpha}\left(\vec{f_0}, v\right)(x)^q dx. \tag{4.14}$$

By (1.20),

$$\begin{aligned} &\tilde{\mathcal{M}}_{\vec{I}_{m,q},\alpha}\left(\vec{f_0}, v\right)(x) \\ &\lesssim \sup_{x\in Q\in\mathcal{D}(Q_0)} \ell(Q)^\alpha \left(\prod_{i=1}^m \|f_i w_i\|_{A_i,3Q}\right) \|w_i^{-1}\|_{\overline{A_i},3Q} \cdot m_Q\left(v^q\right)^{\frac{1}{q}} \\ &\cong \sup_{x\in Q\in\mathcal{D}(Q_0)} \left(\frac{|Q|}{|3Q|}\right)^{\frac{1}{q_0}} |3Q|^{\frac{1}{q_0}-\frac{1}{p_0}+\frac{\alpha}{n}} m_Q\left(v^q\right)^{\frac{1}{q}} \prod_{i=1}^m \|w_i^{-1}\|_{\overline{A_i},3Q} \\ &\times \ell(Q)^{\frac{n}{p_0}-\frac{n}{q_0}} \prod_{i=1}^m \|f_i w_i\|_{A_i,3Q}. \end{aligned} \tag{4.15}$$

By using $[v,\vec{w}]_{p_0,q_0,q_0,\alpha,q,\vec{A}}$, for every $x \in Q_0$,

$$\tilde{\mathcal{M}}_{\vec{I}_{m,q},\alpha}\left(\vec{f_0}, v\right)(x) \lesssim [v,\vec{w}]_{p_0,q_0,q_0,\alpha,q,\vec{A}} \cdot \mathcal{M}_{\vec{A},\frac{n}{p_0}-\frac{n}{q_0}}\left(\vec{f}\right)(x). \tag{4.16}$$

Estimates (4.14)-(4.16) give the following:

$$|Q_0|^{\frac{1}{q_0}} \left(\fint_{Q_0} \mathcal{M}_{\alpha,m}\left(\vec{f_0}\right)(x)^q v(x)^q dx\right)^{\frac{1}{q}} \le [v,\vec{w}]_{p_0,q_0,q_0,\alpha,q,\vec{A}} \left\|\mathcal{M}_{\vec{A},\frac{n}{p_0}-\frac{n}{q_0}}\left(\vec{f_w}\right)\right\|_{\mathcal{M}^{q_0}_q}. \tag{4.17}$$

Therefore, estimates (4.13) and (4.17) give the desired result.

(2) Firstly, we evaluate $I_{\alpha,m}\left(\vec{f_\ell}\right)(x)$. By a geometric observation, for $x \in Q_0$, $\{y_j : |x-y_j| \le 2^k\ell(Q_0)\} \subset 3\cdot 2^k Q_0$. Then,

$$I_{\alpha,m}\left(\vec{f_\ell}\right)(x) \lesssim \sum_{k=1}^\infty |3\cdot 2^k Q_0|^{\frac{\alpha}{n}} \prod_{j=1}^m \left(\fint_{3\cdot 2^k Q_0} f_j(y_j)dy_j\right). \tag{4.18}$$

By (1.20),

$$\begin{aligned} &\sum_{k=1}^\infty |3\cdot 2^k Q_0|^{\frac{\alpha}{n}} \prod_{j=1}^m \left(\fint_{3\cdot 2^k Q_0} f_j(y_j)dy_j\right) \\ &\lesssim \sum_{k=1}^\infty |3\cdot 2^k Q_0|^{\frac{\alpha}{n}} \prod_{j=1}^m \|f_j w_j\|_{A_j,3\cdot 2^k Q_0} \|w_j^{-1}\|_{\overline{A_j},3\cdot 2^k Q_0} \\ &= \sum_{k=1}^\infty |3\cdot 2^k Q_0|^{\frac{\alpha}{n}-\frac{1}{p_0}} \prod_{j=1}^m \|w_j^{-1}\|_{\overline{A_j},3\cdot 2^k Q_0} \\ &\times |3\cdot 2^k Q_0|^{\frac{1}{q_0}} \left(\fint_{3\cdot 2^k Q_0} \left(\ell\left(3\cdot 2^k Q_0\right)^{\frac{n}{p_0}-\frac{n}{q_0}} \prod_{j=1}^m \|f_j w_j\|_{A_j,3\cdot 2^k Q_0}\right)^q dx\right)^{\frac{1}{q}}. \end{aligned} \tag{4.19}$$

By (1.12) and (1.25),

$$I_{\alpha,m}\left(\vec{f_\ell}\right)(x) \lesssim \sum_{k=1}^\infty |3\cdot 2^k Q_0|^{\frac{\alpha}{n}-\frac{1}{p_0}} \left\|\mathcal{M}_{\vec{A},\frac{n}{p_0}-\frac{n}{q_0}}\left(\vec{f_w}\right)\right\|_{\mathcal{M}^{q_0}_q} \prod_{j=1}^m \|w_j^{-1}\|_{\overline{A_j},3\cdot 2^k Q_0}. \tag{4.20}$$

Estimate (4.20) entails

$$
\begin{aligned}
&|Q_0|^{\frac{1}{q_0}} \left(\fint_{Q_0} I_{\alpha,m}\left(\vec{f}_\ell\right)(x)^q v(x)^q dx \right)^{\frac{1}{q}} \\
&\lesssim \left\| \mathcal{M}_{\vec{A},\frac{n}{p_0}-\frac{n}{q_0}}\left(\vec{f}_w\right) \right\|_{\mathcal{M}^{q_0}_q} \\
&\times \sum_{k=1}^{\infty} \left(\frac{|Q_0|}{|3\cdot 2^k Q_0|} \right)^{\frac{1}{aq_0}} \left|3\cdot 2^k Q_0\right|^{\frac{1}{q_0}+\frac{\alpha}{n}-\frac{1}{p_0}} \left(\fint_{Q_0} v(x)^q dx \right)^{\frac{1}{q}} \prod_{i=1}^{m} \|w_i^{-1}\|_{\overline{A_i},3\cdot 2^k Q_0} \\
&\times \left(\frac{|Q_0|}{|3\cdot 2^k Q_0|} \right)^{\frac{1}{q_0}\left(1-\frac{1}{a}\right)}.
\end{aligned} \tag{4.21}
$$

By using $[v,\vec{w}]_{p_0,q_0,aq_0,\alpha,q,\vec{A}}$,

$$
\begin{aligned}
&|Q_0|^{\frac{1}{q_0}} \left(\fint_{Q_0} I_{\alpha,m}\left(\vec{f}_\ell\right)(x)^q v(x)^q dx \right)^{\frac{1}{q}} \\
&\le [v,\vec{w}]_{p_0,q_0,aq_0,\alpha,q,\vec{A}} \left\| \mathcal{M}_{\vec{A},\frac{n}{p_0}-\frac{n}{q_0}}\left(\vec{f}_w\right) \right\|_{\mathcal{M}^{q_0}_q} \sum_{k=1}^{\infty} \left(\frac{|Q_0|}{|3\cdot 2^k Q_0|} \right)^{\frac{1}{q_0}\left(1-\frac{1}{a}\right)}
\end{aligned} \tag{4.22}
$$

Since the series $\sum_{k=1}^{\infty} \left(\frac{|Q_0|}{|3\cdot 2^k Q_0|} \right)^{\frac{1}{q_0}\left(1-\frac{1}{a}\right)}$ is convergent,

$$
\begin{aligned}
&|Q_0|^{\frac{1}{q_0}} \left(\fint_{Q_0} I_{\alpha,m}\left(\vec{f}_\ell\right)(x)^q v(x)^q dx \right)^{\frac{1}{q}} \\
&\lesssim [v,\vec{w}]_{p_0,q_0,aq_0,\alpha,q,\vec{A}} \left\| \mathcal{M}_{\vec{A},\frac{n}{p_0}-\frac{n}{q_0}}\left(\vec{f}_w\right) \right\|_{\mathcal{M}^{q_0}_q}.
\end{aligned} \tag{4.23}
$$

Secondly, we evaluate $I_{\alpha,m}\left(\vec{f}_0\right)(x)$. By Lemma 3.5,

$$
\int_{Q_0} I_{\alpha,m}\left(\vec{f}_0\right)(x)^q v(x)^q dx \le C \int_{Q_0} \tilde{\mathcal{M}}_{\vec{I}_{m,q},\alpha}\left(\vec{f}_0, v\right)(x)^q dx. \tag{4.24}
$$

By Lemma 1.20,

$$
\begin{aligned}
&\tilde{\mathcal{M}}_{\vec{I}_{m,q},\alpha}\left(\vec{f}_0, v\right)(x) \\
&\lesssim \sup_{Q\ni x} \ell(Q)^{\alpha} \prod_{i=1}^{m} \|f_i w_i\|_{A_i,3Q} \cdot \|w_i^{-1}\|_{\overline{A_i},3Q}\, m_Q\left(v^q\right)^{\frac{1}{q}} \\
&\lesssim \sup_{Q:\text{cube}} \left(\frac{|Q|}{|3Q|} \right)^{\frac{1}{aq_0}} |3Q|^{\frac{\alpha}{n}+\frac{1}{q_0}-\frac{1}{p_0}} m_Q\left(v^q\right)^{\frac{1}{q}} \prod_{i=1}^{m} \|w_i^{-1}\|_{\overline{A_i},3Q} \\
&\times \left(\ell(Q)^{\frac{n}{p_0}-\frac{n}{q_0}} \prod_{i=1}^{m} \|f_i w_i\|_{A_i,3Q} \right) \chi_Q(x).
\end{aligned} \tag{4.25}
$$

By using (1.25) and $[v,\vec{w}]_{p_0,q_0,aq_0,\alpha,q,\vec{A}}$, we have

$$
\tilde{\mathcal{M}}_{\vec{I}_{m,q},\alpha}\left(\vec{f}_0, v\right)(x) \lesssim [v,\vec{w}]_{p_0,q_0,aq_0,\alpha,q,\vec{A}}\, \mathcal{M}_{\vec{A},\frac{n}{p_0}-\frac{n}{q_0}}\left(\vec{f}_w\right)(x). \tag{4.26}
$$

Estimates (4.24)-(4.26) give the following.

$$|Q_0|^{\frac{1}{q_0}} \left(⨍_{Q_0} I_{\alpha,m}\left(\vec{f_0}\right)(x)^q v(x)^q dx \right)^{\frac{1}{q}} \lesssim [v,\vec{w}]_{p_0,q_0,aq_0,\alpha,q,\vec{A}} \left\| \mathcal{M}_{\vec{A},\frac{n}{p_0}-\frac{n}{q_0}}\left(\vec{f_w}\right) \right\|_{\mathcal{M}^{q_0}_q}. \tag{4.27}$$

Therefore, estimates (4.23) and (4.27) give the desired result.

(3) Firstly, we evaluate $I_{\alpha,m}\left(\vec{f_\ell}\right)(x)$. By Lemma 3.8, $\left(⨍_{Q_0} v(x)^q dx\right)^{\frac{1}{q}} \lesssim \|v\|_{B,Q_0}$ occurs. Hence, by $[v,\vec{w}]_{p_0,q_0,aq_0,\alpha,B,\vec{A}}$,

$$\left(\frac{|Q_0|}{|3\cdot 2^k Q_0|}\right)^{\frac{1}{aq_0}} |3\cdot 2^k Q_0|^{\frac{1}{q_0}+\frac{\alpha}{n}-\frac{1}{p_0}} \left(⨍_{Q_0} v(x)^q dx\right)^{\frac{1}{q}} \prod_{i=1}^m \|w_i^{-1}\|_{\overline{A}_i,3\cdot 2^k Q_0} \lesssim [v,\vec{w}]_{p_0,q_0,aq_0,\alpha,B,\vec{A}}. \tag{4.28}$$

Since $\sum_{k=1}^\infty \left(\frac{|Q_0|}{|3\cdot 2^k Q_0|}\right)^{\frac{1}{q_0}\left(1-\frac{1}{a}\right)}$ is convergent, (4.21) and (4.28) imply that

$$|Q_0|^{\frac{1}{q_0}} \left(⨍_{Q_0} I_{\alpha,m}\left(\vec{f_\ell}\right)(x)^q v(x)^q dx \right)^{\frac{1}{q}} \lesssim [v,\vec{w}]_{p_0,q_0,aq_0,\alpha,B,\vec{A}} \left\| \mathcal{M}_{\vec{A},\frac{n}{p_0}-\frac{n}{q_0}}\left(\vec{f_w}\right) \right\|_{\mathcal{M}^{q_0}_q}. \tag{4.29}$$

Secondly, we evaluate $I_{\alpha,m}\left(\vec{f_0}\right)(x)$. By Lemma 3.7,

$$\int_{Q_0} I_{\alpha,m}\left(\vec{f_0}\right)(x)^q v(x)^q dx \lesssim \int_{Q_0} \tilde{\mathcal{M}}_{\vec{I}_{m,B},\alpha}\left(\vec{f_0},v\right)(x)^q dx. \tag{4.30}$$

On the other hand, by using $[v,\vec{w}]_{p_0,q_0,aq_0,\alpha,B,\vec{A}}$, we have

$$\tilde{\mathcal{M}}_{\vec{I}_{m,B},\alpha}\left(\vec{f_0},v\right)(x) \lesssim [v,\vec{w}]_{p_0,q_0,aq_0,\alpha,B,\vec{A}} \mathcal{M}_{\vec{A},\frac{n}{p_0}-\frac{n}{q_0}}\left(\vec{f_w}\right)(x). \tag{4.31}$$

Estimates (4.30) and (4.31) imply that

$$|Q_0|^{\frac{1}{q_0}} \left(⨍_{Q_0} I_{\alpha,m}\left(\vec{f_0}\right)(x)^q v(x)^q dx \right)^{\frac{1}{q}} \lesssim [v,\vec{w}]_{p_0,q_0,aq_0,\alpha,B,\vec{A}} \left\| \mathcal{M}_{\vec{A},\frac{n}{p_0}-\frac{n}{q_0}}\left(\vec{f_w}\right) \right\|_{\mathcal{M}^{q_0}_q}. \tag{4.32}$$

Therefore, estimates (4.29) and (4.32) give the desired result.

□

Proof of Theorem 2.10. By the same as the proof of Theorem 2.7, for every dyadic cube Q_0, let $f_i\chi_{3Q_0} = f_i^0$ and $f_i\chi_{(3Q_0)^C} = f_i^\infty$. Then,

$$\mathcal{M}_{\vec{C},\alpha}\left(\vec{f}\right)(x) \le \mathcal{M}_{\vec{C},\alpha}\left(\vec{f_0}\right)(x) + \sum_{\vec{\ell}\neq\vec{0}} \mathcal{M}_{\vec{C},\alpha}\left(\vec{f_\ell}\right)(x),$$

where $\vec{f_0} = (f_1^0,\ldots,f_m^0)$, $\vec{f_\ell} = (f_1^{\ell_1},\ldots,f_m^{\ell_m})$ and $\vec{\ell} = (\ell_1,\ldots,\ell_m) \in \{0,\infty\}^m$.

Firstly, we evaluate $\mathcal{M}_{\vec{C},\alpha}\left(\vec{f_\ell}\right)(x)$. For $x \in Q_0$, note that

$$\mathcal{M}_{\vec{C},\alpha}\left(\vec{f_\ell}\right)(x) \lesssim \sup_{Q_0 \subset Q} \ell(Q)^\alpha \prod_{i=1}^m \|f_i\|_{C_i,Q}. \tag{4.33}$$

By (1.21),

$$\begin{aligned} &\sup_{Q_0 \subset Q} \ell(Q)^\alpha \prod_{i=1}^m \|f_i\|_{C_i,Q} \\ &\lesssim \sup_{Q_0 \subset Q} \ell(Q)^\alpha \prod_{i=1}^m \|f_i w_i\|_{A_i,Q} \left\|w_i^{-1}\right\|_{B_i,Q} \\ &= \sup_{Q_0 \subset Q} \ell(Q)^{\alpha-\frac{n}{p_0}} |Q|^{\frac{1}{q_0}} \left(\fint_Q \left(\ell(Q)^{\frac{n}{p_0}-\frac{n}{q_0}} \prod_{i=1}^m \|f_i w_i\|_{A_i,Q}\right)^q dx\right)^{\frac{1}{q}} \prod_{i=1}^m \left\|w_i^{-1}\right\|_{B_i,Q}. \end{aligned} \tag{4.34}$$

By (1.12) and (1.25),

$$\begin{aligned} &\sup_{Q_0 \subset Q} \ell(Q)^{\alpha-\frac{n}{p_0}} |Q|^{\frac{1}{q_0}} \left(\fint_Q \left(\ell(Q)^{\frac{n}{p_0}-\frac{n}{q_0}} \prod_{i=1}^m \|f_i w_i\|_{A_i,Q}\right)^q dx\right)^{\frac{1}{q}} \prod_{i=1}^m \left\|w_i^{-1}\right\|_{B_i,Q} \\ &\leq \left\|\mathcal{M}_{\vec{A},\frac{n}{p_0}-\frac{n}{q_0}}\left(\vec{f_w}\right)\right\|_{\mathcal{M}^{q_0}_q} \cdot \sup_{Q_0 \subset Q} |Q|^{\frac{\alpha}{n}-\frac{1}{p_0}} \prod_{i=1}^m \left\|w_i^{-1}\right\|_{B_i,Q}. \end{aligned} \tag{4.35}$$

Estimates (4.33)-(4.35) give

$$\begin{aligned} &|Q_0|^{\frac{1}{q_0}} \left(\fint_{Q_0} \mathcal{M}_{\vec{C},\alpha}\left(\vec{f_\ell}\right)(x)^q v(x)^q dx\right)^{\frac{1}{q}} \\ &\lesssim \left\|\mathcal{M}_{\vec{A},\frac{n}{p_0}-\frac{n}{q_0}}\left(\vec{f_w}\right)\right\|_{\mathcal{M}^{q_0}_q} \\ &\times \sup_{Q_0 \subset Q} \left(\frac{|Q_0|}{|Q|}\right)^{\frac{1}{q_0}} |Q|^{\frac{1}{q_0}-\frac{1}{p_0}+\frac{\alpha}{n}} \left(\fint_{Q_0} v(x)^q dx\right)^{\frac{1}{q}} \prod_{i=1}^m \left\|w_i^{-1}\right\|_{B_i,Q}. \end{aligned} \tag{4.36}$$

By $[v,\vec{w}]_{p_0,q_0,q_0,\alpha,q,\vec{B}}$,

$$\begin{aligned} &|Q_0|^{\frac{1}{q_0}} \left(\fint_{Q_0} \mathcal{M}_{\vec{C},\alpha}\left(\vec{f_\ell}\right)(x)^q v(x)^q dx\right)^{\frac{1}{q}} \\ &\lesssim [v,\vec{w}]_{p_0,q_0,q_0,\alpha,q,\vec{B}} \cdot \left\|\mathcal{M}_{\vec{A},\frac{n}{p_0}-\frac{n}{q_0}}\left(\vec{f_w}\right)\right\|_{\mathcal{M}^{q_0}_q}. \end{aligned} \tag{4.37}$$

Secondly, we evaluate $\mathcal{M}_{\vec{C},\alpha}\left(\vec{f_0}\right)(x)$. By Lemma 3.14,

$$\int_{Q_0} \mathcal{M}_{\vec{C},\alpha}\left(\vec{f_0}\right)(x)^q v(x)^q dx \lesssim \int_{Q_0} \tilde{\mathcal{M}}_{\vec{C}_q,\alpha}\left(\vec{f_0},v\right)(x)^q dx. \tag{4.38}$$

By using (1.21), (1.25) and $[v,\vec{w}]_{p_0,q_0,q_0,\alpha,q,\vec{B}}$, we have

$$\begin{aligned} &\tilde{\mathcal{M}}_{\vec{C}_q,\alpha}\left(\vec{f_0},v\right)(x) \\ &\lesssim \sup_{Q \ni x} \left(\frac{|Q|}{|3Q|}\right)^{\frac{1}{q_0}} |3Q|^{\frac{\alpha}{n}+\frac{1}{q_0}-\frac{1}{p_0}} m_Q\left(v^q\right)^{\frac{1}{q}} \prod_{i=1}^m \left\|w_i^{-1}\right\|_{B_i,3Q} \ell(3Q)^{\frac{n}{p_0}-\frac{n}{q_0}} \prod_{i=1}^m \|f_i w_i\|_{A_i,3Q} \\ &\leq [v,\vec{w}]_{p_0,q_0,q_0,\alpha,q,\vec{B}} \mathcal{M}_{\vec{A},\frac{n}{p_0}-\frac{n}{q_0}}\left(\vec{f_w}\right)(x). \end{aligned} \tag{4.39}$$

Estimates (4.38) and (4.39) give

$$\int_{Q_0} \mathcal{M}_{\vec{C},\alpha}\left(\vec{f_0}\right)(x)^q v(x)^q dx \lesssim [v,\vec{w}]^q_{p_0,q_0,q_0,\alpha,q,\overline{\vec{B}}} \int_{Q_0} \mathcal{M}_{\vec{A},\frac{n}{p_0}-\frac{n}{q_0}}\left(\vec{f_w}\right)(x)^q dx. \tag{4.40}$$

Estimates (4.37) and (4.40) give the desired result. □

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NORM INEQUALITIES ON MORREY SPACES FOR THE OSCILLATION AND VARIATION OPERATORS

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Abstract. This paper is devoted to investigating the bounded behaviors of the oscillation and variation operators for the family of multilinear singular integrals with Lipschitz functions on the Morrey spaces. We establish several criterions of boundedness, which are applied to obtain the corresponding bounds for the oscillation and variation operators on Morrey spaces when the m-th derivative of b belongs to the homogenous Lipschitz space.

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1. Introduction

Given m is a positive integer, and b is a function on $\mathbb{R}$. Let $R_{m+1}(b;x,y)$ be the $m+1$-th order Taylor series remainder of b at x about y, that is,

$$R_{m+1}(b;x,y) = b(x) - \sum_{\gamma\leq m} \frac{1}{\gamma!} b^{(\gamma)}(y)(x-y)^{\gamma}.$$

In this paper, we consider the family of operators $T^b := \{T^b_\epsilon\}_{\epsilon>0}$ given by [4], where T^b_ϵ are the multilinear singular integral operators of T_ϵ as follows

$$T^b_\epsilon f(x) = \int\limits_{|x-y|>\epsilon} \frac{R_{m+1}(b;x,y)}{|x-y|^m} K(x,y) f(y)\, dy. \tag{1.1}$$

Thus, if $m=0$, then T^b_ϵ is just the commutator of T_ϵ and b, which is given by

$$T_{\epsilon,b} f(x) = \int\limits_{|x-y|>\epsilon} (b(x)-b(y)) K(x,y) f(y)\, dy,$$

where K is said to be a Calderón-Zygmund standard kernel such that

$$|K(x,y)| \leq \frac{C}{|x-y|}, \qquad \text{for } x\neq y \tag{1.2}$$

and for all x, x_0, $y \in \mathbb{R}$ with $|x-y| > 2|x-x_0|$

$$\begin{aligned} &|K(x,y) - K(x_0,y)| + |K(y,x) - K(y,x_0)| \\ &\leq \frac{C}{|x-y|}\left(\frac{|x-x_0|}{|x-y|}\right)^{\delta}, \end{aligned} \tag{1.3}$$

where $1 > \delta > 0$. But, if $m > 0$, then T_{ϵ}^{b} are non-trivial generation of the commutators.

The theory of multilinear analysis was received extensive studies in the last 3 decades (see [3, 4, 6] for example). Hu and Wang [4] proved that the weighted (L^p, L^q)-boundedness of the oscillation and variation operators for T^b when the m-th derivative of b belongs to the homogenous Lipschitz space $\dot{\Lambda}_{\beta}$. In this sense, we recall the definitions of homogenous Lipschitz space $\dot{\Lambda}_{\beta}$ and bounded mean oscillation space BMO as follows:

Definition 1.1. (***Homogenous Lipschitz space***) *Let $0 < \beta \leq 1$. The homogeneous Lipschitz space $\dot{\Lambda}_{\beta}$ is defined by*

$$\dot{\Lambda}_{\beta}(\mathbb{R}) = \left\{ b : \|b\|_{\dot{\Lambda}_{\beta}} = \sup_{x,h\in\mathbb{R},h\neq 0} \frac{|b(x+h) - b(x)|}{|h|^{\beta}} < \infty \right\}.$$

Obviously, if $\beta > 1$, then $\dot{\Lambda}_{\beta}(\mathbb{R})$ only includes constant. So we restrict $0 < \beta \leq 1$.

Definition 1.2. (***Bounded Mean Oscillation (BMO)***) *Let $|I|$ denote the Lebesgue measure of the interval I. We denote the mean value of b on the interval $I = I(x,y) \subset \mathbb{R}$ by*

$$b_I = M(b,I) = M(b,x,y) = \frac{1}{|I|}\int_I b(y)dy,$$

and the mean oscillation of b on the interval $I = I(x,y)$ by

$$MO(b,I) = MO(b,x,y) = \frac{1}{|I|}\int_I |b(y) - b_I|dy.$$

We also define for a non-negative function ϕ on $\mathbb{R}$

$$MO_{\phi}(b,I) = MO_{\phi}(b,x,y) = \frac{1}{\phi(|I|)\,|I|}\int_I |b(y) - b_I|dy.$$

Now, we define

$$BMO_{\phi}(\mathbb{R}) = \left\{ b \in L_1^{loc}(\mathbb{R}) : \sup_I MO_{\phi}(b,I) < \infty \right\}$$

and

$$\|f\|_{BMO_{\phi}} = \sup_I MO_{\phi}(b,I).$$

The real importance comes when $\phi = 1$, in which case $BMO_{\phi}(\mathbb{R}) = BMO(\mathbb{R})$.

Now, we recall the definition of basic space such as Morrey space. The Morrey space is a generalization of Lebesgue space. It was introduced by Morrey in [5] to study the solutions of some quasi-linear elliptic partial differential equations. A number of results from Lebesgue spaces had been extended to Morrey spaces [1].

The Morrey space $M_p^q(\mathbb{R})$ is defined as follows:

Definition 1.3. (***Morrey space***) *For $1 \leq p \leq q < \infty$, the Morrey space $M_p^q(\mathbb{R})$ is the collection of all measurable functions f whose Morrey space norm is*

$$\|f\|_{M_p^q(\mathbb{R})} = \sup_{\substack{I\subset\mathbb{R} \\ I:Interval}} \frac{1}{|I|^{\frac{1}{p}-\frac{1}{q}}} \|f\chi_I\|_{L^p(\mathbb{R})} < \infty.$$

Remark 1.4. [3] $\cdot$ *If* $p = q$*, then*

$$\|f\|_{M_q^q(\mathbb{R})} = \|f\|_{L^q(\mathbb{R})}.$$

$\cdot$ *if* $q < p$*, then* $M_p^q(\mathbb{R})$ *is strictly larger than* $L^q(\mathbb{R})$*. For example,* $f(x) := |x|^{-\frac{1}{q}} \in M_p^q(\mathbb{R})$ *but* $f(x) := |x|^{-\frac{1}{q}} \notin L^q(\mathbb{R})$*.*

In 2016, Zhang and Wu [6] gave the boundedness of the oscillation and variation operators for Calderón-Zygmund singular integrals and the corresponding commutators on the weighted Morrey spaces. In 2017, Hu and Wang [4] established the weighted (L^p, L^q)-inequalities of the variation and oscillation operators for the multilinear Calderón-Zygmund singular integral with a Lipschitz function in $\mathbb{R}$. In 2020, Gürbüz [3] has proved the boundedness of the oscillation and variation operators for the multilinear singular integrals with Lipschitz functions on weighted Morrey spaces.

Inspired of these results [3, 4, 6], we study the boundedness of the oscillation and variation operators for the family of the multilinear singular integral defined by (1.1) on Morrey spaces $M_p^q(\mathbb{R})$ when the m-th derivative of b belongs to the homogenous Lipschitz space $\dot{\Lambda}_\beta$ in this work.

Suppose that K satisfies (1.2) and (1.3). Then, Zhang and Wu [6] considered the family of operators $T := \{T_\epsilon\}_{\epsilon>0}$ and a related the family of commutator operators $T_b := \{T_{\epsilon,b}\}_{\epsilon>0}$ generated by T_ϵ and b which are given by

$$T_\epsilon f(x) = \int\limits_{|x-y|>\epsilon} K(x,y) f(y)\, dy \tag{1.4}$$

and

$$T_{\epsilon,b} f(x) = \int\limits_{|x-y|>\epsilon} (b(x) - b(y)) K(x,y) f(y)\, dy. \tag{1.5}$$

In this sense, following [6], the definition of the oscillation operator of T is given by

$$\mathcal{O}(Tf)(x) := \left(\sum_{i=1}^{\infty} \sup_{t_{i+1} \le \epsilon_{i+1} < \epsilon_i \le t_i} \left| T_{\epsilon_{i+1}} f(x) - T_{\epsilon_i} f(x) \right|^2 \right)^{\frac{1}{2}},$$

where $\{t_i\}$ is a decreasing fixed sequence of positive numbers converging to 0 and a related ρ-variation operator is defined by

$$\mathcal{V}_\rho(Tf)(x) := \sup_{\epsilon_i \searrow 0} \left(\sum_{i=1}^{\infty} \left| T_{\epsilon_{i+1}} f(x) - T_{\epsilon_i} f(x) \right|^\rho \right)^{\frac{1}{\rho}}, \qquad \rho > 2,$$

where the supremum is taken over all sequences of real number $\{\epsilon_i\}$ decreasing to 0. We also take into account the operator

$$\mathcal{O}'(Tf)(x) := \left(\sum_{i=1}^{\infty} \sup_{t_{i+1} < \eta_i < t_i} \left| T_{t_{i+1}} f(x) - T_{\eta_i} f(x) \right|^2 \right)^{\frac{1}{2}}.$$

On the other hand, it is obvious that

$$\mathcal{O}'(Tf) \approx \mathcal{O}(Tf).$$

That is,

$$\mathcal{O}'(Tf) \le \mathcal{O}(Tf) \le 2\mathcal{O}'(Tf).$$

Recently, Campbell et al. in [2] proved the oscillation and variation inequalities for the Hilbert transform in $L^p(1 < p < \infty)$ and then following [2], we denote by E the mixed norm Banach space of two-variable function h defined on $\mathbb{R}\times\mathbb{N}$ such that

$$\|h\|_E \equiv \left(\sum_i \left(\sup_s |h(s,i)|\right)^2\right)^{1/2} < \infty.$$

Given $T := \{T_\epsilon\}_{\epsilon>0}$ is a family operators such that $\lim_{\epsilon\to 0} T_\epsilon f(x) = Tf(x)$ exists almost everywhere for certain class of functions f, where T_ϵ defined as (1.4). For a fixed decreasing sequence $\{t_i\}$ with $t_i \searrow 0$, let $J_i = (t_{i+1}, t_i]$ and define the E-valued operator $U(T) : f \to U(T) f$ given by

$$U(T) f(x) = \left\{T_{t_{i+1}} f(x) - T_s f(x)\right\}_{s\in J_i, i\in\mathbb{N}} = \left\{ \int_{\{t_{i+1}<|x-y|<s\}} K(x,y) f(y)\,dy \right\}_{s\in J_i, i\in\mathbb{N}}.$$

Then

$$\begin{aligned}\mathcal{O}'(Tf)(x) &= \|U(T) f(x)\|_E = \left\|\left\{T_{t_{i+1}} f(x) - T_s f(x)\right\}_{s\in J_i, i\in\mathbb{N}}\right\|_E \\ &= \left\|\left\{ \int_{\{t_{i+1}<|x-y|<s\}} K(x,y) f(y)\,dy \right\}_{s\in J_i, i\in\mathbb{N}}\right\|_E.\end{aligned}$$

Let $\Phi = \{\beta : \beta = \{\epsilon_i\}, \epsilon_i \in \mathbb{R}, \epsilon_i \searrow 0\}$. We denote by F_ρ the mixed norm space of two variable functions $g(i,\beta)$ such that

$$\|g\|_{F_\rho} \equiv \sup_\beta \left(\sum_i |g(i,\beta)|^\rho\right)^{1/\rho}.$$

We also take into account the F_ρ-valued operator $V(T) : f \to V(T) f$ such that

$$V(T) f(x) = \left\{T_{\epsilon_{i+1}} f(x) - T_{\epsilon_i} f(x)\right\}_{\beta=\{\epsilon_i\}\in\Phi}.$$

Thus,

$$V_\rho(T) f(x) = \|V(T) f(x)\|_{F_\rho}.$$

Throughout this paper, C always means a positive constant independent of the main parameters involved, and may change from one occurrence to another. We also use the notation $F \lesssim G$ to mean $F \le CG$ for an appropriate constant $C > 0$, and $F \approx G$ to mean $F \lesssim G$ and $G \lesssim F$.

2. Main result

We are now ready to present and establish the main result of this paper.

Theorem 2.1. *Let $K(x,y)$ satisfies (1.2) and (1.3), $\rho > 2$, and $T := \{T_\epsilon\}_{\epsilon>0}$ and $T^b := \{T^b_\epsilon\}_{\epsilon>0}$ be given by (1.1) and (1.4), respectively. If $\mathcal{O}(T)$ and $\mathcal{V}_\rho(T)$ are bounded on $L^{p_0}(\mathbb{R}, dx)$ for some $1 < p_0 < \infty$, and $b^{(m)} \in \dot{\Lambda}_\beta(\mathbb{R})$ with $m \in \mathbb{N}$ for $0 < \beta < 1$, then $\mathcal{O}(T^b)$ and $\mathcal{V}_\rho(T^b)$ are bounded from $M^q_p(\mathbb{R})$ to $BMO(\mathbb{R})$ for any $1 < p \le q = \frac{1}{\beta}$.*

Corollary 2.2. [6] *Let $K(x,y)$ satisfies (1.2) and (1.3), $\rho > 2$, and $T := \{T_\epsilon\}_{\epsilon>0}$ and $T_b := \{T_{\epsilon,b}\}_{\epsilon>0}$ be given by (1.4) and (1.5), respectively. If $\mathcal{O}(T)$ and $\mathcal{V}_\rho(T)$ are bounded on $L^{p_0}(\mathbb{R}, dx)$ for some $1 < p_0 < \infty$, and $b \in \dot{\Lambda}_\beta$ for $0 < \beta < 1$, then $\mathcal{O}(T_b)$ and $\mathcal{V}_\rho(T_b)$ are bounded from $M^q_p(\mathbb{R})$ to $BMO(\mathbb{R})$ for any $1 < p \le q = \frac{1}{\beta}$.*

2.1. **The Proof of Theorem 2.1.**

Proof. We consider the proof related to $\mathcal{O}\left(T^b\right)$ firstly. Fix an interval $I=(x_0-l,x_0+l)$ satisfying $|I|=2l$, and we write as $f=f_1+f_2$, where $f_1=f\chi_{4I}$, χ_{4I} denotes the characteristic function of $4I$.

Let

$$C_I=\frac{1}{|I|}\int_I\left\{\int_{\{t_{i+1}<|z-y|<s\}}\frac{R_{m+1}(b;z,y)}{|z-y|^m}K(z,y)\,f_2(y)\,dy\right\}_{s\in J_i,i\in\mathbb{N}}dz.$$

Thus, it is sufficient to show that the conclusion

$$\frac{1}{|I|}\int_I\left|\mathcal{O}'\left(T^b\right)(f)(x)-C_I\right|dx=\frac{1}{|I|}\int_I\left\|\mathcal{U}\left(T^b\right)(f)(x)-C_I\right\|_E dx\lesssim\|b\|_{\dot{\Lambda}_\beta}\|f\|_{M_p^q}$$

holds for every interval $I\subset\mathbb{R}$. Then

$$\begin{aligned}&\frac{1}{|I|}\int_I\left\|\mathcal{U}\left(T^b\right)(f)(x)-C_I\right\|_E dx\\&\lesssim\frac{1}{|I|}\int_I\left\|\mathcal{U}\left(T^b\right)(f_1)(x)+\mathcal{U}\left(T^b\right)(f_2)(x)-C_I\right\|_E dx\\&\lesssim\frac{1}{|I|}\int_I\left\|\mathcal{U}\left(T^b\right)(f_1)(x)\right\|_E dx+\frac{1}{|I|}\int_I\left\|\mathcal{U}\left(T^b\right)(f_2)(x)-C_I\right\|_E dx\\&=:F_1+F_2.\end{aligned}$$

First, we choose $1<p_1<\min\left\{\frac{1}{\beta},p\right\}$ and q_1 with $\frac{1}{q_1}=\frac{1}{p_1}-\beta$ and to estimate F_1, and use (9) in [3] (by taking $w=1$ there), also following [6], we obtain

$$\begin{aligned}F_1&=\frac{1}{|I|}\int_I\mathcal{O}'\left(T^bf_1\right)(x)\,dx\\&\lesssim\frac{1}{|I|}\left(\int_I\left|\mathcal{O}'\left(T^bf_1\right)(x)\right|^{q_1}dx\right)^{\frac{1}{q_1}}|I|^{1-\frac{1}{q_1}}\\&\lesssim\left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta}\frac{1}{|I|}\left(\int_{\mathbb{R}}|f_1(x)|^{p_1}dx\right)^{\frac{1}{p_1}}|I|^{1-\frac{1}{q_1}}\\&=\left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta}\frac{1}{|I|}\left(\int_{4I}|f(x)|^{p_1}dx\right)^{\frac{1}{p_1}}|I|^{1-\frac{1}{q_1}}\\&=\left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta}\frac{1}{|4I|^{\frac{1}{p}-\frac{1}{q}}}\left(\int_{4I}|f(x)|^{p}dx\right)^{\frac{1}{p}}|4I|^{\frac{1}{p}-\frac{1}{q}}|4I|^{\frac{1}{p_1}-\frac{1}{p}}|I|^{-\frac{1}{q_1}}\\&\lesssim\|b\|_{\dot{\Lambda}_\beta}\|f\|_{M_p^q}.\end{aligned}$$

Thus,

$$\frac{1}{|I|}\int_I\left\|\mathcal{U}\left(T^b\right)(f_1)(x)\right\|_E dx\lesssim\|b\|_{\dot{\Lambda}_\beta}\|f\|_{M_p^q}.\tag{2.1}$$

Second, we have

$$\begin{aligned} F_2 &= \frac{1}{|I|}\int_I \left\|\mathcal{U}\left(T^b\right)(f_2)(x) - C_I\right\|_E \\ &= \frac{1}{|I|}\int_I \left\|\mathcal{U}\left(T^b\right)(f_2)(x) - \frac{1}{|I|}\int_I \mathcal{U}\left(T^b\right)(f_2)(z)\,dz\right\|_E dx \\ &\le \frac{1}{|I|^2}\iint_{I\times I} \left\|\mathcal{U}\left(T^b\right)(f_2)(x) - \mathcal{U}\left(T^b\right)(f_2)(z)\,dz\right\|_E dzdx. \end{aligned}$$

Thus, following [6], we write

$$\begin{aligned} &\left\|\mathcal{U}\left(T^b\right)(f_2)(x) - \mathcal{U}\left(T^b\right)(f_2)(z)\,dz\right\|_E \\ &= \left\|\left\{\begin{array}{l} \int\limits_{\{t_{i+1}<|x-y|<s\}} \frac{R_{m+1}(b;x,y)}{|x-y|^m} K(x,y)\, f_2(y)\,dy \\ - \int\limits_{\{t_{i+1}<|z-y|<s\}} \frac{R_{m+1}(b;z,y)}{|z-y|^m} K(z,y)\, f_2(y)\,dy \end{array}\right\}_{s\in J_i, i\in\mathbb{N}}\right\|_E \\ &\le \left\|\left\{\int\limits_{\{t_{i+1}<|x-y|<s\}} \left(\frac{R_{m+1}(b;x,y)}{|x-y|^m} K(x,y) - \frac{R_{m+1}(b;z,y)}{|z-y|^m} K(z,y)\right) f_2(y)\,dy\right\}_{s\in J_i, i\in\mathbb{N}}\right\|_E \\ &+ \left\|\left\{\int\limits_{\mathbb{R}} \left(\chi_{\{t_{i+1}<|x-y|<s\}}(y) - \chi_{\{t_{i+1}<|z-y|<s\}}(y)\right) \frac{R_{m+1}(b;z,y)}{|z-y|^m} K(z,y)\, f_2(y)\,dy\right\}_{s\in J_i, i\in\mathbb{N}}\right\|_E \\ &=: G_1 + G_2. \end{aligned}$$

For $k = 0, 1, 2, \ldots$, let $E_k = \left\{y : 2^k.4l \le |y - z| < 2^{k+1}.4l\right\}$, $D_k = \left\{y : |y - z| < 2^k.4l\right\}$, and

$$b_k(z) = b(z) - \frac{1}{m!}\left(b^{(m)}\right)_{D_k} z^m.$$

By Lemma 2 in [3], for any $y \in E_k$, it is obvious that

$$R_{m+1}(b;x,y) = R_{m+1}(b_k;x,y).$$

Note that for $x, z \in I$, $y \in E_k$, we have

$$\begin{aligned} &\frac{R_{m+1}(b;x,y)}{|x-y|^m}K(x,y) - \frac{R_{m+1}(b;z,y)}{|z-y|^m}K(z,y) \\ &= \frac{R_{m+1}(b_k;x,y)}{|x-y|^m}K(x,y) - \frac{R_{m+1}(b_k;z,y)}{|z-y|^m}K(z,y) \\ &= \frac{(R_m(b_k;x,y) - R_m(b_k;z,y))}{|x-y|^m}K(x,y) \\ &+ R_m(b_k;z,y)\left(\frac{1}{|x-y|^m} - \frac{1}{|z-y|^m}\right)K(x,y) \\ &- \frac{1}{m!}b_k^{(m)}(y)\left(\frac{(x-y)^m}{|x-y|^m} - \frac{(z-y)^m}{|z-y|^m}\right)K(x,y) \\ &+ \frac{R_{m+1}(b_k;z,y)}{|z-y|^m}(K(x,y) - K(z,y)). \end{aligned} \tag{2.2}$$

By Minkowski's inequality, $\left\|\left\{\chi_{\{t_{i+1}<|x-y|<s\}}\right\}_{s\in J_i, i\in\mathbb{N}}\right\|_E \le 1$, and (2.2) we get

$$\begin{aligned} G_1 &\le \int_{\mathbb{R}} \left|\frac{R_{m+1}(b;x,y)}{|x-y|^m}K(x,y) - \frac{R_{m+1}(b;z,y)}{|z-y|^m}K(z,y)\right| |f_2(y)| \left\|\left\{\chi_{\{t_{i+1}<|x-y|<s\}}\right\}_{s\in J_i, i\in\mathbb{N}}\right\|_E dy \\ &\le \int_{\mathbb{R}} \left|\frac{R_{m+1}(b;x,y)}{|x-y|^m}K(x,y) - \frac{R_{m+1}(b;z,y)}{|z-y|^m}K(z,y)\right| |f_2(y)|\, dy \\ &\le \sum_{k=0}^{\infty}\int_{E_k} \frac{|R_m(b_k;x,y) - R_m(b_k;z,y)|}{|x-y|^m}|K(x,y)||f(y)|\, dy \\ &+ \sum_{k=0}^{\infty}\int_{E_k} |R_m(b_k;z,y)|\left|\frac{1}{|x-y|^m} - \frac{1}{|z-y|^m}\right| |K(x,y)||f(y)|\, dy \\ &+ \sum_{k=0}^{\infty}\int_{E_k} \frac{1}{m!}\left|b_k^{(m)}(y)\right|\left|\frac{(x-y)^m}{|x-y|^m} - \frac{(z-y)^m}{|z-y|^m}\right| |K(x,y)||f(y)|\, dy \\ &+ \sum_{k=0}^{\infty}\int_{E_k} \left|\frac{R_{m+1}(b_k;z,y)}{|z-y|^m}\right| |K(x,y) - K(z,y)||f(y)|\, dy \\ &=: H_1 + H_2 + H_3 + H_4. \end{aligned}$$

For H_1, from mean value theorem, there exists $\sigma \in I$ such that

$$R_m(b_k;x,y) - R_m(b_k;z,y) = (x-z)R_{m-1}(b_k';\sigma,y). \tag{2.3}$$

Then, for $x, \sigma \in I$, $y \in E_k$, $\mu \in I_\sigma^y$, since $b^{(m)} \in \dot{\Lambda}_\beta(\mathbb{R})$ with $m \in \mathbb{N}$ for $0 < \beta < 1$, then we have

$$\begin{aligned} \left|b_k^{(m)}(\mu)\right| &= \left|b^{(m)}(\mu) - \left(b^{(m)}\right)_{D_k}\right| \\ &\lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta}\left(2^k l\right)^\beta |x-y|^{m-1}. \end{aligned} \tag{2.4}$$

Hence, by $|y-z| \approx |y-x| \approx |y-\sigma|$, Lemma 2 in [3] and (2.4) we get

$$\begin{aligned} R_{m-1}\left(b_k';\sigma,y\right) &\lesssim |\sigma-y|^{m-1}\left(\frac{1}{|I_\sigma^y|}\int_{I_\sigma^y}\left|b_k^{(m)}(\mu)\right|^s d\mu\right)^{\frac{1}{s}} \\ &\lesssim |x-y|^{m-1}\left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta}\left(2^k l\right)^\beta . \end{aligned} \tag{2.5}$$

Later, by (2.3) and (2.5)

$$\left|R_m\left(b_k;x,y\right)-R_m\left(b_k;z,y\right)\right| \lesssim |x-z|\,|x-y|^{m-1}\left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta}\left(2^k l\right)^\beta .$$

Since, for $x,z\in I$, $y\in(4I)^C$, $|x-z|\le 2l\le\frac{2}{3}|z-y|$ and $|K(x,y)|\lesssim\frac{1}{|z-y|}$, then

$$\begin{aligned} H_1 &= \sum_{k=0}^{\infty}\int_{E_k}\frac{\left|R_m\left(b_k;x,y\right)-R_m\left(b_k;z,y\right)\right|}{|x-y|^m}\left|K(x,y)\right|\left|f(y)\right|dy \\ &\lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta}\sum_{k=0}^{\infty}\left(2^k l\right)^\beta\int_{E_k}\frac{1}{\left(2^k.4l\right)^2}\left|f(y)\right|dy \\ &\lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta}\sum_{k=0}^{\infty}\frac{1}{2^k}\frac{\left(2^k l\right)^\beta}{2^k.4l}\int_{|z-y|<2^{k+1}.4l}\left|f(y)\right|dy \\ &\lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta}\sum_{k=0}^{\infty}\frac{1}{2^k}\frac{\left(2^k l\right)^\beta}{2^k.4l}\frac{\left|2^k.4l\right|^{1-\frac{1}{q}}}{\left|2^k.4l\right|^{\frac{1}{p}-\frac{1}{q}}}\left(\int_{2^{k+1}I}\left|f(y)\right|^p dy\right)^{\frac{1}{p}} \\ &\lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta}\|f\|_{M_p^q}\sum_{k=0}^{\infty}\frac{1}{2^k} \\ &\lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta}\|f\|_{M_p^q} . \end{aligned}$$

Second, for H_2, from [4], we know that

$$\begin{aligned} R_m\left(b_k;x,y\right) &\lesssim |x-y|^m\left(\frac{1}{|I_x^y|}\int_{I_x^y}\left|b_k^{(m)}(z)\right|^s dz\right)^{\frac{1}{s}} \\ &\lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta}\left(2^k l\right)^\beta |x-y|^m \end{aligned} \tag{2.6}$$

and

$$\left|\frac{1}{|x-y|^m}-\frac{1}{|z-y|^m}\right| \lesssim \frac{|x-z|}{|x-y|^{m+1}}. \tag{2.7}$$

Hence, by (2.6), (2.7) and the same estimate (for H_1) as above (here we omit the details)

$$\begin{aligned} H_2 &= \sum_{k=0}^{\infty}\int_{E_k} |R_m(b_k;z,y)| \left|\frac{1}{|x-y|^m} - \frac{1}{|z-y|^m}\right| |K(x,y)|\,|f(y)|\,dy \\ &\lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta} \sum_{k=0}^{\infty} \left(2^k l\right)^\beta \int_{E_k} \frac{1}{\left(2^k.4l\right)^2} |f(y)|\,dy \\ &\lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta} \|f\|_{M_p^q}. \end{aligned}$$

As for H_3, from [4], we know that

$$\left|\frac{(x-y)^m}{|x-y|^m} - \frac{(z-y)^m}{|z-y|^m}\right| \lesssim \frac{|x-z|}{|x-y|} \tag{2.8}$$

and

$$\begin{aligned} \left|b_k^{(m)}(y)\right| &= \left|b^{(m)}(y) - \left(b^{(m)}\right)_{E_k}\right| \\ &\lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta} \left|2^k I\right|^\beta. \end{aligned} \tag{2.9}$$

Hence, by (2.8), (2.9), $|y-z| \approx |y-x|$, $|K(x,y)| \lesssim \frac{1}{|z-y|}$ and Hölder's inequality we get

$$\begin{aligned} H_3 &= \sum_{k=0}^{\infty}\int_{E_k} \frac{1}{m!}\left|b_k^{(m)}(y)\right| \left|\frac{(x-y)^m}{|x-y|^m} - \frac{(z-y)^m}{|z-y|^m}\right| |K(x,y)|\,|f(y)|\,dy \\ &\lesssim \sum_{k=0}^{\infty}\int_{E_k} \left|b^{(m)}(y) - \left(b^{(m)}\right)_{E_k}\right| \frac{|x-z|}{|z-y|^2} |f(y)|\,dy \\ &\lesssim \sum_{k=0}^{\infty} \frac{1}{2^k}\frac{1}{2^k.4l} \int_{|z-y|<2^k.4l} \left|b^{(m)}(y) - \left(b^{(m)}\right)_{E_k}\right| |f(y)|\,dy \\ &\lesssim \sum_{k=0}^{\infty} \frac{1}{2^k} \left(\frac{1}{2^k.4l}\int_{|z-y|<2^k.4l} |f(y)|^p\,dy\right)^{\frac{1}{p}} \left(\frac{1}{2^k.4l}\int_{|z-y|<2^k.4l} \left|b^{(m)}(y) - \left(b^{(m)}\right)_{E_k}\right|^{p'} dy\right)^{\frac{1}{p'}} \\ &\lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta} \sum_{k=0}^{\infty} \frac{1}{2^k}\frac{\left|2^k I\right|^\beta}{\left|2^k I\right|} \left(\int_{2^{k+1}I} |f(y)|^p\,dy\right)^{\frac{1}{p}} \\ &\lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta} \|f\|_{M_p^q} \sum_{k=0}^{\infty}\frac{1}{2^k} \\ &\lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta} \|f\|_{M_p^q}. \end{aligned}$$

Finally, we consider the term H_4. From [4], we know that

$$\begin{aligned}\left|R_{m+1}\left(b_k;z,y\right)\right| &\leq \left|R_m\left(b_k;z,y\right)\right| + \frac{1}{m!}\left|b_k^{(m)}\left(y\right)\left(z-y\right)^m\right| \\ &\lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta}\left(2^k l\right)^\beta \left|z-y\right|^m + \left|b^{(m)}\left(y\right) - \left(b^{(m)}\right)_{E_k}\right|\left|z-y\right|^m \\ &\lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta}\left|2^k I\right|^\beta \left|z-y\right|^m \qquad (2.10)\end{aligned}$$

and

$$\left|K\left(x,y\right) - K\left(z,y\right)\right| \lesssim \frac{\left|x-z\right|^\delta}{\left|z-y\right|^{1+\delta}}. \qquad (2.11)$$

Thus, by (2.10), (2.11) and the same estimates (for H_1 and H_3) as above (here we omit the details)

$$\begin{aligned}H_4 &= \sum_{k=0}^{\infty}\int_{E_k}\left|\frac{R_{m+1}\left(b_k;z,y\right)}{\left|z-y\right|^m}\right|\left|K\left(x,y\right)-K\left(z,y\right)\right|\left|f\left(y\right)\right|dy \\ &\lesssim \sum_{k=0}^{\infty}\int_{E_k}\frac{\left|R_m\left(b_k;z,y\right)\right|}{\left|x-y\right|^m}\frac{\left|x-z\right|^\delta}{\left|z-y\right|^{1+\delta}}\left|f\left(y\right)\right|dy \\ &+ \sum_{k=0}^{\infty}\int_{E_k}\frac{\left|b_k^{(m)}\left(y\right)\left(z-y\right)^m\right|}{\left|x-y\right|^m}\frac{\left|x-z\right|^\delta}{\left|z-y\right|^{1+\delta}}\left|f\left(y\right)\right|dy \\ &\lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta}\sum_{k=0}^{\infty}\frac{1}{2^{k\delta}}\frac{\left|2^k I\right|^\beta}{\left|2^k I\right|}\left(\int_{2^{k+1}I}\left|f\left(y\right)\right|^p dy\right)^{\frac{1}{p}} \\ &\lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta}\left\|f\right\|_{M_p^q}\sum_{k=0}^{\infty}\frac{1}{2^{k\delta}} \\ &\lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta}\left\|f\right\|_{M_p^q}.\end{aligned}$$

By the estimates of H_1, H_2, H_3 and H_4 above, we know that

$$G_1 \lesssim \left\|b^{(m)}\right\|_{\dot{\Lambda}_\beta}\left\|f\right\|_{M_p^q}.$$

Now we turn to estimate G_2. Notice that the integral

$$\int_{\mathbb{R}}\left(\chi_{\{t_{i+1}<|x-y|<s\}}\left(y\right) - \chi_{\{t_{i+1}<|z-y|<s\}}\left(y\right)\right)\frac{R_{m+1}\left(b;z,y\right)}{\left|z-y\right|^m}K\left(z,y\right)f_2\left(y\right)dy$$

will be non-zero if either $\left(\chi_{\{t_{i+1}<|x-y|<s\}}\left(y\right)=1,\ \chi_{\{t_{i+1}<|z-y|<s\}}\left(y\right)=0\right)$ or $\left(\chi_{\{t_{i+1}<|x-y|<s\}}\left(y\right)=0,\ \chi_{\{t_{i+1}<|z-y|<s\}}\left(y\right)=1\right)$. That means this integral will only be non-zero in the following cases:

(1) $t_{i+1} < |x-y| < s$ and $|z-y| \leq t_{i+1}$;
(2) $t_{i+1} < |x-y| < s$ and $|z-y| \geq s$;
(3) $t_{i+1} < |z-y| < s$ and $|x-y| \leq t_{i+1}$;
(4) $t_{i+1} < |z-y| < s$ and $|x-y| \geq s$.

In (1), we know that

$$t_{i+1} < |x-y| \leq |x-z| + |z-y| < l + t_{i+1}$$

as $|x-z|<l$. Similarly, in case (3), we have

$$t_{i+1}<|z-y|<l+t_{i+1}$$

as $|x-z|<l$. In (2), we have

$$s<|z-y|\le|z-x|+|x-y|<l+s$$

and in (4), we have

$$s<|x-y|<l+s.$$

By (1.2) and using Hölder's inequality, it follows that

$$\begin{aligned}
&\int\limits_{\mathbb{R}}\left(\chi_{\{t_{i+1}<|x-y|<s\}}(y)-\chi_{\{t_{i+1}<|z-y|<s\}}(y)\right)\frac{R_{m+1}(b;z,y)}{|z-y|^m}K(z,y)f_2(y)\,dy\\
&\lesssim\int\limits_{\mathbb{R}}\chi_{\{t_{i+1}<|x-y|<s\}}(y)\,\chi_{\{t_{i+1}<|x-y|<l+t_{i+1}\}}(y)\left|\frac{R_{m+1}(b;z,y)}{|z-y|^m}\right|\frac{|f_2(y)|}{|z-y|}dy\\
&+\int\limits_{\mathbb{R}}\chi_{\{t_{i+1}<|x-y|<s\}}(y)\,\chi_{\{s<|z-y|<l+s\}}(y)\left|\frac{R_{m+1}(b;z,y)}{|z-y|^m}\right|\frac{|f_2(y)|}{|z-y|}dy\\
&+\int\limits_{\mathbb{R}}\chi_{\{t_{i+1}<|z-y|<s\}}(y)\,\chi_{\{t_{i+1}<|z-y|<l+t_{i+1}\}}(y)\left|\frac{R_{m+1}(b;z,y)}{|z-y|^m}\right|\frac{|f_2(y)|}{|z-y|}dy\\
&+\int\limits_{\mathbb{R}}\chi_{\{t_{i+1}<|z-y|<s\}}(y)\,\chi_{\{s<|x-y|<l+s\}}(y)\left|\frac{R_{m+1}(b;z,y)}{|z-y|^m}\right|\frac{|f_2(y)|}{|z-y|}dy\\
&\lesssim\left(\int\limits_{\mathbb{R}}\chi_{\{t_{i+1}<|x-y|<s\}}(y)\left|\frac{R_{m+1}(b;z,y)}{|z-y|^m}\right|^p\frac{|f_2(y)|^p}{|z-y|^p}dy\right)^{\frac{1}{p}}(2l)^{\frac{1}{p'}}\\
&+\left(\int\limits_{\mathbb{R}}\chi_{\{t_{i+1}<|z-y|<s\}}(y)\left|\frac{R_{m+1}(b;z,y)}{|z-y|^m}\right|^p\frac{|f_2(y)|^p}{|z-y|^p}dy\right)^{\frac{1}{p}}(2l)^{\frac{1}{p'}}.
\end{aligned}$$

Now for G_2, we decompose it into two parts as follows:

$$\begin{aligned}
G_2&=\left\|\left\{\int\limits_{\mathbb{R}}\left(\chi_{\{t_{i+1}<|x-y|<s\}}(y)-\chi_{\{t_{i+1}<|z-y|<s\}}(y)\right)\frac{R_{m+1}(b;z,y)}{|z-y|^m}K(z,y)f_2(y)\,dy\right\}_{s\in J_i,i\in\mathbb{N}}\right\|_E\\
&\lesssim\left\|\left\{\left(\int\limits_{\mathbb{R}}\chi_{\{t_{i+1}<|x-y|<s\}}(y)\left|\frac{R_{m+1}(b;z,y)}{|z-y|^m}\right|^p\frac{|f_2(y)|^p}{|z-y|^p}dy\right)^{\frac{1}{p}}\right\}_{s\in J_i,i\in\mathbb{N}}\right\|_E(2l)^{\frac{1}{p'}}\\
&+\left\|\left\{\left(\int\limits_{\mathbb{R}}\chi_{\{t_{i+1}<|z-y|<s\}}(y)\left|\frac{R_{m+1}(b;z,y)}{|z-y|^m}\right|^p\frac{|f_2(y)|^p}{|z-y|^p}dy\right)^{\frac{1}{p}}\right\}_{s\in J_i,i\in\mathbb{N}}\right\|_E(2l)^{\frac{1}{p'}}\\
&=:J_1+J_2.
\end{aligned}$$

First, by (2.10), it follows that

$$
\begin{aligned}
J_1 &= \left\| \left\{ \left(\int_{\mathbb{R}} \chi_{\{t_{i+1}<|x-y|<s\}}(y) \left| \frac{R_{m+1}(b;z,y)}{|z-y|^m} \right|^p \frac{|f_2(y)|^p}{|z-y|^p} dy \right)^{\frac{1}{p}} \right\}_{s\in J_i, i\in\mathbb{N}} \right\|_E (2l)^{\frac{1}{p'}} \\
&\lesssim \left\{ \sum_{i\in\mathbb{N}} \sup_{s\in J_i} \left(\int_{\mathbb{R}} \chi_{\{t_{i+1}<|x-y|<s\}}(y) \left| \frac{R_{m+1}(b;z,y)}{|z-y|^m} \right|^p \frac{|f_2(y)|^p}{|z-y|^p} dy \right)^{\frac{2}{p}} \right\}^{\frac{1}{2}} (2l)^{\frac{1}{p'}} \\
&\lesssim \left\{ \sum_{i\in\mathbb{N}} \int_{\mathbb{R}} \chi_{\{t_{i+1}<|x-y|<s\}}(y) \left| \frac{R_{m+1}(b;z,y)}{|z-y|^m} \right|^p \frac{|f_2(y)|^p}{|z-y|^p} dy \right\}^{\frac{1}{p}} (2l)^{\frac{1}{p'}} \\
&\lesssim \left\{ \int_{\mathbb{R}} \left| \frac{R_{m+1}(b;z,y)}{|z-y|^m} \right|^p \frac{|f_2(y)|^p}{|z-y|^p} dy \right\}^{\frac{1}{p}} (2l)^{\frac{1}{p'}} \\
&\lesssim \left\{ \sum_{k=0}^{\infty} \int_{E_k} \left| \frac{R_{m+1}(b_k;z,y)}{|z-y|^m} \right|^p \frac{|f(y)|^p}{|z-y|^p} dy \right\}^{\frac{1}{p}} (2l)^{\frac{1}{p'}} \\
&\lesssim \left\| b^{(m)} \right\|_{\dot{\Lambda}_\beta} \left\{ \sum_{k=0}^{\infty} \left|2^k I\right|^{\beta p} \int_{E_k} \frac{|f(y)|^p}{|z-y|^p} dy \right\}^{\frac{1}{p}} (2l)^{\frac{1}{p'}} \\
&\lesssim \left\| b^{(m)} \right\|_{\dot{\Lambda}_\beta} \left\{ \sum_{k=0}^{\infty} \left|2^k I\right|^{\beta p - p} \int_{E_k} |f(y)|^p dy \right\}^{\frac{1}{p}} (2l)^{\frac{1}{p'}} \\
&\lesssim \left\| b^{(m)} \right\|_{\dot{\Lambda}_\beta} \left\{ \sum_{k=0}^{\infty} \left|2^k I\right|^{\beta p - p + 1 - \frac{p}{q}} \|f\|_{M_p^q}^p \right\}^{\frac{1}{p}} (2l)^{\frac{1}{p'}} \\
&\lesssim \left\| b^{(m)} \right\|_{\dot{\Lambda}_\beta} \|f\|_{M_p^q} \left\{ \sum_{k=0}^{\infty} 2^{k(1-p)} \right\}^{\frac{1}{p}} \\
&\lesssim \left\| b^{(m)} \right\|_{\dot{\Lambda}_\beta} \|f\|_{M_p^q}.
\end{aligned}
$$

Similarly, J_2 has the same estimate above. Here we omit the details, thus the inequality

$$
\begin{aligned}
J_2 &= \left\| \left\{ \left(\int_{\mathbb{R}} \chi_{\{t_{i+1}<|z-y|<s\}}(y) \left| \frac{R_{m+1}(b;z,y)}{|z-y|^m} \right|^p \frac{|f_2(y)|^p}{|z-y|^p} dy \right)^{\frac{1}{p}} \right\}_{s\in J_i, i\in\mathbb{N}} \right\|_E (2l)^{\frac{1}{p'}} \\
&\lesssim \left\| b^{(m)} \right\|_{\dot{\Lambda}_\beta} \|f\|_{M_p^q}
\end{aligned}
$$

is valid.

Putting estimates J_1 and J_2 together, we get the desired conclusion

$$G_2 \lesssim \left\| b^{(m)} \right\|_{\dot{\Lambda}_\beta} \|f\|_{M_p^q}.$$

Similarly, $\mathcal{V}_\rho\left(T^b\right)$ has the same estimate as above (here we omit the details), thus the inequality

$$\left\|\mathcal{V}_\rho\left(T^b f\right)(x)\right\|_{BMO} \lesssim \|b\|_{\dot{\Lambda}_\beta}\|f\|_{M_p^q}$$

is valid.

Therefore, Theorem 2.1 is completely proved. □

3. Conclusion

The oscillation and variation for martingales and some families of operators have been studied in many recent papers on probability, ergodic theory, and harmonic analysis. Thus, in this paper, we have established several criterions of boundedness for the oscillation and variation operators related to multilinear singular integrals with Lipschitz functions. The Morrey spaces play important roles both in harmonic analysis and partial differential equation. In particular, the mapping properties of the Calderón-Zygmund singular integral with a Lipschitz function in $\mathbb{R}$ had been obtained for Morrey spaces. Therefore, it motivates us to investigate the extension of these inequalities to the oscillation and variation operators on Morrey spaces. Indeed, the results obtained in this paper are extensions of some known results. So this research is meaningful.

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CARLESON EMBEDDING FROM WEIGHTED DIRICHLET TYPE SPACES TO TENT TYPE SPACES

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ABSTRACT: In this paper we study Carleson embedding from weighted Dirichlet type spaces to Tent type spaces.

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1. INTRODUCTION

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the class of analytic functions in $\mathbb{D}$. The Bloch space $\mathcal{B}$ is the class of all $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}} := |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

The little Bloch space $\mathcal{B}_0$, consists of all $f \in H(\mathbb{D})$ satisfying

$$\lim_{|z|\to 1^-}(1-|z|^2)|f'(z)| = 0.$$

The Hardy space $H^p(\mathbb{D})(0 < p < \infty)$ is the sets of $f \in H(\mathbb{D})$ with

$$\|f\|_{H^p}^p = \sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^p d\theta < \infty.$$

Assume that $K: [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing weighted function. Let weighted Dirichlet type space $\mathcal{D}_{K,\alpha}$ be the spaces of function $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{\mathcal{D}_{K,\alpha}}^2 := |f(0)|^2 + \int_{\mathbb{D}}|f'(z)|^2\frac{(1-|z|^2)}{K(1-|z|^2)}dA_\alpha(z) < \infty,$$

where $dA_\alpha(z) = (1-|z|^2)^\alpha dA(z)$ and $\alpha \geq 0$. When $K(t) = t^p$ and $\alpha = 0$, $0 < p < 1$, it gives classic Dirichlet space $\mathcal{D}_p$. Especially, when $K(t) = t^p$ and $p = \alpha$, it gives the Hardy space H^2; when $K(t) = t^p$ and $\alpha = p + 1$, we have the Bergman spaces A^2. We refer the paper [11] for studying small Hankel operator acting on $\mathcal{D}_p$, and the paper [13] and [14] for studying multipliers on $\mathcal{D}_p$ spaces. When $\alpha = 0$, under Dirichlet conditions on weighted function K, Kerman and Sawyer [5] have characterized Carleson measures and multipliers of $\mathcal{D}_{K,\alpha}$ in terms of a maximal operator. Aleman has given some basic properties of $\mathcal{D}_{K,\alpha}$ in [1]. For more results on $\mathcal{D}_{K,\alpha}$ spaces, we refer to [2], [7] and [8].
Let I be an arc of $\partial\mathbb{D}$ and $|I|$ be the normalized Lebesgue arc length of I. The Carleson square based on I, denoted by $S(I)$, is defined by

$$S(I) := \left\{z = re^{i\theta} \in \mathbb{D}: 1-|I| \leq r < 1, e^{i\theta} \in I\right\}.$$

Let μ be a positive Borel measure on $\mathbb{D}$. For $0 < s < \infty$, μ is called an s −Carleson measure if

$$\sup_{I\subset\partial\mathbb{D}}\frac{\mu(S(I))}{|I|^s} < \infty.$$

That is,

$$\mu(S(I)) \leq C|I|^s$$

for all interval $I \subset \partial\mathbb{D}$.
If μ is an s −Carleson measure, we set

$$\|\mu\|_s := \sup_{I\subset\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^s}.$$

For a nonnegative Borel measure μ on the unit disk $\mathbb{D}$, we define $\mathcal{T}_K^\infty(\mu)$ as the space of all μ −measurable functions f on $\mathbb{D}$ satisfying

$$\|f\|^2_{\mathcal{T}_K^\infty(\mu)} = \sup_{I\subset\partial\mathbb{D}} \frac{1}{K(|I|)} \int_{S(I)} |f(z)|^2 d\mu(z) < \infty.$$

When $K(t) = t^p$ and $0 < p < 1$, it gives the well known tent spaces, which was first introduced by Xiao in [15]. For more information related to tent space, we refer to [[6], [9], [10]].

Given $f,\ g \in H(\mathbb{D})$. The Volterra integral operator T_g and its companion operator I_g are defined by

$$T_g f(z) := \int_0^z g'(w) f(w) dw$$

and

$$I_g f(z) := \int_0^z g(w) f'(w) dw,$$

$z \in \mathbb{D}$, respectively. Both operators have been studied extensively (see [12]).

Recently, the authors [16] have studied the boundedness and essential norm of Volterra type integral operators T_g and I_g from Dirichlet type spaces $\mathcal{D}_{K,\alpha}$ to Morrey type spaces $H^2_{K,\alpha}$ such that $g \in \mathcal{B}$. As a continuation to their work, we consider the Carleson embedding from $\mathcal{D}_{K,\alpha}$ to $\mathcal{T}_K^\infty(\mu)$. We prove that $I: \mathcal{D}_{K,\alpha} \to \mathcal{T}_K^\infty(\mu)$ if and only if μ is a $(\alpha + 1)$ −Carleson measure, when $0 < \alpha < 3$.

Throughout the paper, we assume that the weighted function K satisfies:

$$\int_0^1 \frac{\varphi_K(s)}{s} ds < \infty \tag{1}$$

and

$$\int_1^\infty \frac{\varphi_K(s)}{s^2} ds < \infty, \tag{2}$$

where

$$\varphi_K(s) = \sup_{0\le t\le 1} K(st)/K(t), 0 < s < \infty.$$

Note that K satisfies (2), by [[4], Lemma 2.2], there exists a small $c > 0$ such that

$$\varphi_K(t) \le Ct^{1-c}, t \ge 1. \tag{3}$$

Finally, in the rest of this paper, C expresses unspecified positive constant, possibly different at each occurrence; the symbol $f \preceq g$ means that $f \le Cg$. If $f \preceq g$ and $g \preceq f$, then we write $f \approx g$.

2. MAIN RESULTS

We are now ready to present and establish the main results of this paper.

Theorem 2.1. *Suppose that K satisfies* (1) *and* (2). *Let* $0 < \alpha < 3$ *and μ is a positive Borel measure on $\mathbb{D}$. Then the inclusion mapping $i: \mathcal{D}_{K,\alpha} \to \mathcal{T}_K^\infty(\mu)$ is bounded $\Leftrightarrow$ μ is a $(\alpha + 1)$ −Carleson measure. That is,*

$$\sup_{I\subset\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^{\alpha+1}} < \infty.$$

Theorem 2.2. *Suppose that K satisfies* (1) *and* (2). *Let* $0 < \alpha < 3$ *and μ is a positive Borel measure on $\mathbb{D}$. Then the inclusion mapping $i: \mathcal{D}_{K,\alpha} \to \mathcal{T}_K^\infty(\mu)$ is compact $\Leftrightarrow$ μ is a vanishing $(\alpha + 1)$ −Carleson measure. That is,*

$$\sup_{|I|\to 0} \frac{\mu(S(I))}{|I|^{\alpha+1}} = 0.$$

2.1 The Proof of Theorem 2.1.

Proof.

Necessity:

Assume that $i: \mathcal{D}_{K,\alpha} \to \mathcal{T}_K^\infty(\mu)$ is bounded. Let

$$f_a(z) := \frac{(1-|a|^2)\sqrt{K(1-|a|^2)}}{(1-\bar{a}z)^{\frac{3+\alpha}{2}}}.$$

By Lemma 2 in [16], we get $f_a \in \mathcal{D}_{K,\alpha}$. Fixed an arc $I \subset \partial\mathbb{D}$. Let $e^{i\theta}$ be the center of I and $a = (1 - |I|)e^{i\theta}$. Then

$$|1 - \bar{a}z| \approx 1 - |a| = |I|, |f_a(z)|^2 \approx \frac{K(|I|)}{|I|^{1+\alpha}}, z \in S(I).$$

Therefore,

$$\frac{\mu(S(I))}{|I|^{1+\alpha}} \approx \frac{1}{K(|I|)} \int_{S(I)} |f_a(z)|^2 d\mu(z) < \infty.$$

That is, μ is a $(\alpha + 1)$ −Carleson measure.

Sufficiency:

Assume that μ is a $(\alpha + 1)$ −Carleson measure. Fixed $f \in \mathcal{D}_{K,\alpha}$. Let I be any arc on $\partial\mathbb{D}$ and $a = (1 - |I|)e^{i\theta}$, where $e^{i\theta}$ is the midpoint of I. From Lemma 1 in [16],

$$|f(a)|^2 \lesssim \frac{\|f\|_{\mathcal{D}_{K,\alpha}} K(|I|)}{|I|^{1+\alpha}}.$$

Since

$$\begin{aligned}
&\frac{1}{K(|I|)} \int_{S(I)} |f(z)|^2 d\mu(z) \\
&\lesssim \frac{1}{K(|I|)} \int_{S(I)} |f(a)|^2 d\mu(z) + \frac{1}{K(|I|)} \int_{S(I)} |f(z) - f(a)|^2 d\mu(z) \\
&= F + G.
\end{aligned}$$

It is obvious that

$$F \lesssim \frac{\mu(S(I))}{|I|^{1+\alpha}} \|f\|^2_{\mathcal{D}_{K,\alpha}} \lesssim \|f\|^2_{\mathcal{D}_{K,\alpha}}.$$

By Lemma 3 in [16], we have $A^2_{\alpha-1} \subset L^2(d\mu)$. Note that

$$\|f\|^2_{A^2_{\alpha-1}} \approx \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^{\alpha+1} dA(z) \leq \|f\|^2_{\mathcal{D}_{K,\alpha}}.$$

Thus, $\mathcal{D}_{K,\alpha} \subset A^2_{\alpha-1}$. Based on these facts, we turn to estimate G. The estimate will be divided into two cases.

Case 1: $1 \leq \alpha < 3$.

Let $z = \varphi_a(w)$. Note that

$$|\varphi_a'(w)|(1 - |w|^2) = 1 - |\varphi_a(w)|^2.$$

Then, we obtain

$$\begin{aligned}
G &\approx \frac{(1-|a|^2)^4}{K(1-|a|^2)} \int_{S(I)} \left|\frac{f(z)-f(a)}{(1-\bar{a}z)^2}\right|^2 d\mu(z) \\
&\leq \frac{(1-|a|^2)^4}{K(1-|a|^2)} \int_{\mathbb{D}} \left|\frac{f(z)-f(a)}{(1-\bar{a}z)^2}\right|^2 d\mu(z) \\
&\lesssim \frac{(1-|a|^2)^4}{K(1-|a|^2)} \int_{\mathbb{D}} \left|\frac{f(z)-f(a)}{(1-\bar{a}z)^2}\right|^2 (1 - |z|^2)^{\alpha-1} dA(z) \\
&\leq \frac{(1-|a|^2)^2}{K(1-|a|^2)} \int_{\mathbb{D}} \frac{|f(z)-f(a)|^2 (1-|a|^2)^2}{|1-\bar{a}z|^4} (1 - |z|^2)^{\alpha-1} dA(z) \\
&= \frac{(1-|a|^2)^2}{K(1-|a|^2)} \int_{\mathbb{D}} |(f \circ \varphi_a)(w) - (f \circ \varphi_a)(0)|^2 (1 - |\varphi_a(w)|^2)^{\alpha-1} dA(w) \\
&\lesssim \frac{(1-|a|^2)^2}{K(1-|a|^2)} \int_{\mathbb{D}} |(f \circ \varphi_a)'(w)|^2 (1 - |w|^2)^2 (1 - |\varphi_a(w)|^2)^{\alpha-1} dA(w) \\
&= \frac{(1-|a|^2)^2}{K(1-|a|^2)} \int_{\mathbb{D}} \left|f'(\varphi_a(w))\right|^2 |\varphi_a'(w)|^2 (1 - |w|^2)^2 (1 - |\varphi_a(w)|^2)^{\alpha-1} dA(w)
\end{aligned}$$

$$= \frac{(1-|a|^2)^2}{K(1-|a|^2)} \int_{\mathbb{D}} \left|f'(\varphi_a(w))\right|^2 (1-|\varphi_a(w)|^2)^{\alpha+1} dA(w)$$
$$= \frac{(1-|a|^2)^2}{K(1-|a|^2)} \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{\alpha+1} \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dA(z)$$
$$= \int_{\mathbb{D}} |f'(z)|^2 \frac{(1-|z|^2)^{\alpha+1}}{K(1-|z|^2)} \frac{K(1-|z|^2)}{K(1-|a|^2)} \frac{(1-|a|^2)^4}{|1-\bar{a}z|^4} dA(z)$$
$$\preceq \int_{\mathbb{D}} |f'(z)|^2 \frac{(1-|z|^2)^{\alpha+1}}{K(1-|z|^2)} \left(\frac{K(|1-\bar{a}z|)}{K(1-|a|^2)}\right) \frac{(1-|a|^2)^4}{|1-\bar{a}z|^4} dA(z)$$
$$\preceq \int_{\mathbb{D}} |f'(z)|^2 \frac{(1-|z|^2)^{\alpha+1}}{K(1-|z|^2)} \frac{(|1-\bar{a}z|)^{1-c}}{(1-|a|^2)^{1-c}} \frac{(1-|a|^2)^4}{|1-\bar{a}z|^4} dA(z)$$
$$\preceq \|f\|^2_{\mathcal{D}_{K,\alpha}},$$

where the last second and fourth inequalities are deduced by (3) and Lemma 2.1 in [3], respectively.

Case 2: $0 < \alpha < 1$.

Checking the proof of above, we have

$$G \preceq \frac{(1-|a|^2)^2}{K(1-|a|^2)} \int_{\mathbb{D}} |(f\circ\varphi_a)(w) - (f\circ\varphi_a)(0)|^2 (1-|\varphi_a(w)|^2)^{\alpha-1} dA(w)$$
$$\preceq \frac{(1-|a|^2)^{1+\alpha}}{K(1-|a|^2)} \int_{\mathbb{D}} |(f\circ\varphi_a)(w) - (f\circ\varphi_a)(0)|^2 (1-|w|^2)^{\alpha-1} dA(w)$$
$$\preceq \frac{(1-|a|^2)^{1+\alpha}}{K(1-|a|^2)} \int_{\mathbb{D}} |(f\circ\varphi_a)'(w)|^2 (1-|w|^2)^{\alpha+1} dA(w)$$
$$\preceq \frac{(1-|a|^2)^{1+\alpha}}{K(1-|a|^2)} \int_{\mathbb{D}} \left|f'(\varphi_a(w))\right|^2 (1-|\varphi_a(w)|^2)^2 (1-|w|^2)^{\alpha-1} dA(w)$$
$$\preceq \frac{(1-|a|^2)^{1+\alpha}}{K(1-|a|^2)} \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^2 (1-|\varphi_a(z)|^2)^{\alpha-1} \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dA(w)$$

$$= \int_{\mathbb{D}} |f'(z)|^2 \frac{(1-|z|^2)^{\alpha+1}}{K(1-|z|^2)} \frac{K(1-|z|^2)}{K(1-|a|^2)} \frac{(1-|a|^2)^{2(1+\alpha)}}{|1-\bar{a}z|^{2(1+\alpha)}} dA(w)$$
$$\preceq \|f\|^2_{\mathcal{D}_{K,\alpha}}.$$

Combining the estimates F and G, we conclude that $i: \mathcal{D}_{K,\alpha} \to \mathcal{T}_K^\infty(\mu)$ is bounded. □

2.2 The Proof of Theorem 2.2.

Proof.

Necessity:

Assume that $i: \mathcal{D}_{K,\alpha} \to \mathcal{T}_K^\infty(\mu)$ is compact. Given a sequence of arcs $\{I_n\}$ with $\lim_{n\to\infty} |I_n| = 0$. Denote the center of I_n by $e^{i\theta_n}$ and $a_n = (1-|I_n|)e^{i\theta_n}$. Let

$$f_n(z) := \frac{(1-|a_n|^2)\sqrt{K(1-|a_n|^2)}}{(1-\bar{a}_n z)^{\frac{3+\alpha}{2}}}, z \in \mathbb{D}.$$

It is clear that $\{f_n\}$ is bounded in $\mathcal{D}_{K,\alpha}$ and $\{f_n\}$ converges to zero uniformly on any compact subset of $\mathbb{D}$. Then $\lim_{n\to\infty} \|f_n\|_{\mathcal{T}_K^\infty(\mu)} = 0$. Since

$$|f_n(z)|^2 \approx \frac{K(|I_n|)}{|I_n|^{1+\alpha}}, z \in S(I_n),$$

we obtain

$$\frac{\mu(S(I_n))}{|I_n|^{1+\alpha}} \approx \frac{1}{K(|I_n|)} \int_{S(I_n)} |f_n(z)|^2 d\mu(z) \preceq \|f_n\|^2_{\mathcal{T}_K^\infty(\mu)} \to 0, (n\to\infty).$$

By the arbitrariness of $\{I_n\}$, we deduce that μ is a vanishing $(\alpha+1)$ −Carleson measure.

Sufficiency:

Assume that μ is a vanishing $(\alpha+1)$ −Carleson measure, then μ is also a $(\alpha+1)$ −Carleson measure and $\lim_{r\to 1^-} \|\mu - \mu_r\|_{\alpha+1} = 0$ by Lemma 2.2 in [6]. It follows from the boundedness above, $i:\mathcal{D}_{K,\alpha} \to \mathcal{T}_K^\infty(\mu)$ is bounded. Let $\{f_n\}$ be a bounded sequence in $\mathcal{D}_{K,\alpha}$ such that $\{f_n\}$ converges to zero uniformly on each compact subset of $\mathbb{D}$. We have

$$\begin{aligned}
&\frac{1}{K(|I|)}\int_{S(I)} |f_n(z)|^2 d\mu(z) \\
&\le \frac{1}{K(|I|)}\int_{S(I)} |f_n(z)|^2 d\mu_r(z) + \frac{1}{K(|I|)}\int_{S(I)} |f_n(z)|^2 d(\mu-\mu_r)(z) \\
&\le \frac{1}{K(|I|)}\int_{S(I)} |f_n(z)|^2 d\mu_r(z) + \|\mu-\mu_r\|_{\alpha+1}\|f_n\|_{\mathcal{D}_{K,\alpha}}^2 \\
&\le \frac{1}{K(|I|)}\int_{S(I)} |f_n(z)|^2 d\mu_r(z) + \|\mu-\mu_r\|_{\alpha+1} \to 0 (r \to 1^-, n \to \infty).
\end{aligned}$$

Thus, we get $\lim_{n\to\infty}\|f_n\|_{\mathcal{T}_K^\infty(\mu)} = 0$. That is, $i:\mathcal{D}_{K,\alpha} \to \mathcal{T}_K^\infty(\mu)$ is compact. □

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