STRONG AND WEAK TOTAL DOMINATIONS IN VARIOUS CORONA GRAPHS AND CHARACTERIZATION

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ABSTRACT. Let $G = (V, E)$ be a graph. A subset $S \subseteq V$ is a total dominating set if every vertex in V has a neighbor in S. A total dominating set S is said to be weak if every vertex $v \in V - S$ is adjacent to vertex $u \in S$ such that $d_G(v) \geq d_G(u)$. Analogously, a total dominating set S is said to be strong if every vertex $v \in V - S$ is adjacent to vertex $u \in S$ such that $d_G(v) \leq d_G(u)$. The minimum cardinality of weak total dominating set and strong total dominating set denoted by $\gamma_{wt}(G)$ and $\gamma_{st}(G)$, respectively. In this paper we obtain some results about weak total and strong total domination number of various corona graphs such as corona, neighborhood corona, edge corona, subdivision vertex and edge corona. In addition, $\gamma_{wt}(G) = \gamma_w(G) + k$ and $\gamma_{st}(G) = \gamma_s(G) + k$, $(0 \leq k \leq \gamma_{wt}(G)$ or $\gamma_{st}(G))$ characterizations are investigated for discussed corona operations.

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1. INTRODUCTION

Let G be n order connected simple graphs. $V(G)$ and $E(G)$ are vertex and edge set of G, respectively. The *open neighborhood of* $v \in V$ is $N_G(v) = \{u \in V : uv \in E(G)\}$ and *closed neighborhood* of $v \in V$ is $N_G[v] = N_G(v) \cup \{v\}$. If v is a vertex of $V(G)$, then the *degree* of v denoted by $\deg_G(v)$, is the cardinality of its open neighborhood. The *maximum* and *minimum degree* of a graph G are denoted by $\Delta(G) = \Delta$ and $\delta(G) = \delta$, respectively.

If a vertex and an edge are incident in G then they cover each other in a graph G. A *vertex cover* in G is a set of vertices that covers all the edges of G. The *vertex covering number*, abbreviated $\alpha(G)$, is the minimum cardinality among all the vertex covers. Similarly, an *edge cover* in G is a set of edges that covers all the vertices of G. The *edge covering number*, denoted by $\alpha'(G)$, is the minimum cardinality among all the edge covers. In addition, a *total vertex cover* in G, denoted by TVC, is a vertex cover and also every vertex in TVC has a neighbor in TVC. The minimum cardinality among all the TVCs in G is called the total vertex covering number of G and is denoted by $tvc(G)$. The parameter TVC is introduced by Henning and Jafari Rad [12].

A subset $S \subseteq V$ is a *dominating set of* G if every vertex in $V - S$ has a neighbor in S and the *domination number of* G, denoted by $\gamma(G)$ is the minimum cardinality of a dominating set. For detailed information about domination parameters readers are referred to two books [10, 11]. A *total dominating set*, denoted by TD-set, *of* G with no isolated vertex is a set S of vertices of G and total domination number that is the minimum cardinality of a total dominating set denoted by $\gamma_t(G)$. Every graph without isolated vertices has TD-set. Total domination was introduced by Cockayne et al. [7]. For some α with

$0 < \alpha \leq 1$, it is said that a TD-set S in G is an *α-total dominating set*, abbreviated by αTD-set, if for every vertex $v \in V - S$, $|N(v) \cap S| \geq \alpha|N(v)|$. Thus, every vertex v in $V - S$ has at least $\alpha|N(v)|$ neighbors in S. The minimum cardinality of an α-TD-set of G is called the *α-total domination number* of G and is denoted by $\gamma_{\alpha t}(G)$. An αTD-set of G of cardinality $\gamma_{\alpha t}(G)$ is called a $\gamma_{\alpha t}(G)$-set. This concept is introduced by Henning and Jafari Rad [12]. They obtained several results and bounds about α-total domination number of a graph G. A dominating set $S \subseteq V$ is called a *weak dominating set* (WD-set) if each vertex $v \in V - S$ is dominated by some vertices $u \in S$ with $\deg(v) > \deg(u)$. The *weak domination number*, denoted by $\gamma_w(G)$, is minimum cardinality of a weak dominating set. Similarly, a dominating set $S \subseteq V$ is called a *strong dominating set* (SD-set) if each vertex $v \in V - S$ is dominated by some vertices $u \in S$ with $\deg(v) < \deg(u)$. The *strong domination number*, denoted by $\gamma_s(G)$, is minimum cardinality of a strong dominating set. The concept weak and strong domination number introduced by Sampathkumar and Pushpa Latha in [16]. In addition, there are some studies about effects of some graph operations on strong and weak domination number in the literature [2, 3, 4, 5]. A weak dominating set $S \subseteq V$ induces a subgraph with no isolated vertex is called *weak total dominating set* (WTD-set). The *weak total domination number,* $\gamma_{wt}(G)$ of G is minimum cardinality of WTD-set ($\gamma_{wt} - set$). Chellali et al. have introduced the parameter weak total domination number [6]. Analogously, the parameter *strong total domination number, denoted by* $\gamma_{st}(G)$, have been defined as minimum cardinality of *strong total dominating set* (γ_{st}-set) that is a strong dominating set $S \subseteq V$ induces a subgraph with no isolated vertex [1]. Also, in [1] Akbari and Jafari Rad have obtained Nordhaus-Gaddum bounds for weak and strong total domination number and in [14] complexity of strong and weak total dominations have been considered for some graphs.

R. Frucht and F. Harary introduced the *corona operation* [8]. In addition, a variant of corona operations, *neighborhood corona* and *edge corona* were introduced by Gopalapillai and Hou and Shiu, respectively [9, 13]. Moreover, using *subdivision graph* concept another corona operation *subdivision vertex corona* and *subdivision edge corona* operations was defined by P. Lu and Y. Miao [15].

Firstly, we give definitions of corona operations. Let G_1 and G_2 be two graphs that have n_1 vertices q_1 edges and n_2 vertices q_2 edges, respectively. *The corona* of G_1 and G_2, denoted by $G_1 \circ G_2$, is the graph consisting one copy of G_1 and n_1 copy of G_2 and then joining the i^{th} vertex of G_1 to every vertex of i^{th} copy of G_2. *The edge corona* of G_1 and G_2, denoted by $G_1 \diamond G_2$, is the graph consisting one copy of G_1 and m_1 copy of G_2, and then joining two end vertices of the i^{th} edge of G_1 to every vertex in the i^{th} copy of G_2. *The neighborhood corona* of G_1 and G_2, denoted by $G_1 \star G_2$, is the graph consisting one copy of G_1 and n_1 copy of G_2,and then joining each neighbor of the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2. The *subdivision vertex corona* of G_1 and G_2, denoted by $G_1 \odot G_2$, is the graph obtained from subdivision graph $S(G_1)$ of a graph G_1 is the graph obtained by inserting a new vertex into every edge of G_1 and n_1 copy of G_2, and joining the i^{th} vertex of G_1 to every vertex of the i^{th} copy of G_2. The *subdivision edge corona* of G_1 and G_2, denoted by $G_1 \ominus G_2$, is the graph obtained from subdivision graph $S(G_1)$ of a graph G_1 and m_1 copy of G_2, and joining the i^{th} new subdivision vertex of G_1 to every vertex in the i^{th} copy of G_2. Figure 1 also illustraits the discussed operations.

In this paper, results about weak total domination and strong total domination of corona, edge corona, neighborhood corona, subdivision vertex and edge corona operations will be provided. Also, the characterizations $\gamma_{wt}(G) = \gamma_w(G) + k$ and $\gamma_{st}(G) = \gamma_s(G) + k$, $0 \leq k \leq \gamma_w(G)$ (or $\gamma_s(G)$), will be constructed and the value of k will be determined for the discussed corona operations.

2. Results about Corona Operations

In this section, results about strong total and weak total domination of various corona graph operations will be provided. Corona, edge corona, neighborhood corona, subdivision vertex and subdivision edge corona operations will be taken into consideration.

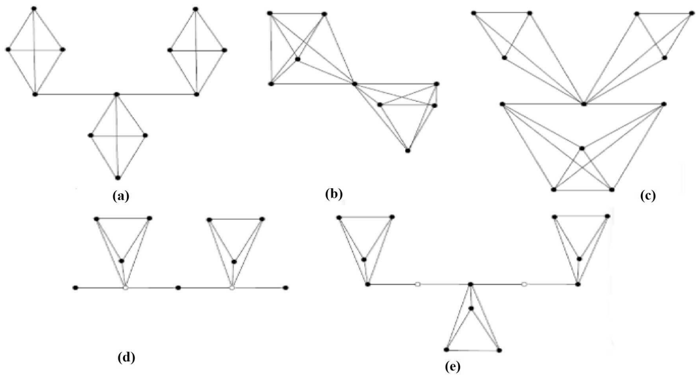

FIGURE 1. The corona graphs of P_3 and C_3. (a) The corona of P_3 and C_3, $P_3 \circ C_3$ (b) The edge corona of P_3 and C_3, $P_3 \diamond C_3$ (c) The neighborhood corona of P_3 and C_3, $P_3 \star C_3$ (d) The subdivision edge corona of P_3 and C_3, $P_3 \ominus C_3$ (e) The subdivision vertex corona of P_3 and C_3, $P_3 \odot C_3$.

Theorem 2.1. *Let G_1 and G_2 be n_1 and n_2 ordered graphs, respectively*

$$\gamma_{wt}(G_1 \circ G_2) = \begin{cases} n_1\gamma_w(G_2) & , \textit{if } \gamma_w(G_2) = \gamma_{wt}(G_2) \\ n_1(\gamma_w(G_2)+1) & , \textit{otherwise} \end{cases}$$
$$\gamma_{st}(G_1 \circ G_2) = n_1.$$

Proof. After corona operation, degree of vertices in G_1 are greater than degree of vertices of G_2. Therefore, $G_1 \circ G_2$ is weakly dominated by some vertices from G_2. If $\gamma_w(G_2) = \gamma_{wt}(G_2)$ then i^{th} copy of G_2 and joining i^{th} vertex of G_1 $(1 \leq i \leq n_1)$ are weakly total dominated by $\gamma_w - set$ of G_2. Thus, $\gamma_{wt}(G_1 \circ G_2) = n_1\gamma_w(G_2)$. If $\gamma_w(G_2) < \gamma_{wt}(G_2)$ then $\gamma_w - set$ of G_2 only weakly dominates to i^{th} copy of G_2 and joining i^{th} vertex of G_1 $(1 \leq i \leq n_1)$. For totality, adding at least one vertex to $\gamma_w - set$ of G_2 for each joining part. Then, i^{th} copy of G_2 and i^{th} vertex of G_1 $(1 \leq i \leq n_1)$ are weakly total dominated by $\gamma_w - set$ of G_2 and i^{th} vertex of G_1. Hence, $\gamma_{wt}(G_1 \circ G_2) = n_1\gamma_w(G_2) + n_1 = n_1(\gamma_w(G_2)+1)$.

Because of the degrees of G_1 after corona operation, all vertices of G_1 compose a minimal strong total dominating set of $G_1 \circ G_2$. Thus, $\gamma_{st}(G_1 \circ G_2) = n_1$. □

Theorem 2.2. *Let G_1 has n_1 vertices q_1 edges and G_2 has n_2 vertices q_2 edges then*

$$\gamma_{wt}(G_1 \diamond G_2) = \begin{cases} q_1\gamma_w(G_2) & , \textit{if } \gamma_w(G_2) = \gamma_{wt}(G_2) \\ q_1\gamma_w(G_2) + \alpha(G_1) & , \textit{otherwise} \end{cases}$$
$$\gamma_{st}(G_1 \diamond G_2) = tvc(G_1).$$

where α is covering number and tvc is total vertex covering number of graph.

Proof. After edge corona operation, degree of vertices in G_1 are greater than or equal to degree of vertices of G_2. If $\gamma_w(G_2) = \gamma_{wt}(G_2)$ then each copy of G_2 are weakly total dominated by $\gamma_w - set$ of G_2. Thus, in this case $\gamma_{wt}(G_1 \diamond G_2) = q_1\gamma_w(G_2)$. In the other case, $\gamma_w - set$ of G_2 is not enough for totality. In order to obtain $\gamma_{wt} - set$ of $G_1 \diamond G_2$, it is needed to at least one vertex for each copy of G_2. Due to the definition of covering number, vertices in a covering set of G_1 is adjacent to all copy of G_2 in $G_1 \diamond G_2$. Hence, $\gamma_{wt}(G_1 \diamond G_2) = q_1\gamma_w(G_2)+ \alpha(G_1)$.

For minimal strong total dominating set, it is needed to construct a set with vertices from G_1 which adjacent to all copies of G_2 without isolated vertex. Then, this correspond total vertex covering number of G_1. Therefore, $\gamma_{st}(G_1 \diamond G_2) = tvc(G_1)$. □

Remark 2.3. *From* [12] *it is known that $tvc(G_1) \leq \gamma_{\alpha t}(G_1)$. Therefore, the relationship between strong total domination and α-total domination can be constructed as $\gamma_{st}(G_1 \diamond G_2) \leq \gamma_{\alpha t}(G_1)$.*

Proposition 2.4. *Let G be a graph and $\alpha > 1$. Then, $\alpha(G) \leq tvc(G) \leq 2\alpha(G) - 1$.*

Proof. According to the definition of total vertex covering set and vertex covering set, left side of inequality is obvious. Let S be minimum vertex covering set of G that includes disjoint elements. Totality of S can be provided with $\alpha(G)$ vertices at most. Due to the definition of vertex covering number, if $\alpha(G)$ vertices added to S to satisfy totality then at least one element adjacent more than one vertex in S. Hence, a total vertex covering set of G can be contained at most $2\alpha(G) - 1$ vertex. Therefore, upper bound of total vertex covering number of G can be expressed in terms of vertex covering number. □

Theorem 2.5. *Let G_1 and G_2 are n_1 and n_2 ordered graphs, respectively*

$$\gamma_{wt}(G_1 \star G_2) = \begin{cases} n_1\gamma_w(G_2) & , \textit{if } \gamma_w(G_2) = \gamma_{wt}(G_2) \\ \gamma_t(G_1) + n_1\gamma_w(G_2) & , \textit{otherwise} \end{cases}$$
$$\gamma_{st}(G_1 \star G_2) = \gamma_t(G_1).$$

Proof. After neighborhood corona operation, degree of vertices in G_1 are greater than or equal to degree of vertices of G_2. If $\gamma_w(G_2) = \gamma_{wt}(G_2)$ then each copy of G_2 are weakly total dominated by $\gamma_w - set$ of G_2. Thus, in this case $\gamma_{wt}(G_1 \star G_2) = n_1\gamma_w(G_2)$. For the situation $\gamma_w(G_2) \neq \gamma_{wt}(G_2)$, $\gamma_w - set$ of G_2 is not enough for totality. In order to obtain a weak total domination set of $G_1 \star G_2$, there are two options. In the first case, $\gamma_{wt} - set$ of G_2 can be chosen for each copy of G_2. Then $\gamma_{wt}(G_1 \star G_2) = n_1\gamma_{wt}(G_2)$. For the other case, to obtain weak total domination set, vertices from G_1 will be added to each $\gamma_w - set$ of G_2. According to form of neighborhood corona graph, a vertex from G_1 adjacent to neighbors and theirs corresponding copy of G_2 but not adjacent to corresponding G_2 copy of itself. Thus, it is enough to choose $\gamma_t - set$ of G_1 to satisfy totality of $G_1 \star G_2$. Then, in this case $\gamma_{wt}(G_1 \star G_2) = \gamma_t(G_1) + n_1\gamma_w(G_2)$. If we compare these two cases, $n_1\gamma_{wt}(G_2)$ is always greater than $\gamma_t(G_1)+n_1\gamma_w(G_2)$. Hence, $\gamma_{wt}(G_1 \star G_2) = \gamma_t(G_1)+n_1\gamma_w(G_2)$ if $\gamma_w(G_2) \neq \gamma_{wt}(G_2)$.

According to form of neighborhood corona graph, the vertices in $\gamma_t - set$ of G_1 strongly total dominates $G_1 \star G_2$. □

Theorem 2.6. *Let G_1 has n_1 vertices q_1 edges and G_2 has n_2 vertices q_2 edges then*

$$\gamma_{wt}(G_1 \odot G_2) = \begin{cases} n_1\gamma_w(G_2) + q_1 + \alpha(G_1) & , \textit{ if } \gamma_w(G_2) = \gamma_{wt}(G_2) \\ n_1(\gamma_w(G_2) + 1) + q_1 & , \textit{ otherwise} \end{cases}$$
$$\gamma_{st}(G_1 \odot G_2) = n_1 + \alpha'(G_1).$$

where α' is edge covering number and α is vertex covering number.

Proof. After subdivision vertex corona operation degree of vertices in G_1 are greater than or equal to degree of vertices of G_2 and also $G_1 \odot G_2$ graph has subdivision vertices that has degree 2. Then, subdivision vertices should be contained by $\gamma_{wt} - set$ of $G_1 \odot G_2$. Let $\gamma_w(G_2) = \gamma_{wt}(G_2)$, if $\gamma_w - set$ of G_2 is chosen from each copy with all subdivision vertices then $\gamma_w - set$ of $G_1 \odot G_2$ can be constructed. For totality of disjoint subdivision vertices, it is needed vertices from G_1 that adjacent to subdivision vertices which is on each the edges of G_1. That corresponds covering number of G_1. Thus, $\gamma_{wt}(G_1 \odot G_2) = n_1\gamma_w(G_2)+q_1+\alpha(G_1)$, if $\gamma_w(G_2) = \gamma_{wt}(G_2)$. Let $\gamma_w(G_2) \neq \gamma_{wt}(G_2)$, each $\gamma_w - set$ of G_2 and all subdivision vertices $\gamma_w - set$ of $G_1 \odot G_2$. For totality, it is needed to at least one vertex for each $G_2 - copy$. Hence, $\gamma_{wt}(G_1 \odot G_2) = n_1(\gamma_w(G_2) + 1) + q_1$ in this case.

For strong total domination, all vertices of G_1 must be included in $\gamma_{st} - set$ of $G_1 \odot G_2$. For totality, it is needed to smallest possible number of subdivision vertices that adjacent to vertex of G_1. Subdivision vertices are placed on edges of G_1. Thus, this number corresponds edge covering number of G_1. Also, totality can be provided from each G_2 copies with n_1 vertices. However, covering number less number of vertices n_1. Hence, $\gamma_{st}(G_1 \odot G_2) = n_1 + \alpha'(G_1)$. □

Theorem 2.7. *Let G_1 has n_1 vertices q_1 edges and G_2 has n_2 vertices q_2 edges and* $\Delta(G_1) < n_2 + 2$ *then*

$$\gamma_{wt}(G_1 \ominus G_2) = \begin{cases} q_1\gamma_w(G_2) + n_1 + \alpha'(G_1) & , \textit{if } \gamma_w(G_2) = \gamma_{wt}(G_2) \\ q_1(\gamma_w(G_2) + 1) + n_1 & , \textit{ otherwise} \end{cases}$$

$$\gamma_{st}(G_1 \ominus G_2) = q_1 + \alpha(G_1).$$

where α' is edge covering number and α is vertex covering number.

Proof. Let S be $\gamma_{wt} - set$ of $G_1 \ominus G_2$. Let $v_i \in V(G_1),\ 1 \leq i \leq n_1$, $z_j \in V(G_2),\ 1 \leq j \leq n_2$ and $u_k \in I(G_1), 1 \leq k \leq q_1$ where $I(G_1)$ is the set of inserted new vertices to $S(G_1)$. From the assumption $n_2 + 2 > \Delta(G_1)$ that $\deg_{G_1 \ominus G_2}(v_i) < \deg_{G_1 \ominus G_2}(u_k)$. This requires that all vertices in $V(G_1)$ should be included by S. According to subdivision edge corona construction, $\deg_{G_1 \ominus G_2}(z_j) < \deg_{G_1 \ominus G_2}(u_k)$. Therefore, $\gamma_w - set$ of G_2, denoted by S_1, must be contained by S. Thus, $V(G_1) \cup S_1 \subseteq S$.

If $\gamma_w(G_2) = \gamma_{wt}(G_2)$ then totality of vertices in S_1 satisfies. Moreover, totality of $V(G_1)$ can be satisfied by some subdivision vertices. Number of the subdivision vertices that requires totality of $V(G_1)$ corresponds edge covering number of G_1, $\alpha'(G_1)$. Hence, $\gamma_{wt}(G_1 \ominus G_2) = q_1\gamma_w(G_2) + n_1 + \alpha'(G_1)$.

If $\gamma_w(G_2) \neq \gamma_{wt}(G_2)$ then totality of S_1 are provided by all subdivision vertices in the set of $I(G_1)$. Thus, totality of $V(G_1)$ also satisfied by $I(G_1)$. Hence, $\gamma_{wt}(G_1 \ominus G_2) = q_1(\gamma_w(G_2) + 1) + n_1$.

Let S' be $\gamma_{st} - set$ of $G_1 \ominus G_2$, subdivision vertices strongly dominates all vertices of $G_1 \ominus G_2$. Totality can be satisfy by vertices of G_1 or G_2. If some vertices are chosen from G_1, according to the definition of vertex covering number, there are at least $\alpha(G_1)$ vertices that are adjacent to the new subdivision vertices. If some vertices are chosen from G_2, it is needed to add q_1 vertices to S'. Then, $q_1 + \alpha(G_1) < 2q_1$. Hence, $\gamma_{st}(G_1 \ominus G_2) = q_1 + \alpha(G_1)$. □

Theorem 2.8. *Let G_1 has n_1 vertices q_1 edges and G_2 has n_2 vertices q_2 edges and let t be the number of $u \in V(G_1)$ such as* $\deg_{G_1 \ominus G_2}(u) \geq n_2 + 2$. *Then*

$$\gamma_{wt}(G_1 \ominus G_2) = \begin{cases} q_1\gamma_w(G_2) + n_1 + \alpha'(G_1) - t & , \textit{ if } \gamma_w(G_2) = \gamma_{wt}(G_2) \\ q_1(\gamma_w(G_2) + 1) + n_1 - t & , \textit{otherwise} \end{cases}$$

$$q_1 + \alpha(G_1) \leq \gamma_{st}(G_1 \ominus G_2) \leq q_1 + n_1.$$

where α' is edge covering number and α is vertex covering number.

Proof. Proof can be done as proof of Theorem 2.7. Let S be $\gamma_{wt} - set$ of $G_1 \ominus G_2$. However, for every $u \in V(G_1)$ that satisfy $\deg_{G_1 \ominus G_2}(u) \geq n_2 + 2$ condition is not included by S. These vertices also weakly dominated by the vertices putting on edge covering set of G_1 as a subdivision vertices. In order to obtain $\gamma_{wt} - set$ of $G_1 \ominus G_2$, the vertices which satisfy condition should not be in the S. Hence,

$$\gamma_{wt}(G_1 \ominus G_2) = \begin{cases} q_1\gamma_w(G_2) + n_1 + \alpha'(G_1) - t & , \text{ if } \gamma_w(G_2) = \gamma_{wt}(G_2) \\ q_1(\gamma_w(G_2) + 1) + n_1 - t & , \text{ otherwise} \end{cases}$$

is obtained.

Let S' be $\gamma_{st} - set$ of $G_1 \ominus G_2$. From the proof of previous theorem, the set of vertices which contains subdivision vertices and vertex covering set of G_1 may be a $\gamma_{st} - set$ of $G_1 \ominus G_2$. However, if there are more vertices than covering number of G_1 whose degree grater than $n_2 + 2$ then $q_1 + \alpha(G_1) \leq |S'|$. Therefore, lower bound can be said. A set which contains all subdivision vertices and all vertices of G_1

is always a strong total dominating set for $G_1 \ominus G_2$. Thus, upper bound can be satisfied for strong total domination case. □

3. Characterization Under Corona Operations

It is well known that totality property increases or at least remains the same the cardinality of a domination parameters. The goal of this section to answer the question that what is the difference between weak total domination number and weak domination number after corona operations (same question will be think for strong version). In other words, the characterizations $\gamma_{wt}(G) = \gamma_w(G) + k$ and $\gamma_{st}(G) = \gamma_s(G) + k$ such that $0 \leq k \leq \gamma_w(G)$ (or $\gamma_s(G)$) will be done under corona operations and we are going to determine the value of "k" in following theorems;

3.1. Characterization for Weak Total Domination.

Theorem 3.1. *Let G_1 and G_2 be n_1 and n_2 ordered graphs, respectively. Let $\gamma_w(G_2) \neq \gamma_{wt}(G_2)$, then $\gamma_{wt}(G_1 \circ G_2) = \gamma_w(G_1 \circ G_2) + k$ where $k = n_1$.*

Proof. Using Theorem 2.1 for the case $\gamma_w(G_2) \neq \gamma_{wt}(G_2)$; $\gamma_{wt}(G_1 \circ G_2) = n_1\gamma_w(G_2) + n_1 = \gamma_w(G_1 \circ G_2) + n_1$. Hence, $k = n_1$. □

Theorem 3.2. *Let G_1 has n_1 vertices q_1 edges and G_2 has n_2 vertices q_2 edges. Let $\gamma_w(G_2) \neq \gamma_{wt}(G_2)$, then $\gamma_{wt}(G_1 \diamond G_2) = \gamma_w(G_1 \diamond G_2) + k$ where $k = \alpha(G_1)$.*

Proof. Using Theorem 2.2 for the case $\gamma_w(G_2) \neq \gamma_{wt}(G_2)$; $\gamma_{wt}(G_1 \diamond G_2) = q_1\gamma_w(G_2) + \alpha(G_1) = \gamma_w(G_1 \diamond G_2) + \alpha(G_1)$ then $k = \alpha(G_1)$. □

Theorem 3.3. *Let G_1 and G_2 are n_1 and n_2 ordered graphs, respectively. Let $\gamma_w(G_2) \neq \gamma_{wt}(G_2)$, then $\gamma_{wt}(G_1 \star G_2) = \gamma_w(G_1 \star G_2) + k$ where $k = \gamma_t(G_1)$.*

Proof. Using Theorem 2.5 for the case $\gamma_w(G_2) \neq \gamma_{wt}(G_2)$; $\gamma_{wt}(G_1 \star G_2) = \gamma_t(G_1) + n_1\gamma_w(G_2) = \gamma_t(G_1) + \gamma_w(G_1 \star G_2)$, then $k = \gamma_t(G_1)$. □

Theorem 3.4. *Let G_1 has n_1 vertices q_1 edges and G_2 has n_2 vertices q_2 edges. Let $\gamma_w(G_2) \neq \gamma_{wt}(G_2)$, then $\gamma_{wt}(G_1 \odot G_2) = \gamma_w(G_1 \odot G_2) + k$ where $k = n_1$.*

Proof. Using Theorem 2.6 for the case $\gamma_w(G_2) \neq \gamma_{wt}(G_2)$; $\gamma_{wt}(G_1 \odot G_2) = n_1(\gamma_w(G_2) + 1) + q_1 = \gamma_w(G_1 \odot G_2) + n_1$, then $k = n_1$. □

Theorem 3.5. *Let G_1 has n_1 vertices q_1 edges and G_2 has n_2 vertices q_2 edges. Let $\gamma_w(G_2) \neq \gamma_{wt}(G_2)$, then $\gamma_{wt}(G_1 \ominus G_2) = \gamma_w(G_1 \ominus G_2) + k$ where $k \leq q_1$.*

Proof. Let $\Delta(G_1) < n_2 + 2$. According to Theorem $\gamma_{wt}(G_1 \ominus G_2) = q_1(\gamma_w(G_2) + 1) + n_1$ for the case $\gamma_w(G_2) \neq \gamma_{wt}(G_2)$. Also, $\gamma_{wt}(G_1 \ominus G_2) = q_1\gamma_w(G_2) + n_1$. Therefore, $k = q_1$. Let $\Delta(G_1) \geq n_2 + 2$ and t be the number of $u \in V(G_1)$ such as $\deg_{G_1 \ominus G_2}(u) \geq n_2 + 2$. According to Theorem $\gamma_{wt}(G_1 \ominus G_2) = q_1(\gamma_w(G_2) + 1) + n_1 - t$ for the case $\gamma_w(G_2) \neq \gamma_{wt}(G_2)$. Also, let S be the $\gamma_w - set$ of $G_1 \ominus G_2$. For the each copy of G_2, $\gamma_w - set$ of G_2 must be contained by S and there are t vertices of G_1 whose degree greater than degree of subdivision vertices. Then remaining vertices of G_1 must be contained by S. In addition, large graded t vertices are weakly dominated by some subdivision vertices. Thus, $\gamma_w(G_1 \ominus G_2) = q_1\gamma_w(G_2) + n_1 - t + x$ where $x \leq q_1$ and it can be said that difference between weak total domination number and weak domination number less than q_1. Hence, $\gamma_{wt}(G_1 \ominus G_2) = \gamma_w(G_1 \ominus G_2) + k$ where $k \leq q_1$. □

3.2. **Characterization for Strong Total Domination.**

Theorem 3.6. *Let G_1 and G_2 be n_1 and n_2 ordered graphs, respectively.* $\gamma_{st}(G_1 \circ G_2) = \gamma_s(G_1 \circ G_2) + k$ *where* $k = 0$.

Proof. Using Theorem 2.1 $\gamma_{st}(G_1 \circ G_2) = n_1$. According to form of $G_1 \circ G_2$, it is easy to see that $\gamma_s(G_1 \circ G_2) = n_1$. Hence, $k = 0$. □

Theorem 3.7. *Let G_1 has n_1 vertices q_1 edges and G_2 has n_2 vertices q_2 edges.* $\gamma_{st}(G_1 \diamond G_2) = \gamma_s(G_1 \diamond G_2) + k$ *where* $0 \leq k \leq \alpha(G_1) - 1$.

Proof. Using Theorem 2.2 $\gamma_{st}(G_1 \diamond G_2) = tvc(G_1) = \gamma_s(G_1 \diamond G_2) + k = \alpha(G_1) + k$ then $k = tvc(G_1) - \alpha(G_1)$. According to Proposition 2.4, $0 \leq k \leq \alpha(G_1) - 1$ is obtained. □

Theorem 3.8. *Let G_1 and G_2 are n_1 and n_2 ordered graphs, respectively.* $\gamma_{st}(G_1 \star G_2) = \gamma_s(G_1 \star G_2) + k$ *where* $k = 0$.

Proof. Using Theorem 2.5, $\gamma_{st}(G_1 \star G_2) = \gamma_t(G_1)$. According to the definition of neighborhood corona operation, $\gamma_s(G_1 \star G_2) = \gamma_t(G_1)$. Hence, $k = 0$. □

Theorem 3.9. *Let G_1 has n_1 vertices q_1 edges and G_2 has n_2 vertices q_2 edges.* $\gamma_{st}(G_1 \odot G_2) = \gamma_s(G_1 \odot G_2) + k$ *where* $k = \alpha'(G_1)$.

Proof. Using Theorem 2.6 $\gamma_{st}(G_1 \odot G_2) = n_1 + \alpha'(G_1)$ and also according to form of $G_1 \odot G_2$, $\gamma_s(G_1 \odot G_2) = n_1$. Hence, $k = \alpha'(G_1)$. □

Theorem 3.10. *Let G_1 has n_1 vertices q_1 edges and G_2 has n_2 vertices q_2 edges.* $\gamma_{st}(G_1 \ominus G_2) = \gamma_s(G_1 \ominus G_2) + k$ *where* $0 \leq k \leq \alpha(G_1)$.

Proof. Using Theorem 2.7 and Theorem 2.8 the characterization under subdivision edge corona operation can be investigated in three cases;

Case 1: In the situation that $\Delta(G_1) < n_2 + 2$; $\gamma_{st}(G_1 \ominus G_2) = q_1 + \alpha(G_1)$ and $\gamma_s(G_1 \ominus G_2) = q_1$. Thus, $k = \alpha(G_1)$.

If $\Delta(G_1) \geq n_2 + 2$ then $q_1 + \alpha(G_1) \leq \gamma_{st}(G_1 \ominus G_2) \leq q_1 + n_1$. Characterization can be done in two cases;

Case 2: Let $q_1 + \alpha(G_1) < \gamma_{st}(G_1 \ominus G_2) < q_1 + n_1$. This corresponds to the situation that there are some $u \in V(G_1)$ such as $\deg_{G_1 \ominus G_2}(u) \geq n_2 + 2$. Let t be the number of vertices that degree of them greater and equal than $n_2 + 2$ provided that $t < n_1$. Let S be $\gamma_{st} - set$ of $G_1 \ominus G_2$. S must contain all subdivision vertices and the all vertices $u \in V(G_1)$ such as $\deg_{G_1 \ominus G_2}(u) \geq n_2 + 2$. In addition, these vertices construct $\gamma_s - set$ of $G_1 \ominus G_2$. Thus, $\gamma_s(G_1 \ominus G_2) = q_1 + t$. For totality, S should include some of the vertices of covering set of G_1. Therefore, $\gamma_{st}(G_1 \ominus G_2) = q_1 + t + k$ where $k < \alpha(G_1)$ and . Hence, the difference between strong total domination and strong domination numbers of $G_1 \ominus G_2$ less than $\alpha(G_1)$.

Case 3: Let $\gamma_{st}(G_1 \ominus G_2) = q_1 + n_1$. This means, the all vertices $u \in V(G_1)$ such as $\deg_{G_1 \ominus G_2}(u) \geq n_2 + 2$. Therefore, $\gamma_s(G_1 \ominus G_2) = q_1 + n_1$. Thus, $k = 0$.

According to the three cases, $\gamma_{st}(G_1 \ominus G_2) = \gamma_s(G_1 \ominus G_2) + k$ where $0 \leq k \leq \alpha(G_1)$. □

4. Conclusion

Domination number and its varieties are one of the most important concept in graph theory. Many parameters of domination are studied frequently and many results have been obtained by many authors. Especially, its association with some important graph operations is common. In this paper, we have discussed strong and weak total domination parameters and the effect of some corona operations on the parameters. Also, both $\gamma_{wt}(G) = \gamma_w(G) + k$ and $\gamma_{st}(G) = \gamma_s(G) + k$, $(0 \leq k \leq \gamma_{wt}(G)$ or $\gamma_{st}(G))$ characterizations are taken consideration for discussed corona operations which is pointed as an open problem only for weak total domination number in [6].

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MIXED CAPUTO ψ-FRACTIONAL OSTROWSKI TYPE INEQUALITIES

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Abstract. Very general univariate mixed Caputo ψ-fractional Ostrowski type inequalities are presented. Estimates are with respect to $\|\cdot\|_p$, $1 \leq p \leq \infty$. We give also applications.

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1. Introduction

In 1938, A. Ostrowski [4] proved the following important inequality.

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] \cdot (b-a) \|f'\|_\infty , \tag{1}$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

Since then there has been a lot of activity around these inequalities with important applications to numerical analysis and probability.

In this article we are greatly motived and inspired by Theorem 1.1, see also [2]. Here we present various ψ-fractional Ostrowski type inequalities and we give interesting applications.

2. Background

Here we follow [1].

Let $\alpha > 0$, $[a,b] \subset \mathbb{R}$, $f : [a,b] \to \mathbb{R}$ which is integrable and $\psi \in C^1([a,b])$ an increasing function such that $\psi'(x) \neq 0$, for all $x \in [a,b]$. Consider $n = \lceil \alpha \rceil$, the ceiling of α. The left and right fractional integrals are defined, respectively, as follows:

$$I_{a+}^{\alpha;\psi} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} f(t)\,dt, \tag{2}$$

and

$$I_{b-}^{\alpha;\psi} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t) (\psi(t) - \psi(x))^{\alpha-1} f(t)\,dt, \tag{3}$$

for any $x \in [a,b]$, where Γ is the gamma function.

The following semigroup property is valid for fractional integrals: if $\alpha, \beta > 0$, then

$$I_{a+}^{\alpha,\psi} I_{a+}^{\beta,\psi} f(x) = I_{a+}^{\alpha+\beta,\psi} f(x), \text{ and } I_{b-}^{\alpha,\psi} I_{b-}^{\beta,\psi} f(x) = I_{b-}^{\alpha+\beta,\psi} f(x).$$

We mention

Definition 2.1. *([1]) Let $\alpha > 0$, $n \in \mathbb{N}$ such that $n = \lceil \alpha \rceil$, $[a,b] \subset \mathbb{R}$ and $f, \psi \in C^n([a,b])$ with ψ being increasing and $\psi'(x) \neq 0$, for all $x \in [a,b]$. The left ψ-Caputo fractional derivative of f of order α is given by*

$$(4) \qquad {}^C D_{a+}^{\alpha,\psi} f(x) := I_{a+}^{n-\alpha,\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n f(x),$$

and the right ψ-Caputo fractional derivative of f is given by

$$(5) \qquad {}^C D_{b-}^{\alpha,\psi} f(x) := I_{b-}^{n-\alpha,\psi} \left(-\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n f(x).$$

To simplify notation, we will use the symbol

$$(6) \qquad f_{\psi}^{[n]}(x) := \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n f(x),$$

with $f_{\psi}^{[0]}(x) = f(x)$.

By the definition, when $\alpha = m \in \mathbb{N}$, we have

$$(7) \qquad \begin{array}{l} {}^C D_{a+}^{\alpha,\psi} f(x) = f_{\psi}^{[m]}(x) \\ \text{and} \\ {}^C D_{b-}^{\alpha,\psi} f(x) = (-1)^m f_{\psi}^{[m]}(x). \end{array}$$

If $\alpha \notin \mathbb{N}$, we have

$$(8) \qquad {}^C D_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{n-\alpha-1} f_{\psi}^{[n]}(t)\, dt,$$

and

$$(9) \qquad {}^C D_{b-}^{\alpha,\psi} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \psi'(t) (\psi(t) - \psi(x))^{n-\alpha-1} f_{\psi}^{[n]}(t)\, dt,$$

$\forall\, x \in [a,b]$.

In particular, when $\alpha \in (0,1)$, we have

$$(10) \qquad \begin{array}{l} {}^C D_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (\psi(x) - \psi(t))^{-\alpha} f'(t)\, dt, \\ \text{and} \\ {}^C D_{b-}^{\alpha,\psi} f(x) = \frac{-1}{\Gamma(1-\alpha)} \int_x^b (\psi(t) - \psi(x))^{-\alpha} f'(t)\, dt \end{array}$$

$\forall\, x \in [a,b]$.

Clearly the above is a generalization of left and right Caputo fractional derivatives.

For more see [1].

Still we need from [1] the following left and right fractional Taylor's formulae:

Theorem 2.2. *([1]) Let $\alpha > 0$, $n \in \mathbb{N}$ such that $n = \lceil \alpha \rceil$, $[a,b] \subset \mathbb{R}$ and $f, \psi \in C^n([a,b])$ with ψ being increasing and $\psi'(x) \neq 0$, for all $x \in [a,b]$. Then, the left fractional Taylor formula follows,*

$$(11) \qquad f(x) = \sum_{k=0}^{n-1} \frac{f_{\psi}^{[k]}(a)}{k!} (\psi(x) - \psi(a))^k + I_{a+}^{\alpha,\psi}\, {}^C D_{a+}^{\alpha,\psi} f(x),$$

and the right fractional Taylor formula follows,

$$f(x)=\sum_{k=0}^{n-1}(-1)^k\frac{f_{\psi}^{[k]}(b)}{k!}(\psi(b)-\psi(x))^k+I_{b-}^{\alpha,\psi}\ {}^{C}D_{b-}^{\alpha,\psi}f(x), \tag{12}$$

$\forall\ x\in[a,b]$.

In particular, given $\alpha\in(0,1)$, *we have*

$$\begin{aligned}&f(x)=f(a)+I_{a+}^{\alpha,\psi}\ {}^{C}D_{a+}^{\alpha,\psi}f(x),\\&\textit{and}\\&f(x)=f(b)+I_{b-}^{\alpha,\psi}\ {}^{C}D_{b-}^{\alpha,\psi}f(x),\end{aligned} \tag{13}$$

$\forall\ x\in[a,b]$.

Remark 2.3. *For convenience we can rewrite (11)-(13) as follows:*

$$f(x)=\sum_{k=0}^{n-1}\frac{f_{\psi}^{[k]}(a)}{k!}(\psi(x)-\psi(a))^k+ \tag{14}$$

$$\frac{1}{\Gamma(\alpha)}\int_a^x\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\ {}^{C}D_{a+}^{\alpha,\psi}f(t)\,dt,$$

and

$$f(x)=\sum_{k=0}^{n-1}\frac{(-1)^k f_{\psi}^{[k]}(b)}{k!}(\psi(b)-\psi(x))^k+ \tag{15}$$

$$\frac{1}{\Gamma(\alpha)}\int_x^b\psi'(t)(\psi(t)-\psi(x))^{\alpha-1}\ {}^{C}D_{b-}^{\alpha,\psi}f(t)\,dt,$$

$\forall\ x\in[a,b]$.

When $\alpha\in(0,1)$, *we get:*

$$\begin{aligned}&f(x)=f(a)+\tfrac{1}{\Gamma(\alpha)}\textstyle\int_a^x\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\ {}^{C}D_{a+}^{\alpha,\psi}f(t)\,dt,\\&\textit{and}\\&f(x)=f(b)+\tfrac{1}{\Gamma(\alpha)}\textstyle\int_x^b\psi'(t)(\psi(t)-\psi(x))^{\alpha-1}\ {}^{C}D_{b-}^{\alpha,\psi}f(t)\,dt,\end{aligned} \tag{16}$$

$\forall\ x\in[a,b]$.

Again from [1] we have the following:

Consider the norms $\|\cdot\|_{\infty}:C([a,b])\to\mathbb{R}$ and $\|\cdot\|_{C_{\psi}^{[n]}}:C^n([a,b])\to\mathbb{R}$, where $\|f\|_{C_{\psi}^{[n]}}:=\sum_{k=0}^{n}\left\|f_{\psi}^{[k]}\right\|_{\infty}$.

We have

Theorem 2.4. ([1]) *The ψ-Caputo fractional derivatives are bounded operators. For all* $\alpha>0$ $(n=\lceil\alpha\rceil)$

$$\left\|{}^{C}D_{a+}^{\alpha,\psi}\right\|_{\infty}\leq K\|f\|_{C_{\psi}^{[n]}} \tag{17}$$

and

$$\left\|{}^{C}D_{b-}^{\alpha,\psi}\right\|_{\infty}\leq K\|f\|_{C_{\psi}^{[n]}}, \tag{18}$$

where

$$K=\frac{(\psi(b)-\psi(a))^{n-\alpha}}{\Gamma(n+1-\alpha)}>0. \tag{19}$$

3. Main Results

We present the following ψ-fractional Ostrowski type inequalities:

Theorem 3.1. *Let $\alpha > 0$, $n \in \mathbb{N} : n = \lceil \alpha \rceil$, $[a,b] \subset \mathbb{R}$ and $f, \psi \in C^n([a,b])$ with ψ being increasing and $\psi'(x) \neq 0$, for all $x \in [a,b]$. Let $x_0 \in [a,b]$ and assume that $f_{\psi}^{[k]}(x_0) = 0$, for $k = 1, ..., n-1$. Then*

$$\left| \frac{1}{(\psi(b) - \psi(a))} \int_a^b f(x)\psi'(x)\,dx - f(x_0) \right| \leq \tag{20}$$

$$\frac{1}{(\psi(b)-\psi(a))\Gamma(\alpha+2)} \left\{ (\psi(x_0) - \psi(a))^{\alpha+1} \left\| {}^C D_{x_0-}^{\alpha,\psi} f \right\|_{\infty,[a,x_0]} + (\psi(b) - \psi(x_0))^{\alpha+1} \left\| {}^C D_{x_0+}^{\alpha,\psi} f \right\|_{\infty,[x_0,b]} \right\} \leq$$

$$\frac{1}{\Gamma(\alpha+2)} \max \left\{ \left\| {}^C D_{x_0-}^{\alpha,\psi} f \right\|_{\infty,[a,x_0]}, \left\| {}^C D_{x_0+}^{\alpha,\psi} f \right\|_{\infty,[x_0,b]} \right\} (\psi(b) - \psi(a))^{\alpha}. \tag{21}$$

In case of $0 < \alpha \leq 1$, (20)-(21) are still valid without any initial conditions.

Proof. By Theorem 2.2 we have that

$$f(x) - f(x_0) = \frac{1}{\Gamma(\alpha)} \int_{x_0}^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} \; {}^C D_{x_0+}^{\alpha,\psi} f(t)\,dt, \tag{22}$$

$\forall\, x \in [x_0, b]$,
and

$$f(x) - f(x_0) = \frac{1}{\Gamma(\alpha)} \int_x^{x_0} \psi'(t)(\psi(t) - \psi(x))^{\alpha-1} \; {}^C D_{x_0-}^{\alpha,\psi} f(t)\,dt, \tag{23}$$

$\forall\, x \in [a, x_0]$.

Hence

$$|f(x) - f(x_0)| \leq \frac{1}{\Gamma(\alpha)} \int_{x_0}^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} \left| {}^C D_{x_0+}^{\alpha,\psi} f(t) \right| dt \leq$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_{x_0}^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1}\,dt \right) \left\| {}^C D_{x_0+}^{\alpha,\psi} f \right\|_{\infty,[x_0,b]} =$$

$$\frac{\left\| {}^C D_{x_0+}^{\alpha,\psi} f \right\|_{\infty,[x_0,b]}}{\Gamma(\alpha+1)} (\psi(x) - \psi(x_0))^{\alpha}. \tag{24}$$

That is

$$|f(x) - f(x_0)| \leq \frac{\left\| {}^C D_{x_0+}^{\alpha,\psi} f \right\|_{\infty,[x_0,b]}}{\Gamma(\alpha+1)} (\psi(x) - \psi(x_0))^{\alpha}, \tag{25}$$

$\forall\, x \in [x_0, b]$.

Similarly, it holds

$$|f(x) - f(x_0)| \leq \frac{1}{\Gamma(\alpha)} \int_x^{x_0} \psi'(t)(\psi(t) - \psi(x))^{\alpha-1} \left| {}^C D_{x_0-}^{\alpha,\psi} f(t) \right| dt \leq$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_x^{x_0} \psi'(t)(\psi(t) - \psi(x))^{\alpha-1}\,dt \right) \left\| {}^C D_{x_0-}^{\alpha,\psi} f \right\|_{\infty,[a,x_0]} =$$

$$\frac{\left\| {}^C D_{x_0-}^{\alpha,\psi} f \right\|_{\infty,[a,x_0]}}{\Gamma(\alpha+1)} (\psi(x_0) - \psi(x))^{\alpha}.$$

That is

$$|f(x)-f(x_0)| \leq \frac{\left\| {}^{C}D^{\alpha,\psi}_{x_0-}f \right\|_{\infty,[a,x_0]}}{\Gamma(\alpha+1)} (\psi(x_0)-\psi(x))^{\alpha}, \tag{26}$$

$\forall\, x \in [a, x_0]$.

We observe that

$$\left| \frac{1}{(\psi(b)-\psi(a))} \int_a^b f(x)\psi'(x)\,dx - f(x_0) \right| =$$

$$\left| \frac{1}{(\psi(b)-\psi(a))} \left[\int_a^b f(x)\psi'(x)\,dx - f(x_0)(\psi(b)-\psi(a)) \right] \right| =$$

$$\left| \frac{1}{(\psi(b)-\psi(a))} \left[\int_a^b f(x)\psi'(x)\,dx - \int_a^b f(x_0)\psi'(x)\,dx \right] \right| =$$

$$\left| \frac{1}{(\psi(b)-\psi(a))} \left[\int_a^b (f(x)-f(x_0))\psi'(x)\,dx \right] \right| \leq$$

$$\frac{1}{(\psi(b)-\psi(a))} \int_a^b |f(x)-f(x_0)|\psi'(x)\,dx = \tag{27}$$

$$\frac{1}{(\psi(b)-\psi(a))} \left\{ \int_a^{x_0} |f(x)-f(x_0)|\psi'(x)\,dx + \int_{x_0}^b |f(x)-f(x_0)|\psi'(x)\,dx \right\}$$

$$\overset{\text{(by (25), (26))}}{\leq}$$

$$\frac{1}{(\psi(b)-\psi(a))\Gamma(\alpha+1)} \left\{ \left(\int_a^{x_0} (\psi(x_0)-\psi(x))^{\alpha}\psi'(x)\,dx \right) \left\| {}^{C}D^{\alpha,\psi}_{x_0-}f \right\|_{\infty,[a,x_0]} \right.$$

$$\left. + \left(\int_{x_0}^b (\psi(x)-\psi(x_0))^{\alpha}\psi'(x)\,dx \right) \left\| {}^{C}D^{\alpha,\psi}_{x_0+}f \right\|_{\infty,[x_0,b]} \right\} = \tag{28}$$

$$\frac{1}{(\psi(b)-\psi(a))\Gamma(\alpha+2)} \left\{ \left\| {}^{C}D^{\alpha,\psi}_{x_0-}f \right\|_{\infty,[a,x_0]} (\psi(x_0)-\psi(a))^{\alpha+1} \right.$$

$$\left. + \left\| {}^{C}D^{\alpha,\psi}_{x_0+}f \right\|_{\infty,[x_0,b]} (\psi(b)-\psi(x_0))^{\alpha+1} \right\} \leq$$

$$\frac{1}{\Gamma(\alpha+2)} \max\left\{ \left\| {}^{C}D^{\alpha,\psi}_{x_0-}f \right\|_{\infty,[a,x_0]}, \left\| {}^{C}D^{\alpha,\psi}_{x_0+}f \right\|_{\infty,[x_0,b]} \right\} (\psi(b)-\psi(a))^{\alpha}. \tag{29}$$

□

We make

Remark 3.2. *In our setting, clearly, it is* $f^{[n]}_{\psi} \in C([a,b])$. *Given* $f \in C([a,b])$, *by Theorem 4.10, p. 98 of* [3], *we get that* $I^{\alpha,\psi}_{a+}f \in C([a,b])$, *and by Theorem 4.11, p. 101 of* [3], *we get that* $I^{\alpha,\psi}_{b-}f \in C([a,b])$. *Therefore, we obtain that* ${}^{C}D^{\alpha,\psi}_{a+}f$, ${}^{C}D^{\alpha,\psi}_{b-}f \in C([a,b])$.

We continue with

Theorem 3.3. *All as in Theorem 3.1 and* $\alpha \geq 1$. *Then*

$$\left| \frac{1}{(\psi(b) - \psi(a))} \int_a^b f(x)\, \psi'(x)\, dx - f(x_0) \right| \leq \tag{30}$$

$$\frac{1}{(\psi(b) - \psi(a))\, \Gamma(\alpha+1)} \left\{ (\psi(x_0) - \psi(a))^{\alpha} \left\| {}^C D_{x_0-}^{\alpha,\psi} f \right\|_{L_1([a,x_0],\psi)} \right.$$

$$\left. + (\psi(b) - \psi(x_0))^{\alpha} \left\| {}^C D_{x_0+}^{\alpha,\psi} f \right\|_{L_1([x_0,b],\psi)} \right\} \leq$$

$$\frac{1}{\Gamma(\alpha+1)} \max \left\{ \left\| {}^C D_{x_0-}^{\alpha,\psi} f \right\|_{L_1([a,x_0],\psi)}, \left\| {}^C D_{x_0+}^{\alpha,\psi} f \right\|_{L_1([x_0,b],\psi)} \right\} (\psi(b) - \psi(a))^{\alpha-1}. \tag{31}$$

Proof. By (22) we obtain:

$$|f(x) - f(x_0)| \leq \frac{(\psi(x) - \psi(x_0))^{\alpha-1}}{\Gamma(\alpha)} \int_{x_0}^{x} \left| {}^C D_{x_0+}^{\alpha,\psi} f(t) \right| d\psi(t) \leq$$

$$\frac{(\psi(x) - \psi(x_0))^{\alpha-1}}{\Gamma(\alpha)} \int_{x_0}^{b} \left| {}^C D_{x_0+}^{\alpha,\psi} f(t) \right| d\psi(t) =$$

$$\frac{(\psi(x) - \psi(x_0))^{\alpha-1}}{\Gamma(\alpha)} \left\| {}^C D_{x_0+}^{\alpha,\psi} f \right\|_{L_1([x_0,b],\psi)}. \tag{32}$$

That is, we get

$$|f(x) - f(x_0)| \leq \frac{(\psi(x) - \psi(x_0))^{\alpha-1}}{\Gamma(\alpha)} \left\| {}^C D_{x_0+}^{\alpha,\psi} f \right\|_{L_1([x_0,b],\psi)}, \tag{33}$$

$\forall\, x \in [x_0, b]$.

Similarly, by (23), we get:

$$|f(x) - f(x_0)| \leq \frac{(\psi(x_0) - \psi(x))^{\alpha-1}}{\Gamma(\alpha)} \int_{x}^{x_0} \left| {}^C D_{x_0-}^{\alpha,\psi} f(t) \right| d\psi(t) \leq$$

$$\frac{(\psi(x_0) - \psi(x))^{\alpha-1}}{\Gamma(\alpha)} \int_{a}^{x_0} \left| {}^C D_{x_0-}^{\alpha,\psi} f(t) \right| d\psi(t) = \tag{34}$$

$$\frac{(\psi(x_0) - \psi(x))^{\alpha-1}}{\Gamma(\alpha)} \left\| {}^C D_{x_0-}^{\alpha,\psi} f \right\|_{L_1([a,x_0],\psi)}.$$

That is, we derive

$$|f(x) - f(x_0)| \leq \frac{(\psi(x_0) - \psi(x))^{\alpha-1}}{\Gamma(\alpha)} \left\| {}^C D_{x_0-}^{\alpha,\psi} f \right\|_{L_1([a,x_0],\psi)}, \tag{35}$$

$\forall\, x \in [a, x_0]$.

As in the proof of Theorem 3.1, we have

$$\left| \frac{1}{(\psi(b) - \psi(a))} \int_a^b f(x)\, \psi'(x)\, dx - f(x_0) \right| \leq$$

$$\frac{1}{(\psi(b) - \psi(a))} \left\{ \int_a^{x_0} |f(x) - f(x_0)|\, \psi'(x)\, dx + \int_{x_0}^{b} |f(x) - f(x_0)|\, \psi'(x)\, dx \right\} \tag{36}$$

$$\overset{\text{(by (33), (35))}}{\leq}$$

$$\frac{1}{\left(\psi\left(b\right)-\psi\left(a\right)\right)\Gamma\left(\alpha\right)}\left\{\left(\int_{a}^{x_{0}}\left(\psi\left(x_{0}\right)-\psi\left(x\right)\right)^{\alpha-1}d\psi\left(x\right)\right)\left\|^{C}D_{x_{0}-}^{\alpha,\psi}f\right\|_{L_{1}\left(\left[a,x_{0}\right],\psi\right)}\right.$$
$$\left.+\left(\int_{x_{0}}^{b}\left(\psi\left(x\right)-\psi\left(x_{0}\right)\right)^{\alpha-1}d\psi\left(x\right)\right)\left\|^{C}D_{x_{0}+}^{\alpha,\psi}f\right\|_{L_{1}\left(\left[x_{0},b\right],\psi\right)}\right\}=$$
$$\frac{1}{\left(\psi\left(b\right)-\psi\left(a\right)\right)\Gamma\left(\alpha+1\right)}\left\{\left(\psi\left(x_{0}\right)-\psi\left(a\right)\right)^{\alpha}\left\|^{C}D_{x_{0}-}^{\alpha,\psi}f\right\|_{L_{1}\left(\left[a,x_{0}\right],\psi\right)}\right.$$
$$\left.+\left(\psi\left(b\right)-\psi\left(x_{0}\right)\right)^{\alpha}\left\|^{C}D_{x_{0}+}^{\alpha,\psi}f\right\|_{L_{1}\left(\left[x_{0},b\right],\psi\right)}\right\}\leq \tag{37}$$
$$\frac{1}{\Gamma\left(\alpha+1\right)}\max\left\{\left\|^{C}D_{x_{0}-}^{\alpha,\psi}f\right\|_{L_{1}\left(\left[a,x_{0}\right],\psi\right)},\left\|^{C}D_{x_{0}+}^{\alpha,\psi}f\right\|_{L_{1}\left(\left[x_{0},b\right],\psi\right)}\right\}\left(\psi\left(b\right)-\psi\left(a\right)\right)^{\alpha-1}.$$

□

Next we present

Theorem 3.4. *All as in Theorem 3.1. Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, *and* $\alpha\geq1$. *Then*

$$\left|\frac{1}{\left(\psi\left(b\right)-\psi\left(a\right)\right)}\int_{a}^{b}f\left(x\right)\psi'\left(x\right)dx-f\left(x_{0}\right)\right|\leq$$
$$\frac{1}{\left(\psi\left(b\right)-\psi\left(a\right)\right)\Gamma\left(\alpha\right)\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)} \tag{38}$$
$$\left\{\left(\psi\left(x_{0}\right)-\psi\left(a\right)\right)^{\alpha+\frac{1}{p}}\left\|^{C}D_{x_{0}-}^{\alpha,\psi}f\right\|_{L_{q}\left(\left[a,x_{0}\right],\psi\right)}+\right.$$
$$\left.\left(\psi\left(b\right)-\psi\left(x_{0}\right)\right)^{\alpha+\frac{1}{p}}\left\|^{C}D_{x_{0}+}^{\alpha,\psi}f\right\|_{L_{q}\left(\left[x_{0},b\right],\psi\right)}\right\}\leq$$
$$\frac{1}{\Gamma\left(\alpha\right)\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)} \tag{39}$$
$$\max\left\{\left\|^{C}D_{x_{0}-}^{\alpha,\psi}f\right\|_{L_{q}\left(\left[a,x_{0}\right],\psi\right)},\left\|^{C}D_{x_{0}+}^{\alpha,\psi}f\right\|_{L_{q}\left(\left[x_{0},b\right],\psi\right)}\right\}\left(\psi\left(b\right)-\psi\left(a\right)\right)^{\alpha-\frac{1}{q}}.$$

Proof. By (22) and Hölder's inequality we have

$$\left|f\left(x\right)-f\left(x_{0}\right)\right|\leq\frac{1}{\Gamma\left(\alpha\right)}\int_{x_{0}}^{x}\left(\psi\left(x\right)-\psi\left(t\right)\right)^{\alpha-1}\left|^{C}D_{x_{0}+}^{\alpha,\psi}f\left(t\right)\right|d\psi\left(t\right)\leq$$
$$\frac{1}{\Gamma\left(\alpha\right)}\left(\int_{x_{0}}^{x}\left(\psi\left(x\right)-\psi\left(t\right)\right)^{p\left(\alpha-1\right)}d\psi\left(t\right)\right)^{\frac{1}{p}}\left(\int_{x_{0}}^{x}\left|^{C}D_{x_{0}+}^{\alpha,\psi}f\left(t\right)\right|^{q}d\psi\left(t\right)\right)^{\frac{1}{q}}\leq \tag{40}$$
$$\frac{1}{\Gamma\left(\alpha\right)}\frac{\left(\psi\left(x\right)-\psi\left(x_{0}\right)\right)^{\alpha-1+\frac{1}{p}}}{\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}}\left\|^{C}D_{x_{0}+}^{\alpha,\psi}f\right\|_{L_{q}\left(\left[x_{0},b\right],\psi\right)}.$$

That is

$$\left|f\left(x\right)-f\left(x_{0}\right)\right|\leq\frac{1}{\Gamma\left(\alpha\right)}\frac{\left(\psi\left(x\right)-\psi\left(x_{0}\right)\right)^{\alpha-1+\frac{1}{p}}}{\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}}\left\|^{C}D_{x_{0}+}^{\alpha,\psi}f\right\|_{L_{q}\left(\left[x_{0},b\right],\psi\right)}, \tag{41}$$

$\forall\ x\in\left[x_{0},b\right].$

By (23) and Hölder's inequality we have

$$|f(x)-f(x_0)| \leq \frac{1}{\Gamma(\alpha)} \int_x^{x_0} (\psi(t)-\psi(x))^{\alpha-1} \left| {}^C D_{x_0-}^{\alpha,\psi} f(t) \right| d\psi(t) \leq$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_x^{x_0} (\psi(t)-\psi(x))^{p(\alpha-1)} d\psi(t) \right)^{\frac{1}{p}} \left(\int_x^{x_0} \left| {}^C D_{x_0-}^{\alpha,\psi} f(t) \right|^q d\psi(t) \right)^{\frac{1}{q}} \leq \tag{42}$$

$$\frac{1}{\Gamma(\alpha)} \frac{(\psi(x_0)-\psi(x))^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| {}^C D_{x_0-}^{\alpha,\psi} f \right\|_{L_q([a,x_0],\psi)}.$$

That is

$$|f(x)-f(x_0)| \leq \frac{1}{\Gamma(\alpha)} \frac{(\psi(x_0)-\psi(x))^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| {}^C D_{x_0-}^{\alpha,\psi} f \right\|_{L_q([a,x_0],\psi)}, \tag{43}$$

$\forall\, x \in [a, x_0]$.

As in the proof of Theorem 3.1, we have

$$\left| \frac{1}{(\psi(b)-\psi(a))} \int_a^b f(x)\, \psi'(x)\, dx - f(x_0) \right| \leq$$

$$\frac{1}{(\psi(b)-\psi(a))} \left\{ \int_a^{x_0} |f(x)-f(x_0)|\, \psi'(x)\, dx + \int_{x_0}^b |f(x)-f(x_0)|\, \psi'(x)\, dx \right\} \tag{44}$$

$$\overset{\text{(by (41), (43))}}{\leq} \frac{1}{(\psi(b)-\psi(a))\,\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}}}$$

$$\left\{ \left(\int_a^{x_0} (\psi(x_0)-\psi(x))^{\alpha-1+\frac{1}{p}} d\psi(x) \right) \left\| {}^C D_{x_0-}^{\alpha,\psi} f \right\|_{L_q([a,x_0],\psi)} \right.$$

$$\left. + \left(\int_{x_0}^b (\psi(x)-\psi(x_0))^{\alpha-1+\frac{1}{p}} d\psi(x) \right) \left\| {}^C D_{x_0+}^{\alpha,\psi} f \right\|_{L_q([x_0,b],\psi)} \right\} =$$

$$\frac{1}{(\psi(b)-\psi(a))\,\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha+\frac{1}{p}\right)}$$

$$\left\{ (\psi(x_0)-\psi(a))^{\alpha+\frac{1}{p}} \left\| {}^C D_{x_0-}^{\alpha,\psi} f \right\|_{L_q([a,x_0],\psi)} + \right.$$

$$\left. (\psi(b)-\psi(x_0))^{\alpha+\frac{1}{p}} \left\| {}^C D_{x_0+}^{\alpha,\psi} f \right\|_{L_q([x_0,b],\psi)} \right\}$$

$$\leq \frac{1}{\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha+\frac{1}{p}\right)} \tag{45}$$

$$\max\left\{ \left\| {}^C D_{x_0-}^{\alpha,\psi} f \right\|_{L_q([a,x_0],\psi)}, \left\| {}^C D_{x_0+}^{\alpha,\psi} f \right\|_{L_q([x_0,b],\psi)} \right\} (\psi(b)-\psi(a))^{\alpha-\frac{1}{q}}.$$

□

Corollary 3.5. *(to Theorem 3.4) All as in Theorem 3.1. Here* $p = q = 2$ *and* $\alpha \geq 1$. *Then*

$$\left|\frac{1}{(\psi(b)-\psi(a))}\int_a^b f(x)\psi'(x)\,dx - f(x_0)\right| \leq$$

$$\frac{1}{(\psi(b)-\psi(a))\,\Gamma(\alpha)\sqrt{(2\alpha-1)}\left(\alpha+\frac{1}{2}\right)} \tag{46}$$

$$\left\{(\psi(x_0)-\psi(a))^{\alpha+\frac{1}{2}}\left\|{}^{C}D_{x_0-}^{\alpha,\psi}f\right\|_{L_2([a,x_0],\psi)}+\right.$$

$$\left.(\psi(b)-\psi(x_0))^{\alpha+\frac{1}{2}}\left\|{}^{C}D_{x_0+}^{\alpha,\psi}f\right\|_{L_2([x_0,b],\psi)}\right\}$$

$$\leq \frac{1}{\Gamma(\alpha)\sqrt{(2\alpha-1)}\left(\alpha+\frac{1}{2}\right)} \tag{47}$$

$$\max\left\{\left\|{}^{C}D_{x_0-}^{\alpha,\psi}f\right\|_{L_2([a,x_0],\psi)},\left\|{}^{C}D_{x_0+}^{\alpha,\psi}f\right\|_{L_2([x_0,b],\psi)}\right\}(\psi(b)-\psi(a))^{\alpha-\frac{1}{2}}.$$

Some applications of Theorem 3.1 follow.

In the case of $\psi(x) = e^x$ we get:

Proposition 3.6. *Let* $\alpha > 0$, $n \in \mathbb{N} : n = \lceil\alpha\rceil$, $[a,b] \subset \mathbb{R}$, $f \in C^n([a,b])$. *Let* $x_0 \in [a,b]$ *and assume that* $f_{e^x}^{[k]}(x_0) = 0$, $k = 1,...,n-1$. *Then*

$$\left|\frac{1}{(e^b-e^a)}\int_a^b f(x)e^x dx - f(x_0)\right| \leq \frac{1}{(e^b-e^a)\,\Gamma(\alpha+2)} \tag{48}$$

$$\left\{(e^{x_0}-e^a)^{\alpha+1}\left\|{}^{C}D_{x_0-}^{\alpha,e^x}f\right\|_{\infty,[a,x_0]}+\left(e^b-e^{x_0}\right)^{\alpha+1}\left\|{}^{C}D_{x_0+}^{\alpha,e^x}f\right\|_{\infty,[x_0,b]}\right\} \leq$$

$$\frac{1}{\Gamma(\alpha+2)}\max\left\{\left\|{}^{C}D_{x_0-}^{\alpha,e^x}f\right\|_{\infty,[a,x_0]},\left\|{}^{C}D_{x_0+}^{\alpha,e^x}f\right\|_{\infty,[x_0,b]}\right\}\left(e^b-e^a\right)^{\alpha}. \tag{49}$$

In case of $0 < \alpha \leq 1$, *(48)-(49) are still valid without any initial conditions.*

In case of $\psi(x) = \ln x$ we obtain:

Proposition 3.7. *Let* $\alpha > 0$, $n \in \mathbb{N} : n = \lceil\alpha\rceil$, $[a,b] \subset (0,+\infty)$, $f \in C^n([a,b])$. *Let* $x_0 \in [a,b]$ *and assume that* $f_{\ln x}^{[k]}(x_0) = 0$, $k = 1,...,n-1$. *Then*

$$\left|\frac{1}{\ln\frac{b}{a}}\int_a^b \frac{f(x)}{x}dx - f(x_0)\right| \leq \frac{1}{\left(\ln\frac{b}{a}\right)\Gamma(\alpha+2)} \tag{50}$$

$$\left\{\left(\ln\frac{x_0}{a}\right)^{\alpha+1}\left\|{}^{C}D_{x_0-}^{\alpha,\ln x}f\right\|_{\infty,[a,x_0]}+\left(\ln\frac{b}{x_0}\right)^{\alpha+1}\left\|{}^{C}D_{x_0+}^{\alpha,\ln x}f\right\|_{\infty,[x_0,b]}\right\} \leq$$

$$\frac{1}{\Gamma(\alpha+2)}\max\left\{\left\|{}^{C}D_{x_0-}^{\alpha,\ln x}f\right\|_{\infty,[a,x_0]},\left\|{}^{C}D_{x_0+}^{\alpha,\ln x}f\right\|_{\infty,[x_0,b]}\right\}\left(\ln\frac{b}{a}\right)^{\alpha}. \tag{51}$$

In case of $0 < \alpha \leq 1$, *(50)-(51) are still valid without any initial conditions.*

Note. Many other interesting examples of our theorems could follow but we skip this task.

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ON DOUBLE ALMOST LACUNARY SUMMABLE SEQUENCES OF ORDER θ DEFINED VIA ORLICZ FUNCTION

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Abstract. The primary aim of this article is to present the double almost lacunary strong P-convergence of order θ via Orlicz function and study some characteristics of the resulting sequence spaces.

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1. First Section

Kuttner [3] examined spaces of strongly summable sequences and later on it was discussed by Maddox [5], and others. Also Maddox [4] studied the set of sequences which are strongly Cesàro summable with respect to a modulus as a generalization of the notion of strongly Cesàro summable sequences. Further, Connor [1] considered the strong A-summability with regard to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and examined some connections between strong A-summability and strong A-summability with respect to a modulus.

We recall [7] in that $y = (y_{r,s})$ is said to be convergent in Pringsheim sense to some complex number ϖ if for every $\epsilon > 0$ there exists $n_0 = n_0 \in \mathbf{N}$ such that

$$|y_{r,s} - \varpi| < \epsilon \text{ if } min\{r, s\} > n_0.$$

We shall describe such an y more shortly as "**P-convergent**".

Furthermore, Moricz and Rhoades [6] presented P-almost convergent sequences as below:

Definition 1.1. *A double sequence $y = (y_{r,s})$ of real numbers is called almost P-convergent to a limit ϖ if*

$$P - \lim_{p,q\to\infty} \sup_{t,z\geq 0} \frac{1}{pq} \sum_{r=t}^{t+p-1} \sum_{s=z}^{z+q-1} |y_{r,s} - \varpi| = 0$$

that is the average value of $(y_{r,s})$ taken over any rectangle $\{(r, s) : t \leq r \leq t + p - 1, z \leq s \leq z + q - 1\}$ tends to ϖ as both p and q tend to ∞, and this P-convergences is uniform in t and z. We denote the set of sequence with this property by $[\hat{c}^2]$.

Later on the following definition was given by Savaş and Patterson [8].

Definition 1.2. *The double sequence $\Phi_{\xi,\eta} = \{(r_\xi, s_\eta)\}$ is called* **double lacunary** *if there exist two increasing of integers such that*

$$r_0 = 0, \gamma_\xi = r_\xi - r_{\xi-1} \to \infty \quad as \quad \xi \to \infty$$

and

$$s_0 = 0, \overline{\gamma}_\eta = s_\eta - s_{\eta-1} \to \infty \quad as \quad \eta \to \infty.$$

Also, $r_{\xi,\eta} = r_\xi s_\eta$, $\gamma_{\xi,\eta} = \gamma_\xi \overline{\gamma}_\eta$, $\Phi_{\xi,\eta}$ is determine by $J_{\xi,\eta} = \{(r,s) : r_{\xi-1} < r \le r_\xi \ \& \ s_{\eta-1} < s \le s_\eta\}$, $\zeta_\xi = \frac{r_\xi}{r_{\xi-1}}$, $\overline{\zeta}_\eta = \frac{s_\eta}{s_{\eta-1}}$, and $\zeta_{\xi,\eta} = \zeta_\xi \overline{\zeta}_\eta$. We will denote the set of all double lacunary sequences by $\mathbf{N}_{\Phi_{\xi,\eta}}$.

Additionally, Savas [9] presented some results by using double sequence and Orlicz functions.

Recall in [2] that an Orlicz function $\mathbf{F}$ is continuous, convex, nondecreasing function such that $\mathbf{F}(0) = 0$ and $\mathbf{F}(y) > 0$ for $y > 0$.

2. Some New Definitions and Notations

In this section, we will present some new definitions and notations that will be needed in main result.

Definition 2.1. *Let $\mathbf{F}$ be an Orlicz function and $\theta \in (0,1]$ be any real number and $\tau = (\tau_{r,s})$ be any factorable double sequence of strictly positive real numbers, we consider the following sequence space:*

$$[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta = \{y = (y_{r,s}) : P - \lim_{\xi,\eta} \frac{1}{\gamma^\theta_{\xi,\eta}} \sum_{(r,s)\in J_{\xi,\eta}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z} - \varpi|}{\rho}\right)\right]^{\tau_{r,s}} = 0,$$
$$\textit{uniformly in } t \textit{ and } z \textit{ for some } \varpi \textit{ and } \rho > 0\}.$$

If y is in $[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta$, we say that y is almost lacunary strongly P-convergent of order θ with regard to the Orlicz function $\mathbf{F}$. If we take $\varpi = 0$, we have $[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta = [\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta_0$.

Also note if $\mathbf{F}(y) = y$, $\tau_{r,s} = 1$ for all r and s, then $[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta = [\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}]^\theta$ which is presented as follows:

$$[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}] = \{y : \text{ for some } \ \varpi, P - \lim_{\xi,\eta} \frac{1}{\gamma^\theta_{\xi,\eta}} \sum_{(r,s)\in J_{\xi,\eta}} |y_{r+t,s+z} - \varpi| = 0,$$
$$\text{uniformly in } \ t \ \text{ and } \ z\}.$$

If $\tau_{r,s} = 1$ for all r and s, then $[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta = [\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}]^\theta$ which is presented as follows:

$$[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}]^\theta = \{y = (y_{r,s}) : P - \lim_{\xi,\eta} \frac{1}{\gamma^\theta_{\xi,\eta}} \sum_{(r,s)\in J_{\xi,\eta}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z} - \varpi|}{\rho}\right)\right] = 0,$$
$$\text{uniformly in } \ t \ \text{ and } \ z \ \text{ for some } \ \varpi \ \text{ and } \ \rho > 0\}.$$

Note that if $\tau_{r,s} = 1$, $\theta = 1$ for all r and s, then $[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta = [\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}]$ which is given as follows:

$$[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}] = \{y = (y_{r,s}) : P - \lim_{\xi,\eta} \frac{1}{\gamma_{\xi,\eta}} \sum_{(r,s)\in J_{\xi,\eta}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z} - \varpi|}{\rho}\right)\right] = 0,$$
$$\text{uniformly in } \ t \ \text{ and } \ z \ \text{ for some } \ \varpi \ \text{ and } \ \rho > 0\}.$$

Almost P-convergent of order θ double sequences to Orlicz function is considered as follows:

Definition 2.2. *The double sequence $y = (y_{r,s})$ of real numbers is called almost P-convergent of order θ to a limit ϖ with regard to the Orlicz function $\mathbf{F}$ if*

$$P - \lim_{u,w} \frac{1}{(uw)^\theta} \sum_{r,s=1,1}^{u,w} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z} - \varpi|}{\rho}\right)\right]^{\tau_{r,s}} = 0,$$
$$\textit{uniformly in } \ t \ \textit{ and } \ z \ \textit{ for some } \ \varpi \ \textit{ and } \ \rho > 0.$$

Almost P-convergent of order θ double sequences with regard to Orlicz function can be shown by a standard argument that $[\hat{c}^2, \mathbf{F}, \tau]^\theta$. An Orlicz function $\mathbf{F}$ is said to fulfill Δ_2-condition for all values of $\widetilde{u}$, if there exists a constant $\widehat{K} > 0$ such that $\mathbf{F}(2\widetilde{u}) \leq \widehat{K}\mathbf{F}(\widetilde{u})$, $\widetilde{u} \geq 0$.

3. Main Results

We first present the following lemma for the next theorem.

Lemma 3.1. *Let $\mathbf{F}$ be an Orlicz function which satisfies Δ_2- condition and let $0 < \widetilde{\delta} < 1$. Then for each $y \geq \widetilde{\delta}$ we have $\mathbf{F}(y) < \widehat{K}\widetilde{\delta}^{-1}\mathbf{F}(2)$ for some constant $\widehat{K} > 0$.*

Theorem 3.2. *For any Orlicz function $\mathbf{F}$ which satisfies Δ_2 condition, we have $[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}]^\theta \subseteq [\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}]^\theta$.*

Proof. Let $y \in \left[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}\right]^\theta$ so that

$$A_{\xi,\eta} = \left\{ y : \text{ for some } \varpi, P - \frac{1}{\gamma^\theta_{\xi,\eta}} \lim_{\xi,\eta} \sum_{(r,s)\in J_{\xi,\eta}} |y_{r+t,s+z} - \varpi| = 0 \right\}.$$

Let $\epsilon > 0$ and choose $\widetilde{\delta}$ with $0 < \widetilde{\delta} < 1$ such that $\mathbf{F}(\varsigma) < \epsilon$ for every ς with $0 \leq \varsigma \leq \widetilde{\delta}$. We obtain the following

$$\begin{aligned}
& \frac{1}{\gamma^\theta_{\xi,\eta}} \sum_{(r,s)\in J_{\xi,\eta}} \mathbf{F}(|y_{r+t,s+z} - \varpi|) \\
= & \frac{1}{\gamma^\theta_{\xi,\eta}} \sum_{(r,s)\in J_{\xi,\eta} \& |y_{r+t,s+z}-\varpi|\leq\widetilde{\delta}} \mathbf{F}(|y_{r+t,s+z} - \varpi|) + \frac{1}{\gamma^\theta_{\xi,\eta}} \sum_{(r,s)\in J_{\xi,\eta} \& |y_{r+t,s+z}-\varpi|>\widetilde{\delta}} \mathbf{F}(|y_{r+t,s+z} - \varpi|) \\
\leq & \frac{1}{\gamma^\theta_{\xi,\eta}} \gamma^\theta_{\xi,\eta}\epsilon + \frac{1}{\gamma^\theta_{\xi,\eta}} \sum_{(r,s)\in J_{\xi,\eta} \& |y_{r+t,s+z}-\varpi|>\widetilde{\delta}} \mathbf{F}(|y_{r+t,s+z} - \varpi|) \\
< & \frac{1}{\gamma^\theta_{\xi,\eta}} (\gamma^\theta_{\xi,\eta}\varepsilon) + \frac{1}{\gamma^\theta_{\xi,\eta}} \widehat{K}\widetilde{\delta}^{-1}\mathbf{F}(2)\gamma_{\xi,\eta}A_{\xi,\eta}.
\end{aligned}$$

Therefore, as ξ and η go to infinity, for each t and z, it is obvious $y \in [\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}]^\theta$. □

In the next theorems we shall interest the connection between $[\hat{c}^2, \mathbf{F}, \tau]^\theta$ and $[\widehat{N}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta$.

Theorem 3.3. *Let $\Phi_{\xi,\eta} = \{(r_\xi, s_\eta)\}$ be a double lacunary sequence, $\mathbf{F}$ is Orlicz function and $\theta \in (0, 1]$. In order for $[\hat{c}^2, \mathbf{F}, \tau]^\theta \subset [\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta$ it is sufficient that $\liminf_\xi \zeta_\xi > 1$ and $\liminf_\eta \overline{\zeta}_\eta > 1$.*

Proof. It is sufficient to show that $[\hat{c}^2, \mathbf{F}, \tau]^\theta_0 \subset [\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta_0$. The general inclusion follows by linearity. Suppose $\liminf_\xi \zeta_\xi > 1$ and $\liminf_\eta \overline{\zeta}_\eta > 1$, then there exists $\widetilde{\delta} > 0$ such that $\zeta_\xi > 1 + \widetilde{\delta}$ and $\overline{\zeta}_\eta > 1 + \widetilde{\delta}$.

This implies $\frac{\gamma_\xi^\theta}{r_\xi^\theta} \geq \frac{\widetilde{\delta}^\theta}{(1+\widetilde{\delta})^\theta}$ and $\frac{\overline{\gamma}_\eta^\theta}{s_\eta^\theta} \geq \frac{\widetilde{\delta}^\theta}{(1+\widetilde{\delta})^\theta}$. Then for $y \in [\hat{c}^2, \mathbf{F}, \tau]_0^\theta$, we can write for each t and z

$$\begin{aligned}
A_{\xi,\eta} &= \frac{1}{\gamma_{\xi,\eta}^\theta} \sum_{(r,s)\in J_{\xi,\eta}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}} \\
&= \frac{1}{\gamma_{\xi,\eta}^\theta} \sum_{r=1}^{r_\xi}\sum_{s=1}^{s_\eta} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}} \\
&- \frac{1}{\gamma_{\xi,\eta}^\theta} \sum_{r=1}^{r_{\xi-1}}\sum_{s=1}^{s_{\eta-1}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}} \\
&- \frac{1}{\gamma_{\xi,\eta}^\theta} \sum_{r=r_{\xi-1}+1}^{r_\xi}\sum_{s=1}^{s_{\eta-1}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}} \\
&- \frac{1}{\gamma_{\xi,\eta}^\theta} \sum_{s=s_{\eta-1}+1}^{s_\eta}\sum_{r=1}^{r_{\xi-1}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}} \\
&= \frac{r_\xi^\theta s_\eta^\theta}{\gamma_{\xi,\eta}^\theta}\left(\frac{1}{r_\xi s_\eta} \sum_{r=1}^{r_\xi}\sum_{s=1}^{s_\eta} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}}\right) \\
&- \frac{r_{\xi-1}^\theta s_{\eta-1}^\theta}{\gamma_{\xi,\eta}^\theta}\left(\frac{1}{r_{\xi-1}^\theta s_{\eta-1}^\theta} \sum_{r=1}^{r_{\xi-1}}\sum_{s=1}^{s_{\eta-1}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}}\right) \\
&- \frac{1}{\gamma_\xi^\theta} \sum_{r=r_{\xi-1}+1}^{r_\xi} \frac{s_\eta^\theta - 1}{\overline{\gamma}_\eta^\theta}\frac{1}{s_\eta^\theta - 1}\sum_{s=1}^{s_\eta - 1} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}} \\
&- \frac{1}{\overline{\gamma}_\eta^\theta} \sum_{s=s_{\eta-1}+1}^{s_\eta} \frac{r_{\xi-1}^\theta}{\gamma_\xi^\theta}\frac{1}{r_{\xi-1}^\theta}\sum_{r=1}^{r_{\xi-1}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}}.
\end{aligned}$$

Since $y \in [\hat{c}^2, \mathbf{F}, \tau]^\theta$ the last two terms tend to zero uniformly in t and z in the Pringsheim sense, thus for each t and z

$$\begin{aligned}
A_{\xi,\eta} &= \frac{r_\xi^\theta s_\eta^\theta}{\gamma_{\xi,\eta}^\theta}\left(\frac{1}{r_\xi^\theta s_\eta^\theta} \sum_{r=1}^{r_\xi}\sum_{s=1}^{s_\eta} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}}\right) \\
&- \frac{r_{\xi-1}^\theta s_{\eta-1}^\theta}{\gamma_{\xi,\eta}^\theta}\left(\frac{1}{r_{\xi-1}^\theta s_{\eta-1}^\theta} \sum_{r=1}^{r_{\xi-1}}\sum_{s=1}^{s_{\eta-1}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}}\right) + o(1).
\end{aligned}$$

Since $\gamma_{\xi,\eta} = r_\xi s_\eta - r_{\xi-1}s_{\eta-1}$ we are granted for each t and z the following:

$$\frac{r_\xi^\theta s_\eta^\theta}{\gamma_{\xi,\eta}^\theta} \leq \frac{(1+\widetilde{\delta})^\theta}{(\widetilde{\delta})^\theta} \text{ and } \frac{r_{\xi-1}^\theta s_{\eta-1}^\theta}{\gamma_{\xi,\eta}^\theta} \leq \frac{1}{(\delta)^\theta}.$$

The terms

$$\frac{1}{r_\xi^\theta s_\eta^\theta} \sum_{r=1}^{r_\xi}\sum_{s=1}^{s_\eta} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}}$$

and

$$\frac{1}{r_{\xi-1}^\theta s_{\eta-1}^\theta} \sum_{r=1}^{r_{\xi-1}}\sum_{s=1}^{s_{\eta-1}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}}$$

are both Pringsheim null sequences. Thus, $A_{\xi,\eta}$ is a Pringsheim null sequence for each t and z. Consequently, y is in $[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]_0^\theta$. □

Theorem 3.4. *Let $\Phi_{\xi,\eta} = \{(r_\xi, s_\eta)\}$ be a double lacunary sequence, $\mathbf{F}$ is Orlicz function and $\theta \in (0,1]$. In order for $[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta \subset [\hat{c}^2, \mathbf{F}, \tau]^\theta$ it is sufficient that $\limsup_\xi \frac{r_\xi}{r_{\xi-1}^\theta} < \infty$ and $\limsup_\eta \frac{s_\eta}{s_{\eta-1}^\theta} < \infty$.*

Proof. Since $\limsup_\xi \frac{r_\xi}{r_{\xi-1}^\theta} < \infty$ and $\limsup_\eta \frac{s_\eta}{s_{\eta-1}^\theta} < \infty$ there exists $\overline{H} > 0$ such that $\frac{r_\xi}{r_{\xi-1}^\theta} < \overline{H}$ and $\frac{s_\eta}{s_{\eta-1}^\theta} < \overline{H}$ for all ξ and η. Let $y \in [\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta$ and $\epsilon > 0$. Also there exist $\xi_0 > 0$ and $\eta_0 > 0$ such that for every $\overline{u} \geq \xi_0$ and $\overline{v} \geq s_0$

$$A_{\overline{u},\overline{v}} = \frac{1}{\gamma_{\xi,\eta}^\theta} \sum_{(r,s)\in J_{\xi,\eta}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}} < \epsilon.$$

Let $M = \max\{A_{\overline{u},\overline{v}} : 1 \leq \xi \leq \xi_0 \text{ and } 1 \leq \eta \leq \eta_0\}$, and u and w be such that $r_{\xi-1} < u \leq r_\xi$ and $s_{\eta-1} < w \leq s_\eta$. Thus we obtain the following:

$$\begin{aligned}
&\frac{1}{(uw)^\theta} \sum_{r,s=1,1}^{u,w} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}} \\
\leq\ & \frac{1}{r_{\xi-1}^\theta s_{\eta-1}^\theta} \sum_{r,s=1,1}^{r_\xi s_\eta} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}} \\
\leq\ & \frac{1}{r_{\xi-1}^\theta s_{\eta-1}^\theta} \sum_{\overline{u},\overline{v}=1,1}^{\xi,\eta} \left(\sum_{(r,s)\in I_{\overline{u},\overline{v}}} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}}\right) \\
=\ & \frac{1}{r_{\xi-1}^\theta s_{\eta-1}^\theta} \sum_{\overline{u},\overline{v}=1,1}^{\xi_0,\eta_0} \gamma_{\overline{u},\overline{v}} A_{\overline{u},\overline{v}} + \frac{1}{r_{\xi-1}^\theta s_{\eta-1}^\theta} \sum_{(\xi_0<\overline{u}\leq\xi)\cup(\eta_0<\overline{v}\leq\eta)} \gamma_{\overline{u},\overline{v}} A_{\overline{u},\overline{v}} \\
\leq\ & \frac{M}{r_{\xi-1}^\theta s_{\eta-1}^\theta} \sum_{\overline{u},\overline{v}=1,1}^{\xi_0,\eta_0} \gamma_{\overline{u},\overline{v}} + \frac{1}{r_{\xi-1}^\theta s_{\eta-1}^\theta} \sum_{(\xi_0<\overline{u}\leq\xi)\cup(\eta_0<\overline{v}\leq\eta)} \gamma_{\overline{u},\overline{v}} A_{\overline{u},\overline{v}} \\
\leq\ & \frac{M\ r_{\xi_0} s_{\eta_0} \xi_0 \eta_0}{r_{\xi-1}^\theta s_{\eta-1}^\theta} + \frac{1}{r_{\xi-1}^\theta s_{\eta-1}^\theta} \sum_{(\xi_0<\overline{u}\leq\xi)\cup(\eta_0<\overline{v}\leq\eta)} \gamma_{\overline{u},\overline{v}} A_{\overline{u},\overline{v}} \\
\leq\ & \frac{M\ r_{\xi_0} s_{\eta_0} \xi_0 \eta_0}{r_{\xi-1}^\theta s_{\eta-1}^\theta} + \left(\sup_{\overline{u}\geq\xi_0\cup\overline{v}\geq\eta_0} A_{\overline{u},\overline{v}}\right) \frac{1}{r_{\xi-1}^\theta s_{\eta-1}^\theta} \sum_{(\xi_0<\overline{u}\leq\xi)\cup(\eta_0<\overline{v}\leq\eta)} \gamma_{\overline{u},\overline{v}} \\
\leq\ & \frac{M\ r_{\xi_0} s_{\eta_0} \xi_0 \eta_0}{r_{\xi-1}^\theta s_{\eta-1}^\theta} + \epsilon \sum_{(\xi_0<\overline{u}\leq\xi)\cup(\eta_0<\overline{v}\leq\eta)} \gamma_{\overline{u},\overline{v}} \\
\leq\ & \frac{M\ r_{\xi_0} s_{\eta_0} \xi_0 \eta_0}{r_{\xi-1}^\theta s_{\eta-1}^\theta} + \epsilon \overline{H}^2.
\end{aligned}$$

Since r_ξ and s_η both approach infinity as both u and w approach infinity. Therefore

$$\frac{1}{(uw)^\theta} \sum_{r,s=1,1}^{u,w} \left[\mathbf{F}\left(\frac{|y_{r+t,s+z}|}{\rho}\right)\right]^{\tau_{r,s}} \to 0, \text{uniformly in } t \text{ and } z.$$

As a result, $y \in [\hat{c}^2, \mathbf{F}, \tau]^\theta$. □

The following theorem is a clear consequence of Theorem 3.3 and Theorem 3.4.

Theorem 3.5. *Let* $\Phi_{\xi,\eta} = \{(r_\xi, s_\eta)\}$ *be a double lacunary sequence with* $1 < \liminf_{\xi,\eta} \zeta_{\xi,\eta} \leq \limsup_{\xi,\eta} \zeta_{\xi,\eta} < \infty$*, then for any Orlicz function* $\mathbf{F}$*,* $[\widehat{\mathbf{N}}_{\Phi_{\xi,\eta}}, \mathbf{F}, \tau]^\theta = [\hat{c}^2, \mathbf{F}, \tau]^\theta$.

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AN INEQUALITY FOR CONTACT CR-SUBMANIFOLDS IN COSYMPLECTIC SPACE FORMS

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ABSTRACT. The notion of CR-submanifold was introduced by Bejancu in [Proc. Amer. Math. Soc. 69, 1978]. In [Internat. J. Math. 23, 2012] B.-Y. Chen introduced the CR δ-invariant for CR-submanifolds. Then, in [Taiwan. J. Math. 18: 199-217, 2014] F. R. Al-Solamy, B.-Y. Chen and S. Deshmukh proved two optimal inequalities for anti-holomorphic submanifolds in complex space forms involving the CR δ-invariant. The aim of this paper is to obtain an optimal inequality for this invariant for contact CR-submanifolds in cosymplectic space forms.

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1. INTRODUCTION

As a generalization of invariant (holomorphic) and anti-invariant (totally real) submanifolds of an almost contact metric manifold, A. Bejancu introduced in [3] the notion of *CR-submanifolds.* The contact CR-submanifolds represent an intersting subject of study followed by several researchers.

The CR δ - invariant $\delta(D)$ on a CR-submanifold M in a Kaehler manifold was defined by Chen in [4] by

$$\delta(D)(x) = \tau(x) - \tau(D_x),$$

where τ is the scalar curvature of M and $\tau(D)$ is the scalar curvature of the holomorphic distribution D of M.

In [1], F. Al-Solamy, B. - Y. Chen and S. Deshmukh proved an optimal inequality for anti-holomorphic submanifolds in complex space forms.

Theorem 1.1. *Let N be an anti-holomorphic submanifold of a complex space form $\widetilde{M}^{h+p}(4c)$, with $h = rank_{\mathbb{C}} D \geq 1$ and $p = rank D^{\perp} \geq 2$. Then we have*

$$\delta(D) \leq \frac{(2h+p)^2}{2} H^2 + \frac{p}{2}(4h+p-1)c - \frac{3p^2}{2(p+2)} |H_{D^{\perp}}|^2. \tag{1.1}$$

The equality sign of (1.1) holds identically if and only if the following three conditions are satisfied:

(a) N is D-minimal, i.e., $H_D = 0$,

(b) N is mixed totally geodesic, and

(c) there exists an orthonormal frame $\{e_{2h+1}, ..., e_n\}$ of $D^{\perp}$ such that the second fundamental form σ of N satisfies

$\sigma^r_{rr} = 3\sigma^r_{ss}$, *for* $2h+1 \leq r \neq s \leq 2h+p$,
$\sigma^r_{st} = 0$, *for distinct* $r, s, t \in \{2h+1, ..., 2h+p\}$.

Afterwards, I. Mihai and I. Presură, in [8] established an optimal inequality for the contact CR-submanifolds in Sasakian space forms. An example for the equality case was given.

A similar inequality was obtained by G. Macsim and A. Mihai in [7] for CR-submanifolds in quaternionic space forms.

The aim of this paper is to obtain some similar results in cosymplectic space forms.

2. Preliminaries

Let $\widetilde{M}$ be a $(2m+1)$-dimensional *almost contact metric manifold* together with an almost contact structure (ϕ, ξ, η), i.e., ξ is a global vector field, ϕ is a $(1,1)$-type tensor field and η is a 1-form on $\widetilde{M}$ such that

$$\phi^2 X = -X + \eta(X)\xi,\ \phi\xi = 0,\ \eta(\phi X) = 0,\ \ \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\ \eta(X) = g(X, \xi), \tag{2.2}$$

for any $X,\ Y \in \Gamma(\widetilde{M})$, where $\Gamma(\widetilde{M})$ denotes the set differentiable vector fields on $\widetilde{M}$.

The fundamental 2-form Φ is defined by

$$\Phi(X, Y) = g(X, \phi Y),$$

for any $X,\ Y \in \Gamma(\widetilde{M})$. Then $\widetilde{M}$ is called an *almost cosymplectic manifold* if η and Φ are closed, i.e., $d\eta = 0$ and $d\Phi = 0$, where d is exterior differentiable operator.

An almost contact metric manifold $\widetilde{M}$ is said to be *normal* if

$$[\phi, \phi](X, Y) = -2d\eta(X, Y)\xi, \tag{2.3}$$

for any X, Y, where $[\phi, \phi]$ denotes the Nijenhuis torsion of ϕ, given by

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]. \tag{2.4}$$

An almost contact metric manifold $\widetilde{M}$ is called *cosymplectic manifold* if it is normal and both η and Φ are closed.

So we have on a cosymplectic manifold $\widetilde{M}$: $(\widetilde{\nabla}_X \phi)Y = 0$, for any vector fields X, Y on $\widetilde{M}$.

Given an almost contact metric manifold $\widetilde{M}$, a ϕ-section of $\widetilde{M}$ at $p \in \widetilde{M}$ is a section $\pi \subseteq T_p\widetilde{M}$ spanned by X_p and ϕX_p, where X_p is a unit tangent vector orthogonal to ξ_p. The sectional curvature of a ϕ-section is called a *ϕ-sectional curvature.*

If a cosymplectic manifold $\widetilde{M}$ has constant ϕ-sectional curvature, then it is said to be a *cosymplectic space form* $\widetilde{M}(c)$. Then the curvature tensor $\widetilde{R}$ is given by

$$\begin{aligned} \widetilde{R}(X, Y)Z = \frac{c}{4}\{&g(Y, Z)X - g(X, Z)Y + \\ &+\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)g(X, Z)\xi - \\ &-\eta(X)g(Y, Z)\xi + g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\}, \end{aligned} \tag{2.5}$$

for any vector fields X, Y, Z tangent to $\widetilde{M}(c)$ [6], [5], [9].

Now, let M be an $(n+1)$-dimensional submanifold isometrically immersed in a cosymplectic manifold $\widetilde{M}$ with induced metric g and ∇ and $\nabla^{\perp}$ the induced connection on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively.

We assume that the submanifold M of $\widetilde{M}$ is tangent to the structure vector field ξ. Then the Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{2.6}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.7}$$

for each X, $Y \in TM$ and $N \in T^{\perp}M$, where σ and A_N are the second fundamental form and the shape operator respectively, for the immersion of M in $\widetilde{M}$, which are related by

$$g(\sigma(X,Y),N) = g(A_N X, Y), \tag{2.8}$$

where g denotes the Riemannian metric on $\widetilde{M}$ as well as on M.

The mean curvature vector H of M is given by

$$H = \frac{1}{n+1} trace\ \sigma. \tag{2.9}$$

If $\sigma(X,Y) = 0$, for each X, $Y \in TM$, then M is said to be *totally geodesic.*

The equation of Gauss is

$$R(X,Y,Z,W) = \widetilde{R}(X,Y,Z,W) - g(\sigma(X,W),\sigma(Y,Z)) + g(\sigma(X,Z),\sigma(Y,W)), \tag{2.10}$$

for each X, Y, Z, $W \in TM$, where R and $\widetilde{R}$ denote the Riemann curvature tensors of M and $\widetilde{M}$, respectively.

The covariant derivative $\overline{\nabla}\sigma$ of σ is defined by

$$(\overline{\nabla}_X\sigma)(Y,Z) = \nabla^{\perp}_X\sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z). \tag{2.11}$$

The normal component of (2.10) is said to be the Codazzi equation and is given by

$$(\widetilde{R}(X,Y)Z)^{\perp} = (\overline{\nabla}_X\sigma)(Y,Z) - (\overline{\nabla}_Y\sigma)(X,Z), \tag{2.12}$$

for X, Y, $Z \in TM$, where $(\widetilde{R}(X,Y)Z)^{\perp}$ denotes the normal component of $\widetilde{R}(X,Y)Z$.

For any $X \in TM$, we write

$$\phi X = PX + FX, \tag{2.13}$$

where PX is the tangential component and FX is the normal component of ϕX. In particular, for $X = \xi$ we get $\phi\xi = P\xi + F\xi$, which implies $P\xi = 0$, $F\xi = 0$.

Similarly, for $N \in T^{\perp}M$, we can write

$$\phi N = tN + fN, \tag{2.14}$$

where tN and fN are the tangential and normal components of ϕN, respectively.

The covariant derivative of the tensors ϕ, P, F, t and f are defined respectively:

$$(\overline{\nabla}_X\phi)Y = \overline{\nabla}_X\phi Y - \phi\overline{\nabla}_X Y,$$

$$(\overline{\nabla}_X P)Y = \nabla_X PY - P\nabla_X Y,$$

$$(\overline{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y,$$

$$(\overline{\nabla}_X t)N = \nabla_X tN - t\nabla^{\perp}_X Y,$$

$$(\overline{\nabla}_X f)N = \nabla^{\perp}_X fN - f\nabla^{\perp}_X N.$$

3. Contact CR-submanifolds

3.1. **Definition of contact CR-submanifold.** Let M be a submanifold isometrically immersed in a cosymplectic manifold $\widetilde{M}$ tangent to the structure field ξ. Then M is called *contact CR-submanifold* if it admits an invariant distribution D whose orthogonal complementary distribution $D^{\perp}$ is anti-invariant, that is,

$$TM = D \oplus D^{\perp} \oplus < \xi > ,$$

where $\phi D \subseteq D$ and $\phi D^{\perp} \subseteq T^{\perp}M$ and $< \xi >$ denotes 1-dimensional distribution which is spanned by ξ.

Remark 3.1. *On a contact CR-submanifold M, we consider ξ tangent to D.*

Invariant and anti-invariant submanifolds are special cases of contact CR-submanifolds.

Let M be a contact CR-submanifold of cosymplectic manifold $\widetilde{M}$. Then:
(a) If $D = \{0\}$, then M is an *anti-invariant* submanifold of $\widetilde{M}$.
(b) If $D^{\perp} = \{0\}$, then M is an *invariant* submanifold of $\widetilde{M}$.
(c) If $\phi D^{\perp} = T^{\perp}M$, then M is said to be a *generic* submanifold of $\widetilde{M}$.

Let M be an $(n+1)$-dimensional submanifold of an $(2m+1)$-dimensional cosymplectic manifold $\widetilde{M}$. If $\dim D = 2n_1 + 1$ and $\dim D^{\perp} = n_2$, the partial mean curvature vectors H_D and $H_{D^{\perp}}$ of M are given by

$$H_D = \frac{1}{2n_1+1} \sum_{i=0}^{2n_1} \sigma(e_i, e_i), \tag{3.1}$$

$$H_{D^{\perp}} = \frac{1}{n_2} \sum_{r=2n_1+1}^{2n_1+n_2} \sigma(e_r, e_r). \tag{3.2}$$

M is called *minimal* (resp., D-minimal or $D^{\perp}$-minimal) if $H = 0$ holds identically (resp., $H_D = 0$ or $H_{D^{\perp}} = 0$ hold identically). M is called *mixed totally geodesic* if $\sigma(X, Z) = 0$, for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.

3.2. **Some basics results on the integrability of distributions for contact CR-submanifolds.** We recall the following results from [5] for later use.

Theorem 3.2. *Let M be a contact CR-submanifold of a cosymplectic manifold $\widetilde{M}$. Then the anti-invariant distribution $D^{\perp}$ is completely integrable and its maximal integral submanifold is anti-invariant submanifold of $\widetilde{M}$.*

Theorem 3.3. *Let M be a contact CR-submanifold of a cosymplectic manifold $\widetilde{M}$. Then the invariant distribution D is completely integrable if and only if the second fundamental form of M satisfies*

$$\sigma(X, \phi Y) = \sigma(\phi X, Y)$$

for any X, $Y \in \Gamma(D)$.

4. An inequality for contact CR-submanifolds in cosymplectic space forms

4.1. *CR* **δ-invariant.** By analogy with Chen's CR δ-invariant (see [4]), on an $(n+1)$-dimensional submanifold of an $(2m+1)$-dimensional cosymplectic manifold $\widetilde{M}$, we define the *contact CR δ-invariant* by

$$\delta(D)(x) = \tau(x) - \tau(D_x),$$

where τ is the scalar curvature of M and $\tau(D)$ is the scalar curvature of the invariant distribution D of M.

4.2. **Optimal inequality.** Next, we will obtain an optimal inequality for generic submanifolds in cosymplectic space forms involving the CR δ-invariant.

Theorem 4.1. *Let M be a $(n+1)$-dimensional generic submanifold isometrically immersed in a cosymplectic space form $\widetilde{M}(c)$ with $\dim D = 2n_1+1$ and $\dim D^{\perp} = n_2$. Then we have*

$$\begin{aligned}\delta(D) \leq & \frac{(n+1)^2}{2}||H||^2 + \frac{n_2}{2}(4n_1+n_2-1)\frac{c}{4} \\ & +n_2 - \frac{3n_2^2}{2(n_2+2)}||H_{D^{\perp}}||^2.\end{aligned} \tag{4.1}$$

The equality sign of (4.1) holds identically if and only if the following statements are satisfied:
(a) M is D-minimal, i.e., $H_D = 0$,
(b) M is mixed totally geodesic, and
(c) there exists an orthonormal frame $\{e_{2n_1+1}, ..., e_{2n_1+n_2}\}$ of $D^{\perp}$ such that the second fundamental form σ of M satisfies
$\sigma^r_{rr} = 3\sigma^r_{ss}$, for $2n_1+1 \leq r \neq s \leq 2n_1+n_2$,
$\sigma^r_{st} = 0$, for distinct r, s, $t \in \{2n_1+1, ..., 2n_1+n_2\}$.

Proof. We assume that M is a generic submanifold isometrically immersed in a cosymplectic space form $\widetilde{M}(c)$. We consider an orthonormal frame $\{e_0 = \xi, e_1, e_2, ..., e_{2n_1+n_2}\}$ on M, such that $e_0 = \xi, e_1, e_2, ..., e_{2n_1}$ are tangent to D and $e_{2n_1+1}, ..., e_{2n_1+n_2}$ are tangent to $D^{\perp}$, where $e_{n_1+1} = \phi e_1, ..., e_{2n_1} = \phi e_{n_1}$. Then, the scalar curvature τ of M is

$$\begin{aligned}2\tau(p) = & \sum_{0\leq i\neq j\leq 2n_1} K(e_i, e_j) + 2\sum_{i=0}^{2n_1}\sum_{r=2n_1+1}^{2n_1+n_2} K(e_i, e_r) \\ & + \sum_{2n_1+1\leq r\neq s\leq 2n_1+n_2} K(e_r, e_s).\end{aligned} \tag{4.2}$$

Using the Gauss equation and the definition of CR δ-invariant we find

$$\begin{aligned}\delta(D) = & \sum_{r=2n_1+1}^{2n_1+n_2} K(\xi, e_r) + \sum_{i=1}^{2n_1}\sum_{r=2n_1+1}^{2n_1+n_2} K(e_i, e_r) \\ & +\frac{1}{2}\sum_{2n_1+1\leq r\neq s\leq 2n_1+n_2} K(e_r, e_s) \\ = & \ n_2 + \sum_{i=1}^{2n_1}\sum_{r=2n_1+1}^{2n_1+n_2} g(\sigma(e_i, e_i), \sigma(e_r, e_r)) \\ & +\frac{1}{2}\sum_{r,s=2n_1+1}^{2n_1+n_2} g(\sigma(e_r, e_r), \sigma(e_s, e_s)) \\ & -\sum_{i=1}^{2n_1}\sum_{r=2n_1+1}^{2n_1+n_2} ||\sigma(e_i, e_r)||^2 - \frac{1}{2}\sum_{r,s=2n_1+1}^{2n_1+n_2} ||\sigma(e_r, e_s)||^2 \\ & +\frac{n_2}{2}(4n_1+n_2-1)\frac{c}{4}.\end{aligned} \tag{4.3}$$

Also, we have

$$\sum_{i=1}^{2n_1}\sum_{r=2n_1+1}^{2n_1+n_2} g(\sigma(e_i,e_i),\sigma(e_r,e_r))$$
$$+\frac{1}{2}\sum_{r,s=2n_1+1}^{2n_1+n_2} g(\sigma(e_r,e_r),\sigma(e_s,e_s)) - \frac{1}{2}\sum_{r,s=2n_1+1}^{2n_1+n_2}||\sigma(e_r,e_s)||^2$$
$$= \frac{(2n_1+n_2+1)^2}{2}||H||^2 - \frac{(2n_1+1)^2}{2}||H_D||^2 - \frac{1}{2}||\sigma_{D^\perp}||^2, \tag{4.4}$$

where $||\sigma_{D^\perp}||^2$ is defined by

$$||\sigma_{D^\perp}||^2 = \sum_{r,s=2n_1+1}^{2n_1+n_2}||\sigma(e_r,e_s)||^2. \tag{4.5}$$

Now, we denote by

$$\sigma_{rs}^t = g(\sigma(e_r,e_s),\phi e_t), \text{ for all } 2n_1+1 \le r,s,t \le 2n_1+n_2. \tag{4.6}$$

We know from [5], page 4793, that $A_{FZ}W = A_{FW}Z$, for all Z, $W \in \Gamma(D^\perp)$. This condition can be written as

$$\sigma_{st}^r = \sigma_{rt}^s = \sigma_{rs}^t, \ \forall\ r,s,t \in \{2n_1+1,...,2n_1+n_2\}. \tag{4.7}$$

Combining (4.3) and (4.4), we obtain

$$\delta(D) = \frac{(2n_1+n_2+1)^2}{2}||H||^2 + n_2 + \frac{n_2}{2}(4n_1+n_2-1)\frac{c}{4}$$
$$-\frac{(2n_1+1)^2}{2}||H_D||^2 - \sum_{i=0}^{2n_1}\sum_{r=2n_1+1}^{2n_1+n_2}||\sigma(e_i,e_r)||^2 - \frac{1}{2}||\sigma_{D^\perp}||^2. \tag{4.8}$$

Using (3.1), (3.2), (4.5) and (4.7), we get

$$(n_2+2)||\sigma_{D^\perp}||^2 - 3n_2^2||H_D||^2 = (n_2-1)\sum_{r=2n_1+1}^{2n_1+n_2}(\sum_{s=2n_1+1}^{2n_1+n_2}\sigma_{ss}^r)^2$$
$$+3(n_2+1)\sum_{2n_1+1\le r\ne s\le 2n_1+n_2}(\sigma_{ss}^r)^2 + 6(n_2+2)\sum_{2n_1+1\le r<s<t\le 2n_1+n_2}(\sigma_{st}^r)^2$$
$$+2(n_2+2)\sum_{r=2n_1+1}^{2n_1+n_2}\sum_{2n_1+1\le s<t\le 2n_1+n_2}\sigma_{ss}^r\sigma_{tt}^r$$
$$= (n_2-1)\sum_{r=2n_1+1}^{2n_1+n_2}(\sigma_{rr}^r)^2 + 3(n_2+1)\sum_{2n_1+1\le r\ne s\le 2n_1+n_2}(\sigma_{ss}^r)^2$$
$$+6(n_2+2)\sum_{2n_1+1\le r<s<t\le 2n_1+n_2}(\sigma_{st}^r)^2 - 6\sum_{r=2n_1+1}^{2n_1+n_2}\sum_{2n_1+1\le s<t\le 2n_1+n_2}\sigma_{ss}^r\sigma_{tt}^r$$
$$= 6(n_2+2)\sum_{2n_1+1\le r<s<t\le 2n_1+n_2}(\sigma_{st}^r)^2 + \sum_{2n_1+1\le s\ne r\le 2n_1+n_2}(\sigma_{rr}^r - 3\sigma_{ss}^r)^2$$
$$+3\sum_{r\ne s,t}\sum_{2n_1+1\le s<t\le 2n_1+n_2}(\sigma_{ss}^r - \sigma_{tt}^r)^2 \ge 0. \tag{4.9}$$

It follows that

$$||\sigma_{D^\perp}||^2 \geq \frac{3n_2^2}{n_2+2}||H_{D^\perp}||^2, \tag{4.10}$$

with equality holding if and only if

$$\begin{aligned} \sigma_{rr}^r &= 3\sigma_{ss}^r, \text{ for } 2n_1+1 \leq r \neq s \leq 2n_1+n_2, \\ \sigma_{st}^r &= 0, \text{ for distinct } r,s,t \in \{2n_1+1,...,2n_1+n_2\}. \end{aligned}$$

Now, by using (4.10) and (4.8), we conclude that

$$\delta(D) \leq \frac{(n+1)^2}{2}||H||^2 + \frac{n_2}{2}(4n_1+n_2-1)\frac{c}{4} + n_2 - \frac{3n_2^2}{2(n_2+2)}||H_{D^\perp}||^2,$$

i.e., the desired inequality.

The equality cases of (4.1) hold identically if the equality cases of (4.8) and (4.10) hold identically, which implies M is D-minimal (statement *(a)*) and mixed totally geodesic (statement *(b)*) and the statement *(c)* of Theorem 4.1 is satisfied.

Conversely, if we suppose that the statements *(a)*, *(b)* and *(c)* are satisfied, then we obtain from (4.9) and (4.10) the equality case of (4.1) from Theorem 4.1. □

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CONTROLLABILITY OF NEUTRAL IMPULSIVE STOCHASTIC INTEGRODIFFERENTIAL EQUATIONS DRIVEN BY A FRACTIONAL BROWNIAN MOTION WITH UNBOUNDED DELAY

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Abstract. This paper studies the controllability of neutral impulsive stochastic integrodifferential systems with infinite delay driven by fractional Brownian motion in separable Hilbert space. The controllability results is obtained by using fixed-point technique and via resolvent operator.

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1. Introduction

The concept of controllability plays a major role in both finite and infinite dimensional spaces for systems represented by ordinary differential equations and partial differential equations. One of the basic qualitative behaviors of a dynamical system is the controllability. The problem of controllability is to show the existence of control function, which steers the solution of the system from its initial state to final state, where the initial and final states may very over the entire space. Conceived by Kalman, the controllability concept has been studied extensively in the fields of finite and infinite-dimensional systems. If a system cannot be controlled completely then different types of controllability can be defined such as approximate, null, local null and local approximate null controllability. For more details the reader may refer to [5, 6, 14, 10, 15] and references therein.

On the other hand, the properties of long/short-range dependence are widely used in describing many phenomena in fields like hydrology and geophysics as well as economics and telecommunications. As extension of Brownian motion, fractional Brownian motion is a self-similar Gaussian process which has the properties of long/short-range dependence. However, fractional Brownian motion is neither a semi martingale nor a Markov process. In [2, 3, 7, 8, 19] studied the general theory for the infinite-dimensional stochastic differential equations driven by a fractional Brownian motion.

Recently, Park et al. [12] investigated the controllability of impulsive neutral integrodifferential systems with infinite delay in Banach spaces using Schauder-fixed point theorem. Very recently, [1, 4] established the existence, uniqueness and asymptotic behaviors of mild solutions to a class of impulsive neutral stochastic integrodifferential equations driven by a fractional Brownian motion with delays. Moreover, several upcoming researchers are keen interest to study the salvation of control problems in the field of stochastic systems. A through survey of literature reveals that a very little work has been done for the fractional Brownian motion in stochastic control problems. Chen [18] concerned the approximate controllability for semilinear stochastic equations with fractional Brownian motion. Several researchers reported the use of fractional Brownian motion in stochastic integrodifferential equations (see refer to [7, 8, 14] and references therein). Moreover, the controllability of neutral impulsive stochastic integrodifferential systems with infinite delay driven by a fractional Brownian motion is an untreated topic in the literature so far. Thus, we will make the first attempt to study such problem in this paper.

The goal of present research work is focus to study the controllability of neutral impulsive stochastic integrodifferential equations of the form:

$$\begin{aligned} d\big[x(t)-g(t,x_t,\int_0^t a_1(t,s,x_s))ds\big] &= A\big[x(t)-g(t,x_t,\int_0^t a_1(t,s,x_s))ds\big]dt + f(t,x_t,\int_0^t a_2(t,s,x_s)ds)dt \\ &+ Bu(t)dt + \big[\int_0^t \gamma(t-s)\big[x(s)-g(s,x_s,\int_0^s a_1(s,r,x_r)dr)\big]ds\big]dt \\ &+ \sigma(t)d\mathtt{B}^{\mathtt{H}}_Q(t),\ t\in I=[0,T],\ t\neq t_k, \end{aligned} \tag{1.1}$$

$$\Delta x\big|_{t-t_k} = x(t_k^+)-x(t_k^-) = I_k(x(t_k^-)),\ k=1,...,m,\ m\in\mathbb{N}, \tag{1.2}$$

$$x(t)=\varphi(t)\in\mathcal{L}_2^0(\Omega,\mathscr{B}_h),\ \text{for } a.e.\ t\in(-\infty,0]. \tag{1.3}$$

Here, A is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ of bounded linear operators in a Hilbert space X; $\mathtt{B}^{\mathtt{H}}$ is a fractional Brownian motion with Hurst parameter $\mathtt{H}>\frac{1}{2}$ on a real and separable Hilbert space Y; and the control function $u(\cdot)$ takes values in $\mathcal{L}^2([0,T],\mathrm{U})$, the Hilbert space of admissionble control functions for a separable Hilbert space U; and B is a bounded linear operator from U into X. The history $x_t:(-\infty,0]\to\mathrm{X}$, $x_t(\theta)=x(t+\theta)$, belongs to an abstract phase space $\mathscr{B}_h$ defined axiomatically, and $f,g:[0,T]\times\mathscr{B}_h\times\mathrm{X}\to\mathrm{X}$, $a_1,a_2:\mathscr{D}\times\mathscr{B}_h\to\mathrm{X}$, $\sigma:[0,T]\to\mathcal{L}_2^0(\mathrm{Y},\mathrm{X})$, are appropriate functions, where $\mathcal{L}_2^0(\mathrm{Y},\mathrm{X})$ denotes the space of all Q-Hilbert-Schmit operators from Y into X and $\mathscr{D}=\{(s,t)\in I\times I: s<t\}$. Moreover, the fixed moments of time t_k satisfy $0<t_1<t_2<...<t_m<T$, $x(t_k^-)$ and $x(t_k^+)$ represent the left and right limits of $x(t)$ at time t_k respectively. $\Delta x(t_k)$ denotes the jump in the state x at time t_k with $I:\mathrm{X}\to\mathrm{X}$ determining the size of the jump.

2. Preliminaries

Let $(\Omega,\Im,\mathbb{P})$ be a complete probability space. A standard fractional Brownian motion $\{\beta^{\mathtt{H}}(t),t\in\mathbb{R}\}$ with Hurst parameter $\mathtt{H}\in(0,1)$ is a zero mean Gaussian process with the covariance function

$$R_{\mathtt{H}(t,s)} = \mathbf{E}\left[\beta^{\mathtt{H}}(t)\beta^{\mathtt{H}}(s)\right] = \frac{1}{2}\left(|t|^{2\mathtt{H}}+|s|^{2\mathtt{H}}-|t-s|^{2\mathtt{H}}\right),\ t,s\in\mathbb{R}.$$

Let X and Y be two real separable Hilbert spaces and let $\mathcal{L}(\mathrm{Y},\mathrm{X})$ be the space of bounded linear operator from Y to X. Let $Q\in\mathcal{L}(\mathrm{X},\mathrm{Y})$ be an operator defined by $Qe_n=\lambda_n e_n$ with finite trace $\mathrm{tr}Q=\sum_{n=1}^{\infty}\lambda_n<\infty$. where $\lambda_n\geq 0$ $(n=1,2,...)$ are non-negative real numbers and $\{e_n\}$ $(n=1,2,...)$ is a complete orthonormal basis in Y. We define the infinite dimensional fractional Brownian motion on Y with covariance Q as

$$\mathtt{B}^{\mathtt{H}}(t)=\mathtt{B}^{\mathtt{H}}_Q(t)=\sum_{n=1}^{\infty}\sqrt{\lambda_n}e_n\beta_n^{\mathtt{H}}(t).$$

where $\beta_n^{\mathtt{H}}$ are real, independent fractional Brownian motion's. This process is Gaussian, it starts from 0, has zero mean and covariance

$$\mathbf{E}\left\langle\mathtt{B}^{\mathtt{H}}(t),x\right\rangle\left\langle\mathtt{B}^{\mathtt{H}}(s),y\right\rangle = R(s,t)\left\langle Q(x),y\right\rangle \text{ for } x,y\in\mathtt{Y} \text{ and } t,s\in[0,T]$$

Now, define the Weiner integrals with respect to the Q-fractional Brownian motion, we introduce the space $\mathcal{L}_2^0=\mathcal{L}_2^0(\mathrm{Y},\mathrm{X})$ of all Q-Hilbert-Schmidt operators $\zeta:\mathrm{Y}\to\mathrm{X}$. We recall that $\zeta\in\mathcal{L}(\mathrm{Y},\mathrm{X})$ is called a Q-Hilbert-Schmidt operator, if

$$\|\zeta\|^2_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty} \left\| \sqrt{\lambda_n}\zeta e_n \right\|^2 < \infty,$$

and that the space $\mathcal{L}_2^0$ equipped with the inner product $<\varphi,\zeta>_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty} <\varphi e_n, \zeta e_n>$ is a separable Hilbert space. Let $\phi(s) : s \in [0,T]$ be a function with values in $\mathcal{L}_2^0(\mathrm{Y},\mathrm{X})$ such that

$$\sum_{n=1}^{\infty} \left\| K^*\phi Q^{1/2} e_n \right\|^2_{\mathcal{L}_2^0} < \infty.$$

The Weiner integral of ϕ with respect to $\mathtt{B}^H$ is defined by

$$\int_0^t \phi(s) d\mathtt{B}^{\mathtt{H}} = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}\phi(s) e_n d\beta_n^{\mathtt{H}}(s). \tag{2.1}$$

Lemma 2.1. *If $\zeta : [0,T] \to \mathcal{L}_2^0(\mathrm{Y},\mathrm{X})$ satisfies $\int_0^t \|\zeta(s)\|^2_{\mathcal{L}_2^0} ds < \infty$, then (4) is well defined as an* X*-valued random variable and*

$$\mathbf{E}\left\| \int_0^t \zeta(s) d\mathtt{B}^{\mathtt{H}}(s) \right\|^2 \leq 2\mathtt{H}t^{2\mathtt{H}-1} \int_0^t \|\zeta\|^2_{\mathcal{L}_2^0} ds.$$

We assume that the phase space $\mathscr{B}_h$ is a linear space of functions mapping $(-\infty,0]$ into X, endowed with a norm $\|\cdot\|_{\mathscr{B}_h}$. First, we present the abstract phase space $\mathscr{B}_h$. Assume that $h : (-\infty,0] \to [0,+\infty)$ is a continuous function with

$$l = \int_{-\infty}^0 h(s) ds < +\infty.$$

We define the abstract phase space $\mathscr{B}_h$ by $\mathscr{B}_h = \Big\{\zeta : (-\infty,0] \to \mathrm{X}$ for any $\tau > 0$, $(\mathbf{E}\|\zeta\|^2)^{1/2}$ is bounded and measurable function $[\tau,0]$ and $\int_{-\infty}^0 h(t) \sup_{t\leq\tau\leq 0} (\mathbf{E}\|\zeta(s)\|^2)^{1/2} dt < +\infty\Big\}$. If this space with the norm

$$\|\zeta\|_{\mathscr{B}_h} = \int_{-\infty}^0 h(t) \sup_{t\leq s\leq 0} (\mathbf{E}\|\zeta\|^2)^{1/2} dt,$$

then it is clear that $(\mathscr{B}_h, \|\cdot\|_{\mathscr{B}_h})$ is a Banach space.

We now consider the space $\mathscr{B}_{\mathscr{D}I}$ [$\mathscr{D}$ and I stand for delay and impulse, respectively] given by $\mathscr{B}_{\mathscr{D}I} = \Big\{x : (-\infty,T] \to \mathrm{X} : x|I_k \in \mathscr{C}(I_k,\mathrm{X})$ and $x(t_k^+), x(t_k^-)$ exist with $x(t_k^+) - x(t_k^-), k - 1,2,...,m$ $x_0 - \varphi \in \mathscr{B}_h$ and $\sup_{0\leq t\leq T} \mathbf{E}(\|x(t)\|^2) < \infty\Big\}$, where $x|I_k$ is the restriction of x to the interval $I_k = (t_k, t_{k+1}]$, $k = 1,2,...,m$. Then the function $\|\cdot\|_{\mathscr{B}_h}$ to be a semi-norm in $\mathscr{B}_{\mathscr{D}I}$, it is defined by

$$\|x\|_{\mathscr{B}_{\mathscr{D}I}} = \|x_0\|_{\mathscr{B}_h} + \sup_{0<t<T} (\mathbf{E}(\|x(t)\|^2))^{1/2}.$$

The following lemma is a common property of phase spaces.

Lemma 2.2. *Suppose $x \in \mathscr{B}_{\mathscr{D}I}$, then for all $t \in [0,T]$, $x_t \in \mathscr{B}_h$ and*

$$l(\mathbf{E}(\|x(t)\|^2))^{\frac{1}{2}} \leq l \sup_{0\leq s\leq t} (\mathbf{E}\|x(s)\|^2)^{\frac{1}{2}} + \|x_0\|_{\mathscr{B}_h},$$

where $l = \int_{-\infty}^0 h(s) ds < \infty$.

2.1. **Partial integrodifferential equations in Banach spaces.** In the present section, we recall some definitions and properties needed in the sequel. In what follows, X will denote a Banach space, A and $\gamma(t)$ are closed linear operators on X. Y represents the Banach space $\mathscr{D}(A)$, the domain of operator A, equipped with the graph norm

$$\|y\|_{\mathrm{Y}} := \|Ay\| + \|y\| \text{ for } y \in \mathrm{Y}.$$

The notation $\mathscr{C}([0,+\infty);\mathrm{Y})$ stands for the space of all continuous functions from $[0,+\infty)$ into Y. We consider the following Cauchy problem

$$\begin{cases} v'(t) = Av(t) + \int_0^t \gamma(t-s)v(s)ds \text{ for } t \geq 0, \\ v(0) = v_0 \in \mathrm{X}. \end{cases} \tag{2.2}$$

Definition 2.3. [11] *A resolvent operator for equation (2.2) is a bounded linear operator valued function $R(t) \in \mathcal{L}(\mathrm{X})$ for $t \geq 0$, satisfying the following properties:*
(i) $R(0) = I$ and $\|R(t)\| \leq Me^{\lambda t}$ for some constants M and λ.
(ii) For each $x \in \mathrm{X}$, $R(t)x$ is strongly continuous for $t \geq 0$.
(iii) For $x \in \mathrm{Y}$, $R(\cdot)x \in \mathscr{C}^1([0,+\infty);\mathrm{X}) \bigcap \mathscr{C}([0,+\infty);\mathrm{Y})$ and

$$\begin{aligned} R^{'}(t)x &= AR(t)x + \int_0^t \gamma(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)B(s)xds \text{ for } t \geq 0. \end{aligned} \tag{2.3}$$

For additional details on resolvent operators, we refer the reader to [11]. In what follows we suppose the following assumptions:
(H1) A is the infinitesimal generator of a C_0-semigroup $(R(t))_{t\geq 0}$ on X.
(H2) For all $t \geq 0$, $\gamma(t)$ is a continuous linear operator from $(\mathrm{Y}, \|\cdot\|_{\mathrm{Y}})$ into $(\mathrm{X}, \|\cdot\|_{\mathrm{X}})$. Moreover, there exists an integrable function $\mathscr{C} : [0,+\infty) \to \mathbb{R}^+$ such that for any $y \in \mathrm{Y}$, $y \to \gamma(t)y$ belongs to $W^{1,1}([0,+\infty);\mathrm{X})$ and

$$\left\| \frac{d}{dt}\gamma(t)(t)y \right\|_{\mathrm{X}} \leq \mathscr{C}(t) \|y\|_{\mathrm{Y}} \text{ for } y \in \mathrm{Y} \text{ and } t \geq 0.$$

Theorem 2.4. *Assume that hypotheses* **(H1)** *and* **(H2)** *hold. Then equation (2.2) admits a resolvent operator $(R(t))_{t\geq 0}$.*

Theorem 2.5. *Assume that hypotheses* **(H1)** *and* **(H2)** *hold. Let $R(t)$ be a compact operator for $t > 0$. Then, the corresponding resolvent operator $R(t)$ of equation (2.2) is continuous for $t > 0$ in the operator norm, for all $t_0 > 0$, it holds that $\lim_{h\to 0} \|R(t_0+h) - R(t_0)\| = 0$.*

In the sequel, we recall some results on existence of solutions for the following integrodifferential equation

$$\begin{cases} v'(t) = Av(t) + \int_0^t \gamma(t-s)v(s)ds + q(t) \text{ for } t \geq 0, \\ v(0) = v_0 \in \mathrm{X}. \end{cases} \tag{2.4}$$

where $q : [0,+\infty[\to \mathrm{X}$ is a continuous function.

Definition 2.6. *A continuous function $v : [0,+\infty) \to \mathrm{X}$ is said to be a strict solution of equation (2.4) if*
(i) $v \in \mathscr{C}^1([0,+\infty);\mathrm{X}) \bigcap \mathscr{C}([0,+\infty);\mathrm{Y})$,
(ii) v satisfies equation (2.4) for $t \geq 0$.

Remark 2.7. *From this definition we deduce that $v(t) \in \mathscr{D}(A)$, and the function $\gamma(t-s)v(s)$ is integrable, for all $t > 0$ and $s \in [0,+\infty)$.*

Theorem 2.8. *Assume that* **(H1)**-**(H2)** *hold. If v is a strict solution of equation (2.4), then the following variation of constants formula holds*

$$v(t) = R(t)v_0 + \int_0^t R(t-s)q(s)ds \text{ for } t \geq 0.$$

Definition 2.9. *An X-valued process $\{x(t) : t \in (-\infty, T]\}$ is a mild solution of (1.1)-(1.3) if*
(i) $x(t)$ is measurable for each $t > 0$, $x(t) = \varphi(t)$ on $(\infty, 0]$,

$$\Delta x\Big|_{t-t_k} = I_k(x(t_k^-)), \ k = 1, 2, ...m$$

the restriction of $x(\cdot)$ to $[0, T] = \{t_1, t_2, ...t_m\}$ is continuous.
(ii) For every $0 \leq s \leq t$, the process x satisfies the following integral equation

$$\begin{aligned} x(t) &= R(t)\left[\varphi(0) - g(0, \varphi, 0)\right] + g(t, x_t, \int_0^t a_1(t, s, x_s)ds) + \int_0^t R(t-s)Bu(s)ds \\ &+ \int_0^t R(t-s)f(s, x_s, \int_0^s a_2(s, r, x_r)dr)ds + \int_0^t R(t-s)\sigma(s)d\mathtt{B}^{\mathtt{H}}(s) \\ &+ \sum_{0<t_k<t} R(t-t_k)I_k(x(t_k^-)), \ -\mathbb{P} \ a.s. \end{aligned} \tag{2.5}$$

3. Controllability Result

Definition 3.1. *System (1.1)-(1.3) is said to be controllable on the interval $(-\infty, T]$ if for every initial stochastic process φ defined on $(-\infty, T]$, there exists a stochastic control $u \in \mathcal{L}^2([0, T]; \mathtt{U})$ such that the mild solution $x(\cdot)$ of (1.1)-(1.3) satisfies $x(T) = x_1$.*

In order to establish the controllability of (1.1)-(1.3), we impose the following hypotheses:

(H3) There exist constants $M \geq 1$ such that $\|R(t)\|^2 \leq M$.

(H4) The mapping $g : I \times \mathscr{D} \times \mathscr{B}_h \to \mathrm{X}$ satisfies the following conditions

(i) The function $a_1 : \mathscr{D} \times \mathscr{B}_h \to \mathrm{X}$ satisfies the following condition. There exists a constant $k_1 > 0$, for $x_1, x_2 \in \mathscr{B}_h$ such that

$$\mathbf{E}\left\|\int_0^t [a_1(t, s, x_1) - a_1(t, s, x_2)]ds\right\|^2 \leq k_1 \|x_1 - x_2\|^2_{\mathscr{B}_h}, \ (t, s) \in \mathscr{D},$$

and

$$\bar{k_1} = \sup_{(t,s)\subset\mathscr{D}} \left\|\int_0^t a_1(t, s, 0)ds\right\|^2.$$

(ii) g is a continuous function and there exists constants $k_2 > 0$ such that for $x_1, x_2 \in \mathscr{B}_h$, $y_1, y_2 \in \mathrm{X}$ and satisfies for all $t \in [0, T]$

$$\mathbf{E}\|g(t, x_1, y_1) - g(t, x_2, y_2)\|^2 \leq k_2\left[\, \|x_1 - x_2\|^2_{\mathscr{B}_h} + \mathbf{E}\|y_1 - y_2\|^2 \,\right],$$

$$\lim_{t\to s} \mathbf{E}\|g(t, x_1, y_1) - g(t, x_2, y_2)\|^2 = 0.$$

and

$$\bar{k_2} = \sup_{t\subset[0,T]} \|g(t, 0, 0)\|^2.$$

(H5) The mapping $f : I \times \mathscr{B}_h \times \mathrm{X} \to \mathrm{X}$ satisfies the following Lipschitz conditions

(i) There exist positive constants $k_3, \bar{k_3}$ for $t \in [0,T]$, $x_1, x_2 \in \mathscr{B}_h$, $y_1, y_2 \in \mathrm{X}$ such that

$$\mathbf{E}\|f(t,x_1,y_1) - f(t,x_2,y_2)\|^2 \leq k_3\left[\|x_1 - x_2\|^2_{\mathscr{B}_h} + \mathbf{E}\|y_1 - y_2\|^2\right],$$

and

$$\bar{k_3} = \sup_{t\in[0,T]} \|f(t,0,0)\|^2.$$

(ii) The function $a_2 : \mathscr{D} \times \mathscr{B}_h \to \mathrm{X}$ satisfies the following condition. There exists a constant $k_k > 0$, for $x_1, x_2 \in \mathscr{B}_h$ such that

$$\mathbf{E}\left\|\int_0^t [a_2(t,s,x_1) - a_2(t,s,x_2)]ds\right\|^2 \leq k_4\|x_1 - x_2\|^2_{\mathscr{B}_h}, \ (t,s) \in \mathscr{D},$$

and

$$\bar{k_4} = \sup_{(t,s)\subset\mathscr{D}} \left\|\int_0^t a_2(t,s,0)ds\right\|^2.$$

(H6) The impulses functions I_k for $k = 1, 2, ..., m$, satisfies the following condition. There exists positive constants M_k, $\widetilde{M_k}$ such that

$$\|I_k(x) - I_k(y)\|^2 \leq M_k\|x - y\|^2 \text{ and } \|I_k(x)\|^2 \leq \widetilde{M_k} \text{ for all } x, y \in \mathscr{B}_h.$$

(H7) The function $\sigma : [0,\infty) \to \mathcal{L}_2^0(\mathrm{Y},\mathrm{X})$ satisfies

$$\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds < \infty, \text{ for } t > 0.$$

(H8) The linear operator W from U into X defined by

$$Wu = \int_0^T R(T-s)Bu(s)ds$$

has an inverse operator W^{-1} that takes values in $\mathcal{L}^2([0,T],\mathrm{U})\ker W$, where $kerW = \{x \in \mathcal{L}^2([0,T],\mathrm{U}) : Wx = 0\}$

(H9) There exists a constant $\lambda > 0$ such that

$$\lambda = 8l^2\left(1 + 3MM_bM_WT^2\right)\left[k_2(1+2k_1) + MT^2k_3(1+k_4) + mM\sum_{k=1}^m M_k\right] < 1.$$

The main result of this paper is given in the next theorem.

Theorem 3.2. *Suppose that* **(H1)**-**(H9)** *hold. Then, the system (1.1)-(1.3) is controllable on* $(-\infty, T]$ *provide that*

$$6l^2\left(1 + 7MM_bM_WT^2\right)\left[8[k_2(1+2k_1)] + 8MT^2[k_3(1+2k_4)]\right] < 1. \tag{3.1}$$

Proof. Using **(H8)** for an arbitrary function $x(\cdot)$, define the control

$$\begin{aligned} u_x(t) &= W^{-1}\Bigg[x_1 - R(T)\left[\varphi(0) - g(0,x_0,0)\right] - g(T,x_T,\int_0^T a_1(T,s,x_s)ds)) \\ &+ \int_0^T R(T-s)f(s,x_s,\int_0^s a_2(s,r,x_r)dr)ds + \int_0^T R(T-s)\sigma(s)d\mathtt{B}^{\mathtt{H}}(s) \\ &+ \sum_{0<t_k<t} R(T-t_k)I_k(x(t_k^-))\Bigg](t). \end{aligned}$$

Now, put the control $u(\cdot)$ into the stochastic control system (2.5) and obtain a nonlinear operator Γ on $\mathscr{B}_{\mathscr{D}I}$ given by

$$\Gamma(x)(t) = \begin{cases} \varphi(t), \text{ for } t \in (-\infty, 0], \\ R(t)\left[\varphi(0) - g(0,\varphi,0)\right] + g(t, x_t, \int_0^t a_1(t,s,x_s)ds) + \int_0^t R(t-s)Bu_x(s)ds \\ + \int_0^t R(t-s)f(s,x_s,\int_0^s a_2(s,r,x_r)dr)ds + \int_0^t R(t-s)\sigma(s)d\mathtt{B}^{\mathtt{H}}(s) \\ + \sum_{0<t_k<t} R(t-t_k)I_k(x(t_k^-)), \text{ if } t \in [0,T]. \end{cases}$$

Then it is clear that to prove the existence of mild solutions to equations (1.1)-(1.3) is equivalent to find a fixed point for the operator. Clearly, $\Gamma x(T) = x_1$, which means that the control u steers the system grow the initial state φ to x_1 in time T, provided we can obtain a fixed point of the operator Γ which implies that the system in controllable. Let $y : (-\infty, T] \to \mathrm{X}$ be the function defined by

$$y(t) = \begin{cases} \varphi(t), \text{ if } t \in (-\infty, 0], \\ R(t)\varphi(0), \text{ if } t \in [0,T]. \end{cases}$$

then, $y_0 = \varphi$. For each function $z \in \mathscr{B}_{\mathscr{D}I}$, set

$$x(t) \quad = \quad z(t) + y(t).$$

It is obvious that x satisfies the stochastic control system (2.5) if and only if z satisfies $z_0 = 0$ and

$$\begin{aligned} z(t) \quad &= \quad g(t, z_t + y_t, \int_0^t a_1(t,s,z_s+y_s)ds) - R(t)g(0,\varphi,o) + \int_0^t R(t-s)B_{z+y}(s)ds \\ &+ \quad \int_0^t R(t-s)f(s,z_s+y_s,\int_0^s a_2(s,r,z_r+y_r)dr)ds + \int_0^t R(t-s)\sigma(s)d\mathtt{B}^{\mathtt{H}}(s) \\ &+ \quad \sum_{0<t_k<t} R(t-t_k)I_k[z(t_k^-) - y(t_k^-)], \text{ if } t \in [0,T], \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} u_{z+y}(t) \quad &= \quad W^{-1}\Bigg[x_1 - R(T)\left[\varphi(0) - g(0, z_0+y_0, 0)\right] - g(T, z_T + y_T, \int_0^T a_1(T,s,z_s+y_s)ds) \\ &- \quad \int_0^T R(T-s)f(s,z_s+y_s,\int_0^s a_2(s,r,z_r+y_r)dr)ds - \int_0^T R(T-s)\sigma(s)d\mathtt{B}^{\mathtt{H}}(s) \\ &- \quad \sum_{0<t_k<T} R(T-t_k)I_k[z(t_k^-) + y(t_k^-)]\Bigg](t). \end{aligned}$$

Set

$$\mathscr{B}^0_{\mathscr{D}I} \quad = \quad \left\{z \in \mathscr{B}_{\mathscr{D}I} : z_0 = 0\right\},$$

for any $z \in \mathscr{B}^0_{\mathscr{D}I}$, we have

$$\|z\|_{\mathscr{B}^0_{\mathscr{D}I}} \quad = \quad \|z_0\|_{\mathscr{B}_h} + \sup_{t\in[0,T]} (\mathbf{E}\,\|z(t)\|^2)^{\frac{1}{2}} = \sup_{t\in[0,T]} (\mathbf{E}\,\|z(t)\|^2)^{\frac{1}{2}}.$$

Then, $(\mathscr{B}^0_{\mathscr{D}I}, \|\cdot\|_{\mathscr{B}^0_{\mathscr{D}I}})$ is a Banach space. Define the operator $\Theta : \mathscr{B}^0_{\mathscr{D}I} \to \mathscr{B}^0_{\mathscr{D}I}$ by

$$(3.3)(\Theta z)(t) = \begin{cases} 0 \text{ if } t \in (-\infty, 0], \\ g(t, z_t + y_t, \int_0^t a_1(t, s, z_s + y_s)ds) - R(t)g(0, \varphi, o) + \int_0^t R(t-s)B_{z+y}(s)ds \\ + \int_0^t R(t-s)f(s, z_s + y_s, \int_0^s a_2(s, r, z_r + y_r)dr)ds + \int_0^t R(t-s)\sigma(s)d\mathtt{B}^{\mathtt{H}}(s) \\ + \sum_{0<t_k<t} R(t-t_k)I_k[z(t_k^-) - y(t_k^-)], \text{ if } t \in [0, T], \end{cases}$$

Set

$$\mathscr{B}_k = \left\{ z \in \mathscr{B}^0_{\mathscr{D}I} : \|z\|^2_{\mathscr{B}^0_{\mathscr{D}I}} \leq k \right\}, \text{ for some } k \geq 0,$$

then $\mathscr{B}_k \subseteq \mathscr{B}^0_{\mathscr{D}I}$ is a bounded closed convex set, and for $z \in \mathscr{B}_k$, we have

$$\begin{aligned} \|z_t + y_t\|_{\mathscr{B}_{\mathscr{D}I}} &\leq 2\left(\|z_t\|^2_{\mathscr{B}_{\mathscr{D}\mathscr{I}}} + \|y_t\|^2_{\mathscr{B}_{\mathscr{D}\mathscr{I}}} \right) \\ &\leq 4\left(l^2 \sup_{0\leq s\leq t} \mathbf{E}\|z(s)\|^2 + \|z_0\|^2_{\mathscr{B}_h} + l^2 \sup_{0\leq s\leq t} \mathbf{E}\|y(s)\|^2 + \|y_0\|^2_{\mathscr{B}_h} \right) \\ &\leq 4l^2\left(k + M\mathbf{E}\|\varphi(0)\|^2 \right) + 4\|y\|^2_{\mathscr{B}_h} \\ &:= r^*. \end{aligned}$$

Next,

$$\begin{aligned} \mathbf{E}\|u_{z+y}\|^2 &\leq 7M_W\Bigg[\|x_1\|^2 + M\mathbf{E}\|\varphi(0)\|^2 + 2M[k_2\|y\|^2_{\mathscr{B}_h} + \bar{k_2}] + 2[k_2(1+2k_1)r^* + 2k_2\bar{k}_1 + \bar{k}_2] \\ &+ 2MT^2[k_3(1+2k_4)r^* + 2k_3\bar{k}_4 + \bar{k}_3] + 2MT^{2\mathtt{H}-1}\int_0^T \|\sigma(s)\|^2_{\mathcal{L}^0_2} ds \\ &+ mM\sum_{k=1}^m \widetilde{M}_k \Bigg] := \mathcal{G} \end{aligned} \tag{3.4}$$

and

$$\mathbf{E}\|u_{z+y} - u_{v+y}\|^2 \leq 3M_W\left[k_2(1+2k_1) + MT^2k_3(1+2k_4) + mM\sum_{k=1}^m M_k \right]\mathbf{E}\|z_t - v_t\|^2_{\mathscr{B}_h}. \tag{3.5}$$

It is clear that the operator Γ has a fixed point if and only if Θ has one, so it turns to prove that Θ has a fixed point. Since all functions involved in the operator are continuous therefore Θ is continuous. The proof will be given in following steps.

Step 1: We claim that there exists a positive number k, such that $\Theta(x) \in \mathscr{B}_k$ whenever $x \in \mathscr{B}_k$. If it is not true, then for each positive number k, there is a function $z^k(\cdot) \in \mathscr{B}_k$, but $\Theta(z^k) \notin \mathscr{B}_k$, that is

$\mathbf{E}\left\|\Theta(z^k)(t)\right\|^2 > k$ for some $t \in [0,T]$. However, on the other hand, we have

$$\begin{aligned}
k &< \mathbf{E}\left\|\Theta(z^k)(t)\right\|^2 \\
&\leq 6\Bigg[2M(k_2\|y\|^2_{\mathscr{B}_h}+\bar{k}_2)+2[k_2(1+2k_1)r^*+2k_2\bar{k_1}+\bar{k_2}]+2MT^2[k_3(1+2k_4)r^*+2k_3\bar{k_4}+\bar{k_3}] \\
&+ 2MT^{2\mathtt{H}-1}\int_0^T\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds+mM\sum_{k=1}^m\widetilde{M_k}+MM_bT^2\mathcal{G}\Bigg] \\
&\leq 6\left(1+7MM_bM_WT^2\right)\Bigg[2M(k_2\|y\|^2_{\mathscr{B}_h}+\bar{k}_2)+2[k_2(1+2k_1)r^*+2k_2\bar{k_1}+\bar{k_2}]+2MT^2[k_3(1+2k_4)r^* \\
&+ 2k_3\bar{k_4}+\bar{k_3}]+2MT^{2\mathtt{H}-1}\int_0^T\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds+mM\sum_{k=1}^m\widetilde{M_k}+MM_bT^2\mathcal{G}\Bigg] \\
&+ 7MM_bM_WT^2\left(\|x_1\|^2+M\mathbf{E}\|\varphi(0)\|^2\right) \\
&\leq \widetilde{\mathcal{G}}+6\left(1+7MM_bM_WT^2\right)\Big[2[k_2(1+2k_1)]r^*+2MT^2[k_3(1+2k_4)r^*]\Big],
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{\mathcal{G}} &= 6\left(1+7MM_bM_WT^2\right)\Bigg[2M(k_2\|y\|^2_{\mathscr{B}_h}+\bar{k}_2)+2[2k_2\bar{k_1}+\bar{k_2}]+2MT^2[2k_3\bar{k_4}+\bar{k_3}] \\
&+ 2MT^{2\mathtt{H}-1}\int_0^T\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds+mM\sum_{k=1}^m\widetilde{M_k}+7MM_bM_WT^2\left(\|x_1\|^2+M\mathbf{E}\|\varphi(0)\|^2\right)\Bigg]
\end{aligned}$$

is independent of k. Dividing both sides by k and taking the limit as $k\to\infty$, we get

$$6l^2\left(1+7MM_bM_WT^2\right)\Big[8[k_2(1+2k_1)]+8MT^2[k_3(1+2k_4)]\Big] \geq 1. \tag{3.6}$$

This contradicts (3.1). Hence for some positive k,

$$(\Theta)(\mathscr{B}_k) \subseteq \mathscr{B}_k.$$

Step 2: Θ is a contraction. Let $t\in[0,T]$ and $z^1,z^2\in\mathscr{B}^0_{\mathscr{D}I}$, we have

$$\begin{aligned}
&\mathbf{E}\left\|\Theta z^1(t)-\Theta z^2(t)\right\|^2 \\
&\leq 4\mathbf{E}\left\|\int_0^t R(t-s)B[u_{z^1+y}(s)-u_{z^2+y}(s)]ds\right\|^2 \\
&+4\mathbf{E}\left\|\sum_{0<t_k<t}R(T-t_k)[I_k(z^1(t_k^-)+y(t_k^-))-I_k(z^2(t_k^-)+y(t_k^-))]\right\|^2 \\
&+4\mathbf{E}\left\|g(t,z_t^1+y_t,\int_0^t a_1(t,s,z_s^1+y_s)ds)-g(t,z_t^2+y_t,\int_0^t a_1(t,s,z_s^2+y_s)ds)\right\|^2 \\
&+4\mathbf{E}\left\|\int_0^t R(t-s)[f(s,z_s^1+y_s,\int_0^s a_2(s,r,z_r^1+y_r)dr)-f(s,z_s^2+y_s,\int_0^s a_2(s,r,z_r^2+y_r)dr)]ds\right\|^2
\end{aligned}$$

On the other hand from **(H1)-(H9)** combined with (3.4), we obtain

$$\begin{aligned}\mathbf{E}\left\|\Theta z^1(t)-\Theta z^2(t)\right\|^2 &\leq 4\left(1+3MM_bM_WT^2\right)\left[k_2(1+2k_1)+MT^2k_3(1+k_4)+mM\sum_{k=1}^{m}M_k\right]\mathbf{E}\left\|z_t^1-z_t^2\right\|^2\\ &\leq 8\left(1+3MM_bM_WT^2\right)\left[k_2(1+2k_1)+MT^2k_3(1+k_4)+mM\sum_{k=1}^{m}M_k\right]\\ &\quad\times\left\{l^2\sup_{0\leq s\leq t}\mathbf{E}\left\|z^1(s)-z^2(s)\right\|^2+\left\|z_0^1-z_0^2\right\|^2_{\mathscr{B}_h}\right\}\\ &\leq \lambda\sup_{0\leq s\leq T}\mathbf{E}\left\|z^1(s)-z^2(s)\right\|^2 \text{ since } (z_0^1=z_0^2=0)\end{aligned}$$

Taking supremum over t,

$$\left\|\Theta z^1-\Theta z^2\right\|_{\mathscr{B}_{\mathscr{D}I}} \leq \lambda\left\|z^1-z^2\right\|_{\mathscr{B}_{\mathscr{D}I}},$$

where

$$\lambda = 8l^2\left(1+3MM_bM_WT^2\right)\left[k_2(1+2k_1)+MT^2k_3(1+k_4)+mM\sum_{k=1}^{m}M_k\right].$$

By condition **(H9)**, we have $\lambda<1$, hence Θ is a contraction mapping on $\mathscr{B}^0_{\mathscr{D}I}$ and therefore has a unique fixed point, which is a mild solution of equation (1.1)-(1.3) on $(-\infty,T]$. Clearly, $(\Theta x)(T)=x_1$ which implies that the system (1.1)-(1.3) is controllable on $(-\infty,T]$. This complete the proof. □

Remark 3.3. *When the impulses disappear, that is $M_k=\widetilde{M_k}=0$, $k=1,2,...,m$ then the system (1.1)-(1.3) reduces to the following neutral stochastic integrodifferential equation:*

$$\begin{aligned}d\left[x(t)-g(t,x_t,\int_0^t a_1(t,s,x_s))\right] &= \left[Ax(t)-g(t,x_t,\int_0^t a_1(t,s,x_s))ds\right]dt+f(t,x_t,\int_0^t a_2(t,s,x_s)ds)dt\\ &+ Bu(t)dt+\left[\int_0^t\gamma(t-s)\left[x(s)-g(s,x_s,\int_0^s a_1(s,r,x_r)dr)\right]ds\right]dt\\ &+ \sigma(t)d\mathtt{B}^{\mathtt{H}}_Q(t),\ t\in I=[0,T],\ t\neq t_k, \qquad (3.7)\\ x(t) &= \varphi(t)\in\mathcal{L}^0_2(\Omega,\mathscr{B}_h),\ \textit{for a.e. } t\in(-\infty,0]. \qquad (3.8)\end{aligned}$$

where the operator A,g,f,a_1,a_2 and σ are defined as same as before. Here $\mathscr{C}=\{x:(-\infty,T]\to \mathrm{X}: x(t)$ is continuous$\}$, Banach space of all stochastic processes $x(t)$ from $(-\infty,T]$ into $\mathscr{X}$, equipped with the supremum norm

$$\|\phi\|^2_{\mathscr{C}} = \sup_{s\in(-\infty,T]}\mathbf{E}\|\phi(s)\|^2,\ \textit{for } \phi\in\mathscr{C}.$$

By using the same technique in Theorem 3.2, we can easily deduce the following corollary.

Corollary 3.4. *Suppose that* **(H1)-(H9)** *hold. Then, the system (3.7)-(3.8) is controllable on $(-\infty,T]$ provide that the condition (3.1) is satisfied.*

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PROPERTIES OF BINOMIAL TRANSFORMS k-JACOBSTHAL LUCAS SEQUENCE

Ş. UYGUN

ABSTRACT. In this study, we define the binomial, $k-$binomial, rising, and falling transforms for $k-$Jacobsthal Lucas sequence. We investigate some properties of these sequence such as recurrence relations, Binet′s formula, generating functions. In the sequel of this paper, Pascal Jacobsthal Lucas triangles for all binomial transformation sequences are denoted.

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1. INTRODUCTION

In the literature, there are a lot of integer sequences, which are used in almost every field of modern sciences. The oldest and the most popular sequence is called Fibonacci sequence. It is important, because the equality of the proportion of consecutive two terms is "Golden Ratio". The mathematicians have been defined and studied special integer sequences from both algebraic and combinatorial prospectives in recent years. One of these sequence is called Jacobsthal Lucas and defined by $c_n = c_{n-1} + 2c_{n-2}$ for $n \geq 2$, beginning with the values $c_0 = 2,\ c_1 = 1$ in [1]. In this paper we use k-Jacobsthal Lucas sequence $\{c_{k,n}\}_{n\in\mathbb{N}}$. It is defined by the recurrence formula $c_{k,n} = kc_{k,n-1} + 2c_{k,n-2},\ \ c_{k,0} = 2,\ c_{k,1} = k$ for $n \geq 2$ and $k \geq 0$ natural numbers. The Binet formula for the k-Jacobsthal Lucas sequence is demonstrated by $c_{k,n} = x_1^n + x_2^n$, where $x_1 = \frac{k+\sqrt{k^2+8}}{2},\ \ x_2 = \frac{k-\sqrt{k^2+8}}{2}$. x_1 and x_2 are the roots of the characteristic equation of the recurrence formula. You can see detailed information about k-Jacobsthal Lucas sequence in [4] .

In the literature Prodinger investigated some properties about the binomial transformation in [5]. Chen found identities about the binomial transform in [6]. Falcon and Plaza gave the properties of k-Fibonacci in [2], k-Fibonacci sequence are defined and the binomial transform of k-Fibonacci sequence are studied in [7]. The authors gave the binomial transform of the k-Lucas sequence in [8]. As a final study of this section, we note that in [9] , some properties of the binomial transform of k-Jacobsthal sequence have been investigated.

This paper represents an investigation about the binomial transform of k-Jacobsthal Lucas sequence and its variations such as k-binomial transform, rising k-binomial transform, falling k-binomial transforms. As a consequence of these transforms we find recurrence relations, generating functions, Binet formulas, sum formulas for these sequences.

2. Binomial Transform of k-Jacobsthal Lucas Sequences

Definition 2.1. *The binomial transform of k-Jacobsthal Lucas sequence* $\{b_{k,n}\}_{n\in\mathbb{N}}$ *is given by*

$$b_{k,n} = \sum_{i=0}^{n} \binom{n}{i} c_{k,i} \tag{2.1}$$

for any integer $k > 0$.

Proposition 2.2. *Let* $n \geq 1$ *be integer, then the terms of binomial transform of k-Jacobsthal Lucas sequence can be denoted by using k-Jacobsthal Lucas sequence as*

$$b_{k,n+1} = \sum_{i=0}^{n} \binom{n}{i} (c_{k,i} + c_{k,i+1}). \tag{2.2}$$

Proof. By using the properties $\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}$ and $\binom{n}{n+1} = 0$ we get

$$\begin{aligned} b_{k,n+1} &= \sum_{i=0}^{n+1} \binom{n+1}{i} c_{k,i} = 2 + \sum_{i=1}^{n+1} \left[\binom{n}{i} + \binom{n}{i-1} \right] c_{k,i} \\ &= 2 + \sum_{i=1}^{n} \binom{n}{i} c_{k,i} + \sum_{i=0}^{n} \binom{n}{i} c_{k,i+1} \\ &= \sum_{i=0}^{n+1} \binom{n}{i} c_{k,i} + \sum_{i=0}^{n+1} \binom{n}{i} c_{k,i+1}. \end{aligned}$$

□

Theorem 2.3. *The recurrence relation for the binomial transform of k-Jacobsthal Lucas sequence is denoted by*

$$b_{k,n+1} = (k+2) b_{k,n} + (1-k) b_{k,n-1}. \tag{2.3}$$

Proof. By using (2.1), we get the initial conditions as $b_{k,0} = 0$ and $b_{k,1} = 1$. By (2.1) and (2.2), we get

$$\begin{aligned} b_{k,n+1} &= \sum_{i=1}^{n} \binom{n}{i} (c_{k,i} + c_{k,i+1}) + (c_{k,0} + c_{k,1}) \\ &= \sum_{i=1}^{n} \binom{n}{i} ((k+1) c_{k,i} + 2 c_{k,i-1}) + (c_{k,0} + c_{k,1}) \\ &= \sum_{i=1}^{n} \binom{n}{i} (k+1) c_{k,i} + 2 \sum_{i=1}^{n} \binom{n}{i} c_{k,i-1} + (k+2) \end{aligned}$$

$$b_{k,n+1} = (k+1) b_{k,n} + 2 \sum_{i=1}^{n} \binom{n}{i} c_{k,i-1} - k. \tag{2.4}$$

If we substitute in (2.4) for n in place of $n+1$,we have

$$\begin{aligned} b_n &= (k+1)b_{n-1} + 2\sum_{i=1}^{n-1}\binom{n-1}{i}c_{k,i-1} - k \\ b_n &= kb_{n-1} + \sum_{i=0}^{n-1}\binom{n-1}{i}c_{k,i} + 2\sum_{i=1}^{n-1}\binom{n-1}{i}c_{k,i-1} - k \\ b_n &= kb_{n-1} + \sum_{i=1}^{n}\binom{n-1}{i-1}c_{k,i-1} + 2\sum_{i=1}^{n-1}\binom{n-1}{i}c_{k,i-1} - k \end{aligned}$$

If we take int oaccount that $\binom{n-1}{n}$=0, then we satisfy

$$\begin{aligned} b_{k,n} &= kb_{k,n-1} - k \\ &+\sum_{i=1}^{n}\left[\binom{n-1}{i-1} + 2\binom{n-1}{i} + 2\binom{n-1}{i-1} - 2\binom{n-1}{i-1}\right]c_{k,i-1} \\ &= kb_{k,n-1} + \sum_{i=1}^{n}\left[-\binom{n-1}{i-1} + 2\binom{n}{i}\right]c_{k,i-1} - k. \end{aligned}$$

By (2.4), we get

$$b_{k,n} = kb_{k,n-1} - \sum_{i=0}^{n-1}\binom{n-1}{i}c_{k,i} + (b_{k,n+1} - (k+1)b_{k,n} + k) - k.$$

So, the proof is completed:

$$b_{k,n} = (k-1)b_{k,n-1} + b_{k,n+1} - (k+1)b_{k,n}.$$

By (2.4), we get

$$b_{k,n} = kb_{k,n-1} - \sum_{i=0}^{n-1}\binom{n-1}{i}c_{k,i} + (b_{k,n+1} - (k+1)b_{k,n} + k) - k.$$

So, the proof is completed:

$$b_{k,n} = (k-1)b_{k,n-1} + b_{k,n+1} - (k+1)b_{k,n}.$$

□

Theorem 2.4. *Any terms of the binomial transform of k-Jacobsthal Lucas sequence can be calculated by means of the Binet formula. It is demonstrated by*

$$b_{k,n} = b_1^n + b_2^n \tag{2.5}$$

where $b_1 = \frac{k+2+\sqrt{k^2+8}}{2}$, $b_2 = \frac{k+2-\sqrt{k^2+8}}{2}$.
The roots satisfies the following relations

$$b_1 + b_2 = k+2, \qquad b_1 - b_2 = \sqrt{k^2+8}, \qquad b_1.b_2 = k-1.$$

Proof. The characteristic polynomial equation of the recurrence formula (2.3) is $x^2-(k+2)x+(k-1)=0$, whose solutions are b_1 and b_2. The Binet formula is given by $b_{k,n} = c_1b_1^n+c_2b_2^n$. For $n=0$, $b_{k,0} = c_1+c_2 = 2$ and for $n=1$, $b_{k,1} = c_1b_1 + c_2b_2 = k$. By these equalities $c_1 = c_2 = 1$. □

Theorem 2.5. *Let us demonstrate the generating function as* $b_k(x) = b_{k,0} + b_{k,1}x + b_{k,2}x^2 + \ldots$ *Then, we get*

$$b_k(x) = \frac{2 - x(k+2)}{1-(k+2)x+(1-k)x^2}. \tag{2.6}$$

Proof. If we multiply the equality by -$(k+2)x$ and $(1-k)x^2$, we get

$$\begin{aligned} -(k+2)xb_k(x) &= -(k+2)xb_{k,0}-(k+2)x^2b_{k,1}+... \\ (1-k)x^2b_k(x) &= (1-k)x^2b_{k,0}+(1-k)x^3b_{k,1}+... \end{aligned}$$

By the equalities and the recurrence relation (2.3), it is obtained that

$$\begin{aligned} \left[1-(k+2)x+(1-k)x^2\right]b_k(x) &= b_{k,0}+x(b_{k,1}-(k+2)b_{k,0})+0 \\ &= 2+x(k+2-2k-4) \end{aligned}$$

So, the generating function for binomial transform of k-Jacobsthal Lucas sequence is obtained. □

Theorem 2.6. *The combinatorial formula for the binomial transform of k-Jacobsthal Lucas sequence is obtained by*

$$b_{k,n}=\sum_{i=0}^{\lfloor \frac{n}{2}\rfloor}\frac{1}{2^{n-1}}\binom{n}{2i}(k+2)^{n-2i}(k^2+8)^i. \tag{2.7}$$

Theorem 2.7. *Let n is a positive integer. Then the sum of the binomial transform of k-Jacobsthal Lucas sequence is given as*

$$\sum_{i=0}^{p-1}b_{k,mi+n}=\frac{b_{k,n}-(k-1)^nb_{k,m-n}-b_{k,mp+n}+[b_{k,m(p-1)+n}](k-1)^m}{1-b_{k,m}+(k-1)^m} \tag{2.8}$$

Proof.

$$\begin{aligned} \sum_{i=0}^{p-1}b_{k,mi+n} &= \sum_{i=0}^{p-1}b_1^{mi+n}+b_2^{mi+n}=b_1^n\sum_{i=0}^{p-1}b_1^{mi}+b_2^n\sum_{i=0}^{p-1}b_2^{mi} \\ &= b_1^n\left(\frac{1-b_1^{mp}}{1-b_1^m}\right)+b_2^n\left(\frac{1-b_2^{mp}}{1-b_2^m}\right) \\ &= \frac{(b_1^n+b_2^n)-b_2^mb_1^n-b_1^mb_2^n-b_1^{mp+n}-b_2^{mp+n}+b_2^{mp+n}b_1^m+b_1^{mp+n}b_2^m}{1-(b_1^m+b_2^m)+(k-1)^m} \\ &= \frac{(b_1b_2)^n[b_2^{m-n}+b_1^{m-n}]-(b_1b_2)^m[b_2^{m(p-1)+n}+b_1^{m(p-1)+n}]}{1-b_{k,m}+(k-1)^m} \\ &= \frac{b_{k,n}-(k-1)^nb_{k,m-n}-b_{k,mp+n}+[b_{k,m(p-1)+n}](k-1)^m}{1-b_{k,m}+(k-1)^m} \end{aligned}$$

□

Triangle of the binomial transform of the k-Jacobsthal Lucas sequence

In this part we introduce a new triangle of numbers for each k by using the following rules: The elements of the left diagonal of the triangle consist of the elements of the k- Jacobsthal Lucas sequences. The other elements of the triangle are the sum of the number to its left and the number diagonally above it to the left. On the right diagonal is the binomial transform of the k-Jacobsthal Lucas sequence.

For example the following triangle is for 3-Jacobsthal Lucas sequence and its binomial transform:

$$\begin{array}{ccccccccc} &&&& 2 &&&& \\ &&& 3 && 5 &&& \\ && 13 && 16 && 21 && \\ & 45 && 58 && 74 && 95 & \\ 161 && 206 && 264 && 338 && 433 \end{array}$$

3. The $k-$Binomial Transforms of k-Jacobsthal Lucas Sequences

Definition 3.1. *The k-binomial transform of the $k-$Jacobsthal Lucas sequence $\{w_{k,n}\}_{n\in N}$ is denoted by*

$$w_{k,n} = \sum_{i=0}^{n} \binom{n}{i} k^n c_{k,i}. \tag{3.1}$$

We can see $\{w_{1,n}\}_{n\in N} = \{b_{k,n}\}_{n\in N}$, and $w_{k,n} = k^n b_{k,n},\ \ w_{k,0} = 2,\ \ w_{k,1} = 2k + k^2$.

Proposition 3.2. *The $k-$binomial transform of the $k-$Jacobsthal Lucas sequence has the following property*

$$w_{k,n+1} = \sum_{i=0}^{n} \binom{n}{i} k^{n+1}(c_{k,i} + c_{k,i+1}). \tag{3.2}$$

Proof.

$$\begin{aligned}
w_{k,n+1} &= \sum_{i=0}^{n+1} \binom{n+1}{i} k^{n+1} c_{k,i} = 2k + \sum_{i=1}^{n+1} \left[\binom{n}{i} + \binom{n}{i-1} \right] k^{n+1} c_{k,i} \\
&= 2k + \sum_{i=1}^{n} k^{n+1} \binom{n}{i} c_{k,i} + \sum_{i=0}^{n} k^{n+1} \binom{n}{i} c_{k,i+1} \\
&= \sum_{i=0}^{n} \binom{n}{i} k^{n+1}(c_{k,i} + c_{k,i+1}).
\end{aligned}$$

□

Theorem 3.3. *The recurrence relation for the $k-$binomial transform of the $k-$Jacobsthal Lucas sequence*

$$w_{k,n+1} = k(k+2)w_{k,n} - k^2(k-1)w_{k,n-1}. \tag{3.3}$$

Proof. By (3.1) the initial conditions are $w_{k,0} = 2$ and $w_{k,1} = k(k+2)$ By using Proposition 3.2, we obtain

$$\begin{aligned}
w_{k,n+1} &= \sum_{i=0}^{n} \binom{n}{i} k^{n+1}(c_{k,i} + c_{k,i+1}) \\
&= (k+2)k^{n+1} + \sum_{i=1}^{n} \binom{n}{i} k^{n+1}(c_{k,i} + c_{k,i+1}) \\
&= (k+2)k^{n+1} + (k+1)\sum_{i=1}^{n} \binom{n}{i} k^{n+1} c_{k,i} + 2\sum_{i=1}^{n} \binom{n}{i} k^{n+1} c_{k,i-1}.
\end{aligned}$$

$$w_{k,n+1} = -k^{n+2} + k(k+1)w_{k,n} + 2\sum_{i=1}^{n} \binom{n}{i} k^{n+1} c_{k,i-1}. \tag{3.4}$$

If we write the equality (3.4) again for n in place of $n+1$

$$\begin{aligned}
w_{k,n} &= k(k+1)w_{k,n-1} + 2\sum_{i=1}^{n-1}\binom{n-1}{i}k^n c_{k,i-1} - k^{n+1} \\
&= k^2 w_{k,n-1} + k\left[\sum_{i=0}^{n-1}\binom{n-1}{i}k^{n-1}c_{k,i}\right] + 2\sum_{i=1}^{n-1}\binom{n-1}{i}k^n c_{k,i-1} - k^{n+1} \\
&= k^2 w_{k,n-1} + \left[\sum_{i=1}^{n}\binom{n-1}{i-1}k^n c_{k,i-1}\right] + 2\sum_{i=1}^{n-1}\binom{n-1}{i}k^n c_{k,i-1} - k^{n+1}
\end{aligned}$$

$$\begin{aligned}
w_{k,n} &= k^2 w_{k,n-1} + \sum_{i=1}^{n}\left[2\binom{n-1}{i} + \binom{n-1}{i-1}\right]k^n c_{k,i-1} - k^{n+1} \\
&= k^2 w_{k,n-1} - k^{n+1} \\
&\quad + \sum_{i=1}^{n}\left[2\binom{n-1}{i} + \binom{n-1}{i-1} + 2\binom{n-1}{i-1} - 2\binom{n-1}{i-1}\right]k^n c_{k,i-1} \\
&= k^2 w_{k,n-1} + \sum_{i=1}^{n}\left[-\binom{n-1}{i-1} + 2\binom{n}{i}\right]k^n c_{k,i-1} - k^{n+1} \\
&= k^2 w_{k,n-1} + 2\sum_{i=1}^{n}\binom{n}{i}k^n c_{k,i-1} - \sum_{i=0}^{n-1}\binom{n-1}{i}k^n c_{k,i} - k^{n+1}
\end{aligned}$$

$$w_{k,n} = k(k-1)w_{k,n-1} + 2\sum_{i=1}^{n}\binom{n}{i}k^n c_{k,i-1} - k^{n+1} \tag{3.5}$$

By substituting the above equality (3.4) into (3.5), we get

$$\begin{aligned}
w_{k,n} &= k(k-1)w_{k,n-1} + 2\sum_{i=1}^{n}\binom{n}{i}k^n c_{k,i-1} - k^{n+1} \\
&= k(k-1)w_{k,n-1} + (w_{k,n+1} - k(k+1)w_{k,n} + k^{n+2})/k - k^{n+1} \\
w_{k,n} &= k(k-1)w_{k,n-1} + w_{k,n+1}/k - (k+1)w_{k,n}.
\end{aligned}$$

The proof is found by doing after some calculations as

$$w_{k,n+1} = k(k+2)w_{k,n} - k^2(k-1)w_{k,n-1}.$$

□

Theorem 3.4. *(Binet formula) The characteristic equation of recurrence formula (3.3) is $x^2 - k(k+2)x + k^2(k-1) = 0$. The roots are $w_1 = k\frac{k+2+\sqrt{k^2+8}}{2}$ and $w_2 = k\frac{k+2-\sqrt{k^2+8}}{2}$. Any terms of the $k-$binomial transform of k-Jacobsthal Lucas sequence are denoted by using the Binet formula as the following*

$$w_{k,n} = w_1^n + w_2^n. \tag{3.6}$$

The roots satisfies the following relations:

$$w_1 + w_2 = k(k+2), \qquad w_1 - w_2 = k\sqrt{k^2+8}, \qquad w_1.w_2 = k^2(k-1).$$

Theorem 3.5. *(Generating function) Let us demonstrate the generating function as* $w_k(x) = w_{k,0} + w_{k,1}x + w_{k,2}x^2 + ...$ *The generating function for the* $k-$*binomial transform of k-Jacobsthal Lucas sequence is obtained as*

$$w_k(x) = \frac{2 - x\,[k(k+2)]}{1 - k(k+2)x + k^2(1-k)x^2}. \tag{3.7}$$

Proof. If we multiply $w_k(x)$ by$-k(k+2)x$ and $k^2(k-1)x^2$,

$$\begin{aligned} -k(k+2)xw_k(x) &= -k(k+2)xw_{k,0} - k(k+2)x^2w_{k,1} + ... \\ k^2(k-1)x^2w_k(x) &= k^2(k-1)x^2w_{k,0} + k^2(k-1)x^3w_{k,1} + ... \end{aligned}$$

From these equalities and the recurrence relation (3.3), $w_{k,0} = 2,\ \ w_{k,1} = 2k + k^2$, we have

$$w_k(x) = \frac{2 - x\,[k(k+2)]}{1 - k(k+2)x + k^2(k-1)x^2}.$$

□

Theorem 3.6. *The combinatorial formula for the* $k-$*binomial transform of k-Jacobsthal Lucas sequence is*

$$w_{k,n} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{k^n}{2^{n-1}} \binom{n}{2i} (k+2)^{n-2i}(k^2+8)^i. \tag{3.8}$$

Theorem 3.7. *Assume that* n *is a positive integer. Then the sum of the* $k-$*binomial transform of k-Jacobsthal Lucas sequence is given as*

$$\sum_{i=0}^{p-1} w_{k,mi+n} = \frac{w_{k,n} - k^2(k-1)^n w_{k,m-n} - w_{k,mp+n} + [w_{k,m(p-1)+n}]k^{2m}(k-1)^m}{1 - w_{k,m} + k^{2m}(k-1)^m} \tag{3.9}$$

Triangle of the k$-$ binomial transform of the k-Jacobsthal Lucas sequence

In this part we introduce a new triangle of numbers for each k by using the following rules: The elements of the left diagonal of the triangle consist of the elements of the k- Jacobsthal Lucas sequences. The other elements of the triangle are k times the sum of the number to its left and the number diagonally above it to the left. On the right diagonal is the $k-$binomial transform of the k-Jacobsthal Lucas sequence.

For example the following triangle is for 3-Jacobsthal Lucas sequence and its 3- binomial transform:

2
3 15
13 48 189
45 174 666 2565
161 618 2376 9126 35073

4. The Rising $k-$Binomial Transform of the k-Jacobsthal Lucas Sequence

Definition 4.1. *The rising k-binomial transform of the* $k-$*Jacobsthal Lucas sequence* $\{r_{k,n}\}_{n \in N}$ *is demonstrated by*

$$r_{k,n} = \sum_{i=0}^{n} \binom{n}{i} k^i c_{k,i}. \tag{4.1}$$

Theorem 4.2. *(Binet Formula) The Binet formula for the rising k-binomial transform of the $k-$Jacobsthal Lucas sequence is*

$$r_{k,n} = (r_1^2 - 1)^n + (r_2^2 - 1)^n \tag{4.2}$$

where $r_1 = \frac{k+\sqrt{k^2+8}}{2}$, $r_2 = \frac{k-\sqrt{k^2+8}}{2}$.

The roots satisfies the following relations

$$r_1 + r_2 = k, \quad r_1 - r_2 = \sqrt{k^2+8}, \quad r_1.r_2 = -2.$$

Proof.

$$\begin{aligned} r_{k,n} &= \sum_{i=0}^{n} \binom{n}{i} k^i c_{k,i} \\ &= \sum_{i=0}^{n} \binom{n}{i} k^i (r_1^i + r_2^i) \\ &= \sum_{i=0}^{n} \binom{n}{i} \left[(r_1 k)^i + (r_2 k)^i\right] \\ &= (r_1 k + 1)^n + (r_2 k + 1)^n \end{aligned}$$

By the property of $kr_1 + 2 = r_1^2$, we get the proof as

$$r_{k,n} = (r_1^2 - 1)^n + (r_2^2 - 1)^n.$$

□

Theorem 4.3. *Let $n \geqslant 1$,and natural number, then the recurrence sequence of the rising k-binomial transform of the $k-$Jacobsthal Lucas sequence is*

$$r_{k,n+1} = (k^2 + 2) r_{k,n} - (1 - k^2) r_{k,n-1}, \tag{4.3}$$

with the initial conditions $r_{k,0} = 2$ and $r_{k,1} = 2 + k^2$.

Proof. By using (4.2) and $k\alpha + 2 = \alpha^2$,

$$\begin{aligned} (k^2+2) r_{k,n} - (1-k^2) r_{k,n-1} &= (k^2+2)[(\alpha^2-1)^n + (\beta^2-1)^n] \\ &\quad -(1-k^2)[(\alpha^2-1)^{n-1} + (\beta^2-1)^{n-1}] \\ &= (\alpha k+1)^{n-1}[\alpha k^3 + 2k^2 + 2\alpha k + 1] \\ &\quad +(\beta k+1)^{n-1}[\beta k^3 + 2k^2 + 2\beta k + 1] \\ &= (\alpha k+1)^{n-1}[k^2(\alpha k+2) + 2\alpha k + 1] \\ &\quad +(\beta k+1)^{n-1}[k^2(\beta k+2) + 2\beta k + 1] \\ &= (\alpha k+1)^{n-1}(\alpha k+1)^2 + (\beta k+1)^{n-1}(\beta k+1)^2 \\ &= (\alpha k+1)^{n+1} + (\beta k+1)^{n+1} = r_{k,n+1}. \end{aligned}$$

□

Theorem 4.4. *(Generating function) The generating function of the rising k-binomial transform of the $k-$Jacobsthal Lucas sequence $R_k(x)$ obtained as*

$$R_k(x) = \frac{2 - x\left[k^2+2\right]}{1 - (k^2+2)x + (1-k^2)x^2}. \tag{4.4}$$

Proof. By following same procedure with Theorem 3.5, we have the result. □

Theorem 4.5. *The combinatorial formula for the rising k-binomial transform of k-Jacobsthal Lucas sequence is obtained as*

$$r_{k,n}=\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}\frac{1}{2^{n-1}}\binom{n}{2i}k^{n-2i}(k^2+8)^i. \tag{4.5}$$

Theorem 4.6. *Assume that n is a positive integer. Then the sum of the rising k-binomial transform of k-Jacobsthal Lucas sequence is given as*

$$\sum_{i=0}^{p-1}r_{k,mi+n}=\frac{r_{k,n}-(-2)^n r_{k,m-n}-r_{k,mp+n}+[r_{k,m(p-1)+n}](-2)^m}{1-r_{k,m}+(-2)^m} \tag{4.6}$$

Triangle of the rising $k-$ binomial transform of the k-Jacobsthal Lucas sequence

In this part weintroduce a new triangle of numbers for each k by using the following rules: The elements of the left diagonal of the triangle consist of the elements of the $k-$ Jacobsthal Lucas sequences. The other elements of the triangle are the sum of k- times the sum of k- times the number to its left and the number diagonally above it to the left. On the right diagonal is the rising $k-$ binomial transform of the k-Jacobsthal Lucas sequence.

For example the following triangle is for rising 3-Jacobsthal Lucas sequence and its 3- binomial transform

2

3 11

13 42 137

45 146 480 1577

161 528 1730 5670 18587

5. The Falling k-Binomial Transform of the $k-$Jacobsthal Lucas Sequence

Definition 5.1. *Assume that k is any positive integer. The falling k-binomial transform of the $k-$Jacobsthal Lucas sequence $\{f_{k,n}\}_{n\in N}$ is defined as*

$$f_{k,n}=\sum_{i=0}^{n}\binom{n}{i}k^{n-i}c_{k,i}. \tag{5.1}$$

Proposition 5.2. *The falling $k-$binomial transform of the $k-$Jacobsthal Lucas sequence has thefollowingrelation*

$$f_{k,n+1}=\sum_{i=0}^{n}\binom{n}{i}k^{n-i}(kc_{k,i}+c_{k,i+1}) \tag{5.2}$$

with the initial conditions $f_{k,0}=2$ and $f_{k,1}=3k$.

Proof.

$$\begin{aligned} f_{k,n+1} &= \sum_{i=0}^{n+1}\binom{n+1}{i}k^{n+1-i}c_{k,i} \\ &= 2k^{n+1}+\sum_{i=1}^{n+1}\binom{n}{i}k^{n+1-i}c_i+\sum_{i=1}^{n+1}\binom{n}{i-1}k^{n+1-i}c_i \\ &= \sum_{i=0}^{n}\binom{n}{i}k^{n+1-i}c_i+\sum_{i=0}^{n}\binom{n}{i}k^{n-i}c_{i+1} \end{aligned}$$

□

Theorem 5.3. *The following recurrence relation is verified by the falling $k-$binomial transform of the $k-$Jacobsthal Lucas sequence*

$$f_{k,n+1} = 3kf_{k,n} - 2(k^2-1)f_{k,n-1}. \tag{5.3}$$

Proof. By using Proposition 5.2 and (2.2), we obtain

$$\begin{aligned} f_{k,n+1} &= \sum_{i=0}^{n}\binom{n}{i}k^{n-i}(kc_{k,i}+c_{k,i+1}) = 3k^{n+1} + \sum_{i=1}^{n}\binom{n}{i}k^{n-i}(kc_{k,i}+c_{k,i+1}) \\ &= 3k^{n+1} + 2k\sum_{i=1}^{n}\binom{n}{i}k^{n-i}c_{k,i} + 2\sum_{i=1}^{n}\binom{n}{i}k^{n-i}c_{k,i-1} \end{aligned}$$

$$f_{k,n+1} = 2kf_{k,n} - k^{n+1} + 2\sum_{i=1}^{n}\binom{n}{i}k^{n-i}c_{k,i-1}. \tag{5.4}$$

If we write this equality again for n in place of $n+1$, we get

$$\begin{aligned} f_{k,n} &= 2kf_{k,n-1} + 2\sum_{i=1}^{n-1}\binom{n-1}{i}k^{n-1-i}c_{k,i-1} - k^n \\ &= kf_{k,n-1} + \left[\sum_{i=0}^{n-1}\binom{n-1}{i}k^{n-i}c_{k,i}\right] + 2\sum_{i=1}^{n-1}\binom{n-1}{i}k^{n-1-i}c_{k,i-1} - k^n \\ &= kf_{k,n-1} - k^n \\ &\quad + \left[\sum_{i=1}^{n}\binom{n-1}{i-1}k^{n+1-i}c_{k,i-1}\right] + 2\sum_{i=1}^{n-1}\binom{n-1}{i}k^{n-1-i}c_{k,i-1}. \end{aligned}$$

If we take into account that $\binom{n-1}{n} = 0$, it is obtained that

$$\begin{aligned} f_{k,n} &= kf_{k,n-1} + \sum_{i=1}^{n}\left[2\binom{n-1}{i} + k^2\binom{n-1}{i-1}\right]k^{n-1-i}c_{k,i-1} - k^n \\ &= kf_{k,n-1} - k^n \\ &\quad + \sum_{i=1}^{n}\left[2\binom{n-1}{i} + k^2\binom{n-1}{i-1} + 2\binom{n-1}{i-1} - 2\binom{n-1}{i-1}\right]k^{n-1-i}c_{k,i-1} \\ &= kf_{k,n-1} + \sum_{i=1}^{n}\left[(k^2-2)\binom{n-1}{i-1} + 2\binom{n}{i}\right]k^{n-1-i}c_{k,i-1} - k^n \\ &= kf_{k,n-1} - k^n \\ &\quad + 2\sum_{i=1}^{n}\binom{n}{i}k^{n-1-i}c_{k,i-1} + (k^2-2)\sum_{i=0}^{n-1}\binom{n-1}{i}k^{n-2-i}c_{k,i} \end{aligned}$$

$$\begin{aligned} f_{k,n} &= kf_{k,n-1} + f_{k,n+1}/k - 2f_{k,n} + k^n + (k^2-2)f_{k,n-1}/k - k^n, \\ kf_{k,n} &= k^2f_{k,n-1} + f_{k,n+1} - 2kf_{k,n} + (k^2-2)f_{k,n-1}. \end{aligned} \tag{5.5}$$

By substituting the above equality (5.4) into (5.5), we get

$$f_{k,n+1} = 3kf_{k,n} - 2(k^2-1)f_{k,n-1}.$$

□

Theorem 5.4. *(Binet formula) The characteristic polynomial equation of recurrence formula (5.3) is* $x^2 - 3kx + 2(k^2 - 1) = 0$, *whose solutions are* f_1 *and* f_2. *The Binet formula for the falling* $k-$*binomial transform of k-Jacobsthal Lucas sequence is demonstrated by*

$$f_{k,n} = f_1^n + f_2^n \tag{5.6}$$

where $f_1 = \frac{3k+\sqrt{k^2+8}}{2}$, $f_2 = \frac{3k-\sqrt{k^2+8}}{2}$.
The roots satisfies the following relations:

$$f_1 + f_2 = 3k, \qquad f_1 - f_2 = \sqrt{k^2+8}, \qquad f_1 .\ f_2 = 2(k^2-1).$$

Theorem 5.5. *(Generating function)*

$$f_k(x) = \frac{x}{1 - 3kx + 2(k^2-1)x^2}. \tag{5.7}$$

Proof. If we multiply the equality $f_k(x)$ by $-3kx$ and $2(k^2-1)x^2$, we get the desired result. □

Theorem 5.6. *The combinatorial formula for the falling k-binomial transform of k-Jacobsthal Lucas sequence is*

$$f_{k,n} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2^{n-1}} \binom{n}{2i} (3k)^{n-2i} (k^2+8)^i. \tag{5.8}$$

Theorem 5.7. *Assume that n is a positive integer. Then the sum of the falling k-binomial transform of k-Jacobsthal Lucas sequence is given as*

$$\sum_{i=0}^{p-1} f_{k,mi+n} = \frac{f_{k,n} - (2k^2-2)^n f_{k,m-n} - f_{k,mp+n} + [f_{k,m(p-1)+n}](2k^2-2)^n}{1 - f_{k,m} + (2k^2-2)^n}. \tag{5.9}$$

Triangle of the falling $k-$ binomial transform of the k-Jacobsthal Lucas sequence

In this part we introduce a new triangle of numbers for each k by using the following rules: The left diagonal of the triangle consists of the elements of the k- Jacobsthal Lucas numbers. The other elements of the triangle are the sum of the number to its left and k- times the number diagonally above it to the left. On the right diagonal is the falling $k-$binomial transform of the k-Jacobsthal Lucas sequence.

For example the following triangle is for 3-Jacobsthal Lucas sequence and its falling 3- binomial transform.

2				
3	9			
13	22	49		
45	84	150	297	
161	296	548	998	1889

Evidently, if $k = 1$, the falling 1-binomial transformof Jacobsthal Lucas sequence coincides with the 1-binomial transform.

All the binomial transformations of the classic Jacobsthal Lucas sequence ($k = 1$) are equal.

6. Matrix Form of the Binomial Transforms

Assume that the elements of k-Jacobsthal Lucas sequence is denoted by $S = [c_{k,0},\ c_{k,1}, ...,]^T$ in matrix form. Let $K = diag\left[k^0,\ k^1, ...,\right]^T$ and the Pascal triangle matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & ... \\ 1 & 1 & 0 & 0 & ... \\ 1 & 2 & 1 & 0 & ... \\ 1 & 3 & 3 & 1 & ... \\ & & & & \end{bmatrix}.$$

We want to find the matrices of binomial transforms of $k-$Jacobsthal Lucas sequences by using S, P, K. We define the matrix of binomial transforms of $k-$Jacobsthal Lucas sequence as $B = [b_{k,0},\ b_{k,1}, ...,]^T$, the matrix of $k-$ binomial transforms of $k-$Jacobsthal Lucas sequence as $W = [w_{k,0},\ w_{k,1}, ...,]^T$, the matrix of rising $k-$binomial transforms of $k-$Jacobsthal Lucas sequence as $R = [r_{k,0},\ r_{k,1}, ...,]^T$, and finally the matrix of falling $k-$binomial transforms of $k-$Jacobsthal Lucas sequence as $F = [f_{k,0},\ f_{k,1}, ...,]^T$.

Binomial transform B, k-binomial transform W, rising $k-$binomial transform R, falling $k-$binomial transform F, are satisfied the following relations

$$B = P.S, \qquad W = K.P.S, \qquad R = P.K.S, \qquad F = K.P.K^{-1}.S$$

Both matrices P and K are invertible. And the inverse of P is matrix P^{-1} of entries $(-1)^{i-j}\binom{i}{j}$ and the inverse of K is the diagonal matrix $K^{-1} = diag(k^0, k^{-1}, k^{-2}, ...)$.Then, S satisfies the following relations, so we get the elements of $k-$Jacobsthal Lucas sequence by using these relations:

$$S = P^{-1}.B = P^{-1}.K^{-1}.W = K^{-1}P^{-1}.R = K.P^{-1}.K^{-1}.F.$$

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